<u>Calculus III Review for EMAG</u>

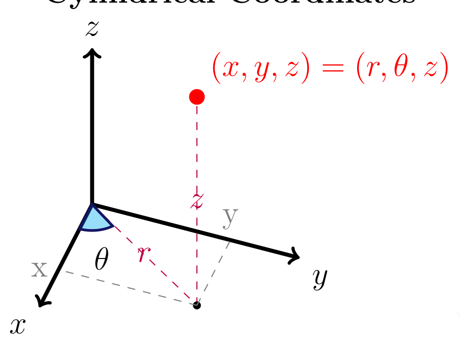
1. **The significant notation difference in cylindrical and spherical coordinates:**

I'd like to mention the logic behind diverging some of notations over time. Read this only if you are curious.

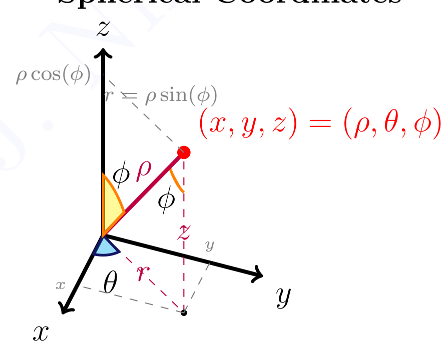
In Math courses, we use θ for azimuthal angle and ϕ for polar angle. In Physics, they use the opposite. There are two reasons for mathematician choice: ① In polar coordinate, you see θ which is the same Greek letter you used in trigonometry and is familiar. ② When finding the Jacobian for spherical coordinates, this notation in **alphabetic order** results in <u>positive</u> $\rho^2 \sin(\phi)$ and switching the order of the angles will create a negative sign which then will be removed by the absolute value sign. (Silly but as far as I know is true.)

In Math courses, we also use r in cylindrical coordinates but in Physics courses, they use ρ.
① You can tell by looking at the two below figures. ② In Physics and Electromagnetism, they prefer using r for the positions vector. In Math we use $\vec{\mathbf{r}}$ for position vector instead.

Cylindrical Coordinates



Spherical Coordinates



https://www.geogebra.org/m/wmhtgy6r

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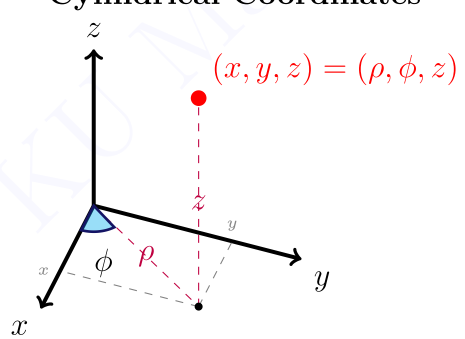
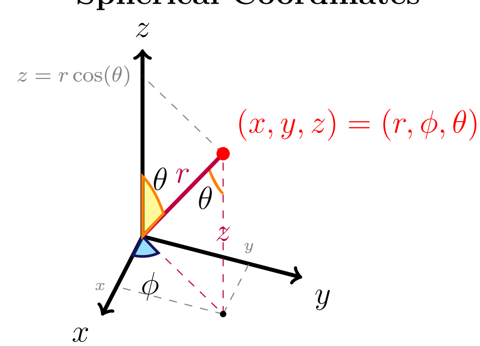
2. **From now on, we use this notation:**

Cylindrical Coordinates



Spherical Coordinates



- Conversion from cylindrical to Cartesian coordinates:

$$x = \rho\cos(\phi) \quad y = \rho\sin(\phi) \quad z = z$$

- Conversion from Cartesian to polar coordinates:

$$\rho^2 = x^2 + y^2 \quad \tan(\phi) = \frac{y}{x} \quad z = z$$

- **Conversion from spherical to Cartesian:**

$$x = r\sin(\theta)\cos(\phi) \quad y = r\sin(\theta)\sin(\phi) \quad z = r\cos(\theta)$$

- **Conversion from Cartesian to spherical:**

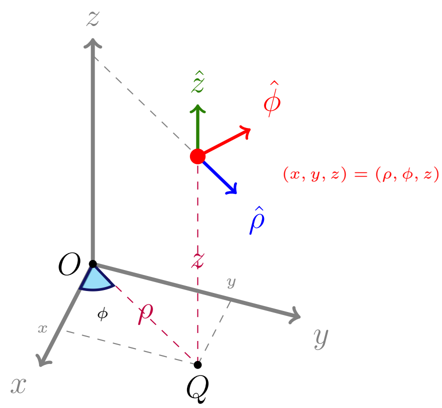
$$r = \sqrt{x^2 + y^2 + z^2} \quad \tan(\phi) = \frac{y}{x} \quad \cos(\theta) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

3. **The base unit vectors in different coordinate Systems:**

In physics, many physical phenomena can be explained using symmetries. Cylindrical coordinates is a major tool for the symmetries around an infinite line and the spherical coordinates is a major tool for the symmetries around a point.

To use the coordinate tools to the fullest (and expedite the computation process), they use unit vectors in the direction of increase in ϕ, r, ρ, θ and z.

Cylindrical Coordinates: The unit vectors in the direction of increase for ρ, ϕ and z:



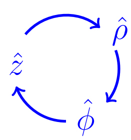
- The base vectors in Cartesian Coordinates:

$$\hat{\rho} = \langle \cos(\phi), \sin(\phi), 0\rangle = \frac{x\vec{\mathbf{i}} + y\vec{\mathbf{j}}}{\sqrt{x^2+y^2}}$$

$$\hat{\phi} = \langle -\sin(\phi), \cos(\phi), 0\rangle = \frac{-y\vec{\mathbf{i}} + x\vec{\mathbf{j}}}{\sqrt{x^2+y^2}}$$

$$\hat{z} = \langle 0, 0, 1\rangle = \vec{\mathbf{k}}$$

- You can easily see the properties:

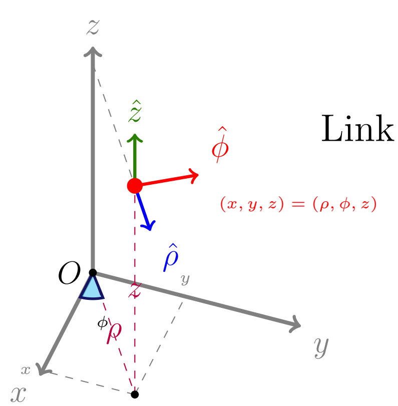


$\hat{\rho}\cdot\hat{\rho} = 1$ $\quad\hat{\phi}\cdot\hat{\phi} = 1$ $\quad\hat{z}\cdot\hat{z} = 1$

$\hat{\rho}\cdot\hat{\phi} = 0$ $\quad\hat{\phi}\cdot\hat{z} = 0$ $\quad\hat{\rho}\cdot\hat{z} = 0$

$\hat{\rho}\times\hat{\phi} = \hat{z}$ $\quad\hat{\phi}\times\hat{z} = \hat{\rho}$ $\quad\hat{z}\times\hat{\rho} = \hat{\phi}$

- Use the above to understand the geometry, e.g. $\hat{\rho}||\overrightarrow{OQ}$, $\hat{\phi}\perp\overrightarrow{OQ}$, $\hat{\phi}||xy$-plane .
- Note that $\hat{\rho}$ and $\hat{\phi}$ depend on the point:



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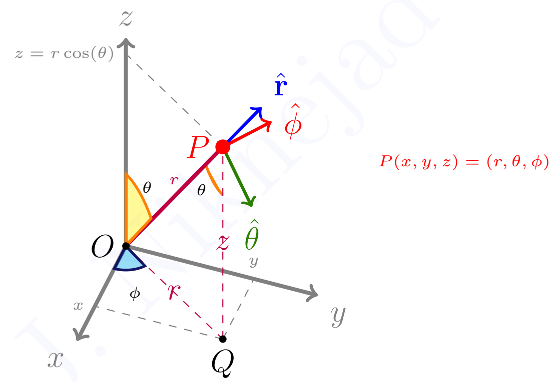
Spherical Coordinates:

The unit vectors in ρ, ϕ and θ directions



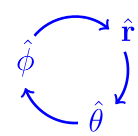
- The base vectors in Cartesian Coordinates:

$\hat{\mathbf{r}} = \langle \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)\rangle = \frac{x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}}{r}$

$\hat{\theta} = \langle \cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), -\sin(\theta)\rangle$

$\hat{\phi} = \langle -\sin(\phi), \cos(\phi), 0\rangle = \frac{-y\vec{\mathbf{i}} + x\vec{\mathbf{j}}}{\sqrt{x^2+y^2}}$

- You can easily see the properties:

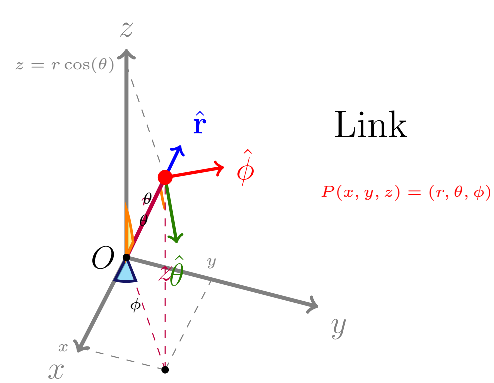


$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1$ $\quad\hat{\phi}\cdot\hat{\phi} = 1$ $\quad\hat{\theta}\cdot\hat{\theta} = 1$

$\hat{\mathbf{r}}\cdot\hat{\phi} = 0$ $\quad\hat{\phi}\cdot\hat{\theta} = 0$ $\quad\hat{\mathbf{r}}\cdot\hat{\theta} = 0$

$\hat{\mathbf{r}}\times\hat{\theta} = \hat{\phi}$ $\quad\hat{\theta}\times\hat{\phi} = \hat{\mathbf{r}}$ $\quad\hat{\phi}\times\hat{\mathbf{r}} = \hat{\theta}$

- Use the above to understand the geometry, e.g. $\hat{\mathbf{r}}||\overrightarrow{OP}$, $\hat{\phi}\perp\overrightarrow{OP}$, $\hat{\phi}||xy$-plane .
- Note that $\hat{\mathbf{r}}$, $\hat{\phi}$ and $\hat{\theta}$ depend on the point:



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4. **Conversion of vector fields from Cartesian to cylindrical or spherical coordinates**

(i) First convert $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$ using these formulas:

Cylindrical:

$$\vec{\mathbf{i}} = \frac{x\hat{\rho} - y\hat{\phi}}{\sqrt{x^2+y^2}}$$

$$\vec{\mathbf{j}} = \frac{y\hat{\rho} + x\hat{\phi}}{\sqrt{x^2+y^2}}$$

$$\vec{\mathbf{k}} = \hat{\mathbf{z}}$$

Spherical:

$$\vec{\mathbf{i}} = \frac{x\left(\sqrt{x^2+y^2}\hat{\mathbf{r}} + z\hat{\theta}\right) - y\sqrt{x^2+y^2+z^2}\hat{\phi}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$$

$$\vec{\mathbf{j}} = \frac{y\left(\sqrt{x^2+y^2}\hat{\mathbf{r}} + z\hat{\theta}\right) + x\sqrt{x^2+y^2+z^2}\hat{\phi}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$$

$$\vec{\mathbf{k}} = \frac{z\hat{\mathbf{r}} - \sqrt{x^2+y^2}\hat{\theta}}{\sqrt{x^2+y^2+z^2}}$$

(ii) Then convert x, y and z using these formulas:

Cylindrical:

$$x = \rho\cos(\phi)$$
$$y = \rho\sin(\phi)$$
$$z = z$$

Spherical:

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

(iii) Last, use algebra to find each component in the directions of $\hat{\mathbf{r}}, \hat{\rho}, \hat{\theta}, \hat{\phi}$, and $\hat{\mathbf{z}}$.

Example 1: Convert $\vec{\mathbf{F}}(x, y, z) = 5xz\vec{\mathbf{j}}$ to spherical presentation.

<u>**Solution**</u>:

$$(i)\ \ 5xz\vec{\mathbf{j}} = 5xz\frac{y\left(\sqrt{x^2+y^2}\hat{\mathbf{r}} + z\hat{\theta}\right) + x\sqrt{x^2+y^2+z^2}\hat{\phi}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$$

$$= 5(r\sin(\theta)\cos(\phi))(r\cos(\theta))\frac{r\sin(\theta)\sin(\phi)(r\sin(\theta)\hat{\mathbf{r}} + r\cos(\theta)\hat{\theta}) + r\sin(\theta)\cos(\phi)r\hat{\phi}}{r\sin(\theta)r}$$

$$(ii)\ \ = \underbrace{5r^2\sin^2(\theta)\cos(\theta)\sin(\phi)\cos(\phi)\hat{\mathbf{r}}}_{F_r} + \underbrace{5r^2\sin(\theta)\cos(\theta)\cos^2(\phi)\hat{\theta}}_{F_\theta} + \underbrace{5r^2\sin(\theta)\cos(\theta)\cos^2(\phi)\hat{\phi}}_{F_\phi}$$

$$\frac{5}{2}r^2\sin^2(\theta)\cos(\theta)\sin(2\phi)\hat{\mathbf{r}} + \frac{5}{2}r^2\sin(2\theta)\frac{1+\cos(2\theta)}{2}\hat{\theta} + \frac{5}{2}\sin(2\theta)\frac{1+\cos(2\theta)}{2}r^2\hat{\phi}$$

Example 2: Convert $\vec{\mathbf{F}}(x, y, z) = 5x\vec{\mathbf{i}}$ to cylindrical presentation.

<u>**Solution**</u>:

$$\underset{(i)}{=} 5x\frac{x\hat{\rho} - y\hat{\phi}}{\sqrt{x^2+y^2}} \underset{(ii)}{=} 5\rho\cos(\phi)\frac{\rho\cos(\phi)\hat{\rho} - \rho\sin(\phi)\hat{\phi}}{\rho} \underset{(iii)}{=} \underbrace{5\rho\cos^2(\phi)\hat{\rho}}_{F_\rho} - \underbrace{2.5\rho\sin(2\phi)\hat{\phi}}_{F_\phi}$$

5. **Components of the vector fields in cylindrical and spherical coordinates:**

   Since vectors $\{\hat{\rho},\, \hat{\phi}, \hat{z}\}$ are mutually orthogonal unit vectors,[1] we can express any vector fields in terms of the vector field's component in the direction of the unit vectors. That is, we can write

$$\vec{\mathbf{F}} = F_\rho \hat{\rho} + F_\phi \hat{\phi} + F_z \hat{\mathbf{z}}$$

   The same is true for any vector field's components in the directions $(\hat{\mathbf{r}},\, \hat{\theta},\, \hat{\phi})$.

$$\vec{\mathbf{F}} = F_r \hat{\mathbf{r}} + F_\theta \hat{\theta} + F_\phi \hat{\phi}.$$

6. **Divergence, Gradient, and Curl in cylindrical coordinates:**

   Divergence: $\nabla \cdot \vec{\mathbf{F}} \;= \dfrac{1}{\rho}\dfrac{\partial\left(\rho F_\rho\right)}{\partial \rho} + \dfrac{1}{\rho}\dfrac{\partial F_\phi}{\partial \phi} + \dfrac{\partial F_z}{\partial z}$

   Gradient: $\nabla f \;= \dfrac{\partial f}{\partial \rho}\hat{\rho} + \dfrac{1}{\rho}\dfrac{\partial f}{\partial \phi}\hat{\phi} + \dfrac{\partial f}{\partial z}\hat{\mathbf{z}}$

$$\text{Curl:}\quad \nabla \times \vec{\mathbf{F}} \;= \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

$$= \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial\left(\rho F_\phi\right)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi}\right)\hat{\mathbf{z}}$$

7. **Divergence, Gradient, and Curl in spherical coordinates:**

   Divergence: $\nabla \cdot \vec{\mathbf{F}} \;= \dfrac{1}{r^2}\dfrac{\partial\left(r^2 F_r\right)}{\partial r} + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}\left(F_\theta \sin\theta\right) + \dfrac{1}{r\sin\theta}\dfrac{\partial F_\phi}{\partial \phi}$

   Gradient: $\nabla f \;= \dfrac{\partial f}{\partial r}\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\hat{\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \phi}\hat{\phi}$

   Curl:

$$\nabla \times \vec{\mathbf{F}} \;= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}\left(F_\phi \sin\theta\right) - \frac{\partial F_\theta}{\partial \varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \varphi} - \frac{\partial}{\partial r}\left(rF_\varphi\right)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{\partial F_r}{\partial \theta}\right)\hat{\phi}$$

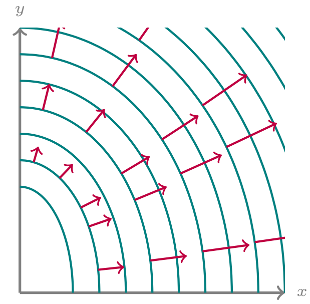
8. **Physical properties of Div., Grad. and Curl:**

   Gradient of a vector field: Let $f(x, y, z)$ be a scalar-valued function. Its gradient is a vector field:

$$\vec{\mathbf{F}} = \nabla f = \left\langle \frac{\partial f}{\partial x},\, \frac{\partial f}{\partial y},\, \frac{\partial f}{\partial z} \right\rangle$$

   - The function f is called a **(scalar) potential function** for $\vec{\mathbf{F}}$.
   - A vector field is called **conservative** if it has a potential function,

[1] This ensures that the three vectors are a basis for the vector space around the point. Refer to your linear algebra course for the definition of basis.



Conservative fields occur naturally in physics, as force fields in which energy is conserved.

Example 1: Consider the scalar function $f(x, y, z) = \dfrac{-k}{\sqrt{x^2 + y^2 + z^2}}$.

(A) Find the gradient of $f(x, y, z)$.

(B) Express the gradient in spherical coordinate notation and in terms of spherical unit vectors $\{\hat{\mathbf{r}}, \hat{\phi}, \hat{\theta}\}$.

(C) What is the potential function for ∇f?

Solution:

(A) $\nabla f = \langle \dfrac{kx}{(x^2+y^2+z^2)^{3/2}}, \dfrac{ky}{(x^2+y^2+z^2)^{3/2}}, \dfrac{kz}{(x^2+y^2+z^2)^{3/2}} \rangle$

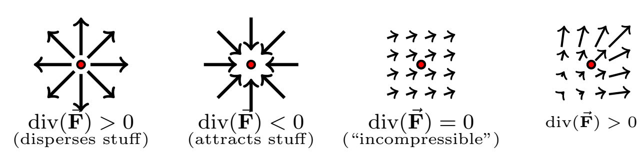
(B)
$$\begin{aligned} \nabla f &= \frac{\langle x, y, z\rangle}{r^3} \\ &= \frac{k\langle r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)\rangle}{r^3} \\ &= \frac{k\langle \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)\rangle}{r^2} \\ &= \frac{k\hat{\mathbf{r}}}{r^2} \end{aligned}$$

(C) $h(x, y, z) = \dfrac{-k}{\sqrt{x^2 + y^2 + z^2}} + C$

If $\vec{\mathbf{F}} = \nabla f$ is a conservative vector field, then at all points P the vector $\vec{\mathbf{F}}(P)$ is orthogonal to the level curve of the potential function f.

Divergence of a vector field:

The divergence of a vector field $\vec{\mathbf{F}}$ at a point P measures how much $\vec{\mathbf{F}}$ disperses "stuff" near P. Div($\vec{\mathbf{F}}$) is a <u>scalar function</u>.



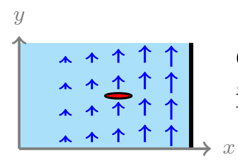
Example 2:

Compute the divergent of $\vec{\mathbf{F}} = \langle x, y, z\rangle$.

Solution:
$div(\vec{\mathbf{F}}) = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$
This answer does not change in different coordinates.

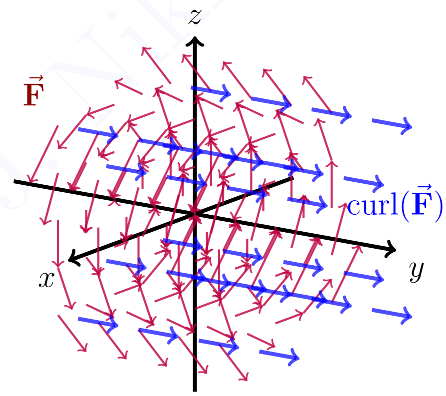
Curl of a vector field: The **curl** of a vector field $\vec{\mathbf{F}}$ measures how $\vec{\mathbf{F}}$ causes objects to rotate.

Example 3: The current in a river is stronger near the banks than in the middle. A boat is anchored near the right bank. What happens to the boat? **It rotates counterclockwise.**



Curl($\vec{\mathbf{F}}$) > 0 is a vector field sticking out of the page toward us.

Example 4: In the figure to the left, you can see that $\vec{\mathbf{F}} = \langle z, 0, -x\rangle$ is rotating and the curl($\vec{\mathbf{F}}$) is a vector field that follows right hand rule. Compute the $\nabla \times \vec{\mathbf{F}}$.



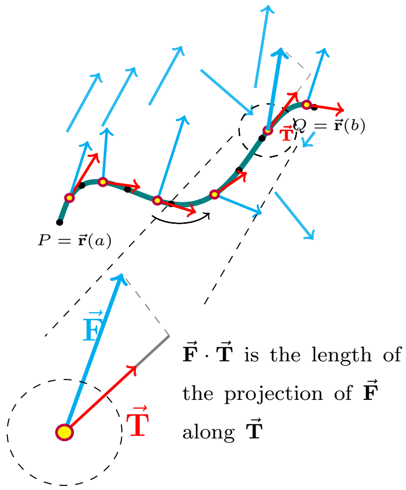
Solution:
$\nabla \times \vec{\mathbf{F}} = \langle 0, 2, 0\rangle$

9. A few properties:

   $\nabla \times (\nabla f) = \vec{0}$ or curl of gradient is zero.

   $\nabla \cdot (\nabla \times \vec{\mathbf{F}}) = 0$ or divergence of the curl is zero.

10. **Line Integral Review:**



Let $\vec{\mathbf{F}}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$ be a vector field and $\mathcal{C}$ a curve parametrized by $\vec{\mathbf{r}}(t)=\langle x(t),y(t),z(t)\rangle$ for $[a,b]$. There are many different ways to write the line integral in Calculus III:

- Vector Differential Form: $\int_{\mathcal{C}}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\int_{\mathcal{C}}\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$
- Parametric Vector Evaluation: $\int_a^b\vec{\mathbf{F}}(x(t),y(t),z(t))\cdot\vec{\mathbf{r}}'(t)\,dt$
- Parametric Scalar Evaluation:

$$\int_a^b(P(x(t),y(t),z(t))x'(t)+Q(x(t),y(t),z(t))y'(t)+R(x(t),y(t),z(t))z'(t))dt$$

- Scalar Differential Form: $\int_{\mathcal{C}}P\,dx+Q\,dy+R\,dz$

Use slightly different notation $\int_{\mathcal{C}}\vec{\mathbf{F}}\cdot d\vec{l}$ or for the contour integral

$$\oint_{\mathcal{C}}\vec{\mathbf{F}}\cdot d\vec{l}$$

Example 1:

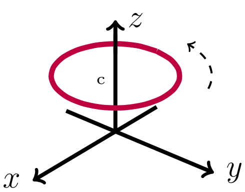
In Electromagnetism, you write $\int Pdx+Qdy+Rdz$ as

$$\begin{aligned}&\int\langle P,Q,R\rangle\cdot\langle dx,dy,dz\rangle\\ =&\int\vec{\mathbf{F}}\cdot\underbrace{(dx\vec{\mathbf{i}}+dy\vec{\mathbf{j}}+dz\vec{\mathbf{k}})}_{\text{Remember this } d\vec{l}.}\end{aligned}$$

In Electromagnetism his notation is mostly used when two of the x,y,z-components are constant. For example, the line element for a line perpendicular to xy-plane, $d\vec{l}=dz\mathbf{k}$. In Calculus III, we used this notation for parametrized curves.

Example 2: A circle in the plane $z=$constant, centered at $(0,0,c)$ with radius r, is shown below.

The circle can be parameterized as $\vec{\mathbf{r}}(t)=\langle r\cos(t),r\sin(t),c\rangle$.



In Calculus III, we find $\vec{\mathbf{r}}'(t)=\langle -r\sin(t),r\cos(t),0\rangle$ and therefore $d\vec{\mathbf{r}}=\langle -r\sin(t),r\cos(t),0\rangle\,dt$.

Then we compute $\int_0^{2\pi}\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\langle -r\sin(t),r\cos(t),0\rangle\,dt$.

In Electromagnetism, we parametrize the circle using the cylindrical coordinates notation, $\vec{l}(\phi)=\langle\rho\cos(\phi),\rho\sin(\phi),c\rangle$. Now the line element in Cartesian base is:

$$\begin{aligned}d\vec{l}(\phi)&=\langle -\rho\sin(\phi),\rho\cos(\phi),0\rangle\,d\phi\\ &=\underbrace{(-\rho\sin(\phi)\,\vec{\mathbf{i}}+\rho\cos(\phi)\,\vec{\mathbf{j}})\,d\phi}_{\text{Remember this one.}}\end{aligned}$$

Then

$$\int_0^{2\pi}\vec{\mathbf{F}}(\vec{l}(\phi))\cdot(-\rho\sin(\phi)\,\vec{\mathbf{i}}+\rho\cos(\phi)\,\vec{\mathbf{j}})\,d\phi$$

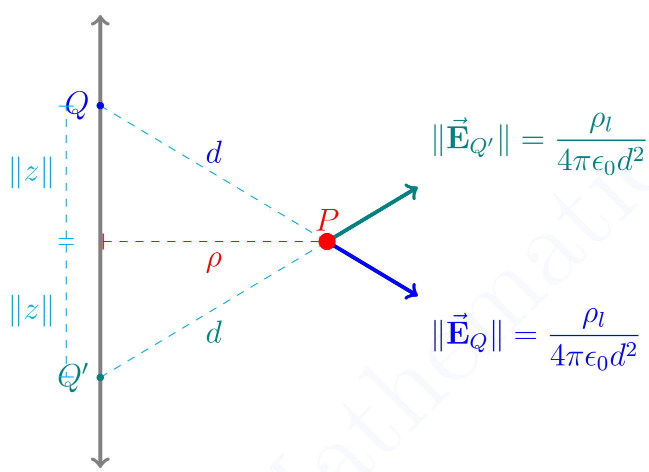
Now for $c=2$ and $\rho=3$, $d\vec{l}=(-3\sin(\phi)\vec{\mathbf{i}}+3\cos(\phi)\vec{\mathbf{j}})d\phi$. This line element is useful for certain vector fields.

Example 3: Express the line element in Example 1 using cylindrical base vectors.

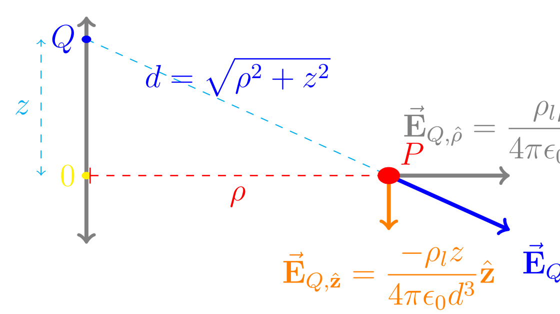
$$
\begin{aligned}
d\vec{l} &= (-\rho\sin(\phi)\underbrace{\frac{x\hat{\rho}-y\hat{\phi}}{\sqrt{x^2+y^2}}}_{\vec{\mathbf{i}}} + \rho\cos(\phi)\underbrace{\frac{y\hat{\rho}+x\hat{\phi}}{\sqrt{x^2+y^2}}}_{\vec{\mathbf{j}}})\,d\phi \\
&= (-\rho\sin(\phi)\frac{\rho\cos(\phi)\hat{\rho}-\rho\sin(\phi)\hat{\phi}}{\rho} + \rho\cos(\phi)\frac{\rho\sin(\phi)\hat{\rho}+\rho\cos(\phi)\hat{\phi}}{\rho})\,d\phi \\
&= (-\rho\sin(\phi)\cos(\phi)\hat{\rho} + \rho\sin^2(\phi)\hat{\phi} + \rho\cos(\phi)\sin(\phi)\hat{\rho} + \rho\cos^2(\phi)\hat{\phi})\,d\phi \\
&= \underbrace{(\rho d\phi)\,\hat{\phi}}_{\text{Remember this } d\vec{l}.}
\end{aligned}
$$

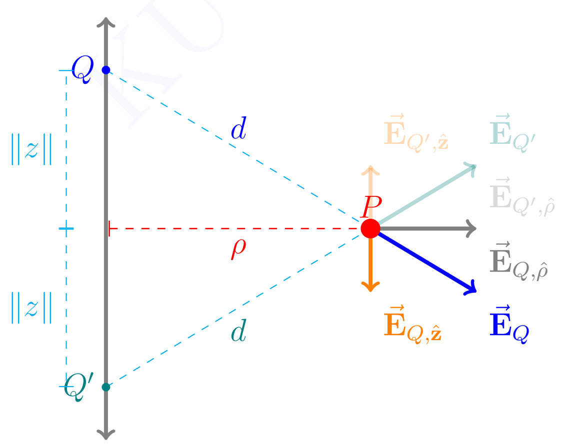
Example 4:

This problem does not need Calculus III to solve but the problem contains the words "line" and "integral" so people wonder. This problem is one of the cases that Columbus Law can be used to evaluate electrical field in closed form (not numerically). Let infinite line l have a uniform charge distribution, ρ_l and point P be a point outside of the line with distance ρ form the line. We use Columbus law for each point charge on the line and we integrate to find the accumulative electric field exerted by those charges at the point P.



Note that the electric field exerted by point charge Q, $\vec{\mathbf{E}}_Q$ decomposes into two component in unit direction perpendicular to the line pointed outward, $\hat{\rho}$, denoted by $\vec{\mathbf{E}}_{Q\hat{\rho}}$ and in unit direction parallel to the line pointed upward, $\hat{\mathbf{z}}$, denoted by $\vec{\mathbf{E}}_{Q\hat{\mathbf{z}}}$:





$$
\begin{aligned}
\int_{-\infty}^{\infty}\vec{\mathbf{E}}_{Q\hat{\mathbf{z}}}dz &= \int_{-\infty}^{\infty}\frac{-\rho_l z}{4\pi\epsilon_0 d^3}\hat{\mathbf{z}}dz \\
&\quad \hat{\mathbf{z}} \text{ does not change w.r.t. } z\text{:} \\
&= \hat{\mathbf{z}}\int_{-\infty}^{\infty}\left(\frac{-\rho_l z}{4\pi\epsilon_0(\rho^2+z^2)^{3/2}}\right)dz \\
&= 0 \qquad \text{Because the integrand is odd.}
\end{aligned}
$$

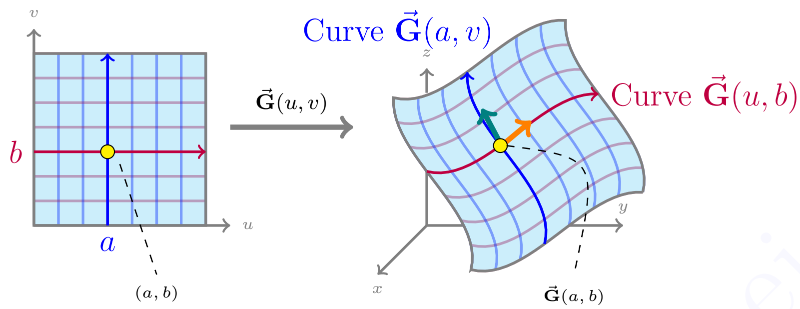
$$
\begin{aligned}
\int_{-\infty}^{\infty}\vec{\mathbf{E}}_{Q\hat{\rho}}dz &= \int_{-\infty}^{\infty}\frac{\rho_l\rho}{4\pi\epsilon_0 d^3}\hat{\rho}dz \\
&\quad \hat{\rho} \text{ does not change w.r.t. } z\text{:} \\
&= \hat{\rho}\int_{-\infty}^{\infty}\left(\frac{\rho_l}{4\pi\epsilon_0(\rho^2+z^2)^{3/2}}\right)dz \\
&= \hat{\rho}\lim_{b\to\infty,c\to\infty}\left(\frac{\rho_l z}{4\pi\epsilon_0\rho^2(\rho^2+z^2)^{1/2}}\right)\Bigg|_{z=-b}^{z=c} = \frac{\rho_l}{2\pi\epsilon_0\rho}\hat{\rho}
\end{aligned}
$$

Finally, $\vec{\mathbf{E}} = \int_{-\infty}^{\infty}\vec{\mathbf{E}}_{Q\hat{\rho}}dz + \int_{-\infty}^{\infty}\vec{\mathbf{E}}_{Q\hat{\mathbf{z}}}dz = \frac{\rho_l}{2\pi\epsilon_0\rho}\hat{\rho}$

11. **Vector Surface Integrals:**

A brief review from Calculus III:

In Calculus 3, we learn how to parametrize general surfaces and find the surface elements. In Electromagnetism, the parametrization of cube, cylinder and sphere are some of the most important ones. So we learn the surface elements for those.



Normal vector: $\vec{\mathbf{N}}(a,b) = \vec{\mathbf{G}}_u(a,b) \times \vec{\mathbf{G}}_v(a,b)$

Unit normal vector: $\vec{\mathbf{n}}(a,b) = \frac{\vec{\mathbf{N}}(a,b)}{\|\vec{\mathbf{N}}(a,b)\|}$

The parametrization $\vec{\mathbf{G}}$ is **regular** if $\vec{\mathbf{n}}$ is well-defined ($\vec{\mathbf{N}} \neq \vec{0}$ always). If a surface has a regular and smooth, then the surface is orientable.

Let $\mathcal{S}$ be an oriented surface with normal vector $\vec{\mathbf{n}}$, and let $\vec{\mathbf{F}}$ be a vector field.

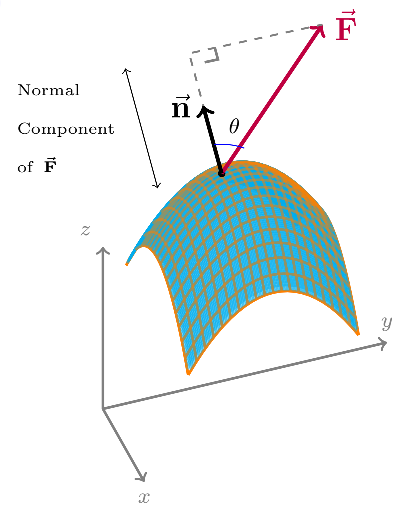
The **normal component** of $\vec{\mathbf{F}}$ with respect to $\mathcal{S}$ is $\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}$.

This is a scalar-valued function on $\mathcal{S}$ that measures the extent to which $\vec{\mathbf{F}}$ is flowing through $\mathcal{S}$ in the direction of $\vec{\mathbf{n}}$.

If $\vec{\mathbf{F}}$ is a continuous vector field defined on an oriented surface $\mathcal{S}$ with unit normal vector $\vec{\mathbf{n}}$, then the **vector surface integral** of $\vec{\mathbf{F}}$ over $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{\mathcal{S}} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS$$

The integral is also called the **flux** of $\vec{\mathbf{F}}$ across $\mathcal{S}$.



If $\mathcal{S}$ has a regular parametrization $\vec{\mathbf{G}}(u,v)$ over $\mathcal{R}$, then

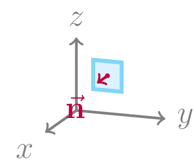
$$\begin{aligned}
\iint_{\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_{\mathcal{S}} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS \\
&= \iint_{\mathcal{R}} \vec{\mathbf{F}}\left(\vec{\mathbf{G}}(u,v)\right) \cdot \frac{\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v}{\|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|} \|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|\, dA \\
&= \iint_{\mathcal{R}} \vec{\mathbf{F}}\left(\vec{\mathbf{G}}(u,v)\right) \cdot (\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v)\, dA \\
&= \iint_{\mathcal{R}} \vec{\mathbf{F}}\left(\vec{\mathbf{G}}(u,v)\right) \cdot \vec{\mathbf{N}}\, dA
\end{aligned}$$

We often call $\vec{\mathbf{N}}dA$ the **Surface element**. Other ways to express a surface elemnt are:
$\vec{\mathbf{N}}\, du\, dv = \vec{\mathbf{N}}\, dv\, du = \|\vec{\mathbf{N}}\|\vec{\mathbf{n}}\, du\, dv = \|\vec{\mathbf{N}}\|\vec{\mathbf{n}}\, dv\, du = \|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|\vec{\mathbf{n}}\, du\, dv$

12. **Cartesian Coordinates**

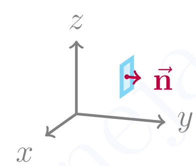
Parametrize a rectangle parallel to yz-plane:

$$\begin{aligned}
\vec{\mathbf{G}}(y,z) &= \langle x, y, z\rangle \quad y\in[a,b],\ z\in[c,d]\\
\vec{\mathbf{G}}_y(y,z) &= \langle 0,\ 1,\ 0\rangle\\
\vec{\mathbf{G}}_z(y,z) &= \langle 0,\ 0,\ 1\rangle\\
\vec{\mathbf{N}}=\vec{\mathbf{G}}_y\times\vec{\mathbf{G}}_z &= \langle 1,\ 0,\ 0\rangle \quad = \boxed{\vec{\mathbf{i}}}
\end{aligned}$$



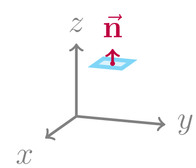
Parametrize a rectangle parallel to xz-plane:

$$\begin{aligned}
\vec{\mathbf{G}}(x,z) &= \langle x, y, z\rangle \quad x\in[a,b],\ z\in[c,d]\\
\vec{\mathbf{G}}_x(x,z) &= \langle 1,\ 0,\ 0\rangle\\
\vec{\mathbf{G}}_z(x,z) &= \langle 0,\ 0,\ 1\rangle\\
\vec{\mathbf{N}}=\vec{\mathbf{G}}_x\times\vec{\mathbf{G}}_z &= \langle 0,\ 1,\ 0\rangle \quad = \boxed{\vec{\mathbf{j}}}
\end{aligned}$$



Parametrize a rectangle parallel to xy-plane:

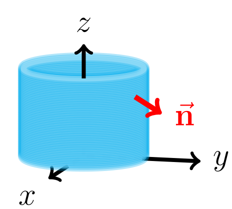
$$\begin{aligned}
\vec{\mathbf{G}}(x,y) &= \langle x, y, z\rangle \quad x\in[a,b],\ y\in[c,d]\\
\vec{\mathbf{G}}_y(x,y) &= \langle 0,\ 1,\ 0\rangle\\
\vec{\mathbf{G}}_z(x,y) &= \langle 1,\ 0,\ 0\rangle\\
\vec{\mathbf{N}}=\vec{\mathbf{G}}_x\times\vec{\mathbf{G}}_y &= \langle 0,\ 0,\ 1\rangle \quad = \boxed{\vec{\mathbf{k}}}
\end{aligned}$$



13. **Cylindrical Coordinates**

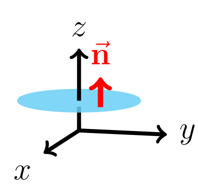
Parametrize the cylindrical shell centered about z-axis, ρ =constant:

$$\begin{aligned}
\vec{\mathbf{G}}(\phi,z) &= \langle \rho\cos(\phi),\ \rho\sin(\phi),\ z\rangle \qquad \phi\in[0,2\pi]\\
\vec{\mathbf{G}}_\phi(\phi,z) &= \langle -\rho\sin(\phi),\ \rho\cos(\phi),\ 0\rangle\\
\vec{\mathbf{G}}_z(\phi,z) &= \langle 0,\ 0,\ 1\rangle\\
\vec{\mathbf{N}}=\vec{\mathbf{G}}_\phi\times\vec{\mathbf{G}}_z &= \langle \rho\cos(\phi),\ \rho\sin(\phi),\ 0\rangle \qquad = \boxed{\rho\hat{\rho}}
\end{aligned}$$



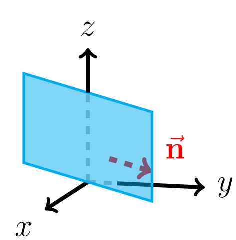
Parametrize the disk centered about z-axis in plane z =constant:

$$\begin{aligned}
\vec{\mathbf{G}}(\rho,\phi) &= \langle \rho\cos(\phi),\ \rho\sin(\phi),\ z\rangle \qquad \phi\in[0,2\pi]\\
\vec{\mathbf{G}}_\rho(\rho,\phi) &= \langle \cos(\phi),\ \sin(\phi),\ 0\rangle\\
\vec{\mathbf{G}}_\phi(\rho,\phi) &= \langle -\rho\sin(\phi),\ \rho\cos(\phi),\ 0\rangle\\
\vec{\mathbf{N}}=\vec{\mathbf{G}}_\rho\times\vec{\mathbf{G}}_\phi &= \langle 0,\ 0,\ \rho\rangle \qquad = \boxed{\rho\hat{\mathbf{z}}}
\end{aligned}$$



Parametrize the rectangle bisected by z-axis, ϕ =constant:

$$\begin{aligned}
\vec{\mathbf{G}}(\rho,z) &= \langle \rho\cos(\phi),\ \rho\sin(\phi),\ z\rangle\\
\vec{\mathbf{G}}_z(\rho,z) &= \langle 0,\ 0,\ 1\rangle\\
\vec{\mathbf{G}}_\rho(\rho,z) &= \langle \cos(\phi),\ \sin(\phi),\ 0\rangle\\
\vec{\mathbf{N}}=\vec{\mathbf{G}}_z\times\vec{\mathbf{G}}_\rho &= \langle -\sin(\phi),\ \cos(\phi),\ 0\rangle \quad = \boxed{\hat{\phi}}
\end{aligned}$$

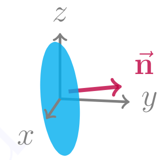


Spherical coordinates

Parametrize sphere centered about origin, r =constant:

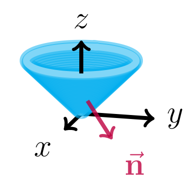
$$\begin{aligned}
\vec{\mathbf{G}}(\phi,\theta) &= \langle r\sin(\theta)\cos(\phi),\, r\sin(\theta)\sin(\phi),\, r\cos(\theta)\rangle\,, && \theta\in[0,\pi], \phi\in[0,2\pi]\\
\vec{\mathbf{G}}_\theta &= \langle r\cos(\theta)\cos(\phi),\, r\cos(\theta)\sin(\phi),\, -r\sin(\theta)\rangle\\
\vec{\mathbf{G}}_\phi &= \langle -r\sin(\theta)\sin(\phi),\, r\sin(\theta)\cos(\phi),\, 0\rangle\\
\vec{\mathbf{N}}=\vec{\mathbf{G}}_\theta\times\vec{\mathbf{G}}_\phi &= \left\langle r^2\sin^2(\theta)\cos(\phi),\, r^2\sin^2(\theta)\sin(\phi),\, r^2\sin(\theta)\cos(\theta)\right\rangle &&= \boxed{r^2\sin(\theta)\hat{\mathbf{r}}}
\end{aligned}$$

Parametrize disk centered about origin in plane ϕ =constant:



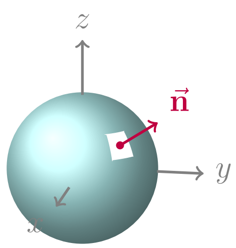
$$\begin{aligned}
\vec{\mathbf{G}}(r,\phi) &= \langle r\sin(\theta)\cos(\phi),\, r\sin(\theta)\sin(\phi),\, r\cos(\theta)\rangle\,, && r\in[0,c], \theta\in[0,\pi]\\
\vec{\mathbf{G}}_r &= \langle \sin(\theta)\cos(\phi),\, \sin(\theta)\sin(\phi),\, \cos(\theta)\rangle\\
\vec{\mathbf{G}}_\phi &= \langle r\cos(\theta)\cos(\phi),\, r\cos(\theta)\sin(\phi),\, -r\sin(\theta)\rangle\\
\vec{\mathbf{N}}=\vec{\mathbf{G}}_\phi\times\vec{\mathbf{G}}_r &= \langle -r\sin(\phi),\, r\cos(\phi),\, 0\rangle &&= \boxed{r\hat{\phi}}
\end{aligned}$$

Parametrize cone/disk centered about z-axis/origin, θ =constant:



$$\begin{aligned}
\vec{\mathbf{G}}(r,\theta) &= \langle r\sin(\theta)\cos(\phi),\, r\sin(\theta)\sin(\phi),\, r\cos(\theta)\rangle\,, && r\in[0,c], \phi\in[0,2\pi]\\
\vec{\mathbf{G}}_\theta &= \langle r\cos(\theta)\cos(\phi),\, r\cos(\theta)\sin(\phi),\, -r\sin(\theta)\rangle\\
\vec{\mathbf{G}}_r &= \langle \sin(\theta)\cos(\phi),\, \sin(\theta)\sin(\phi),\, \cos(\theta)\rangle\\
\vec{\mathbf{N}}=\vec{\mathbf{G}}_\phi\times\vec{\mathbf{G}}_r &= \left\langle r\sin(\theta)\cos(\theta)\cos(\phi),\, r\sin(\theta)\cos(\theta)\sin(\phi),\, -r\sin^2(\theta)\right\rangle &&= \boxed{r\sin(\theta)\,\hat{\theta}}
\end{aligned}$$

Example: Find the flux of the vector field $\vec{\mathbf{F}}(x,y,z)=\langle z,y,x\rangle$ across the unit sphere $x^2+y^2+z^2=1$, oriented outward.



Solution:

Parametrize the unit sphere as usual:

$$\begin{aligned}
\textit{The surface element:}\quad & 1^2\sin(\theta)\,d\theta d\,\phi\,\hat{\mathbf{r}} = \sin(\theta)\,d\theta d\,\phi\hat{\mathbf{r}}\\
\vec{\mathbf{N}}\ & = \sin(\theta)\,\hat{\mathbf{r}}\\
\text{OR } \vec{\mathbf{N}}=\vec{\mathbf{G}}_\theta\times\vec{\mathbf{G}}_\phi\ & = \sin(\theta)\,\langle \sin(\theta)\cos(\phi),\, \sin(\theta)\sin(\phi),\, \cos(\theta)\rangle\\
\vec{\mathbf{F}}(\vec{\mathbf{G}}(\theta,\phi))\ & = \langle \cos(\theta),\, \sin(\theta)\sin(\phi),\, \sin(\theta)\cos(\phi)\rangle
\end{aligned}$$

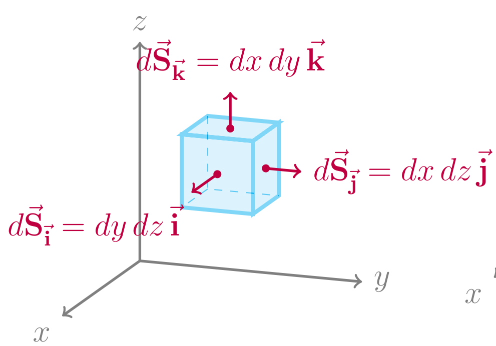
Then compute the vector surface integral:

$$\begin{aligned}
\iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} &= \iint_{\mathcal{R}} \vec{\mathbf{F}}\cdot\left(\vec{\mathbf{G}}_\theta\times\vec{\mathbf{G}}_\phi\right)\,dA\\
&= \int_0^{2\pi}\int_0^{\pi}\left[2\sin^2(\theta)\cos(\theta)\cos(\phi)+\sin^3(\theta)\sin^2(\phi)\right]\,d\theta\,d\phi\\
&= \pi\int_0^{\pi}\sin^3(\theta)\,d\theta \;=\; \frac{4\pi}{3}
\end{aligned}$$

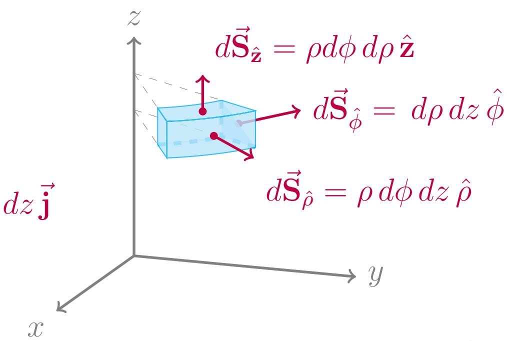
14. **Electromagnetism and surface elements:**

In Electromagnetism, since certain surfaces are used frequently, the surface elements for those surfaces need to be memorized. This is usually done using surfaces of a volume element. Note that $\vec{\mathbf{n}}$ *in these cases are* $\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}, \hat{\mathbf{r}}, \hat{\phi}, \hat{\theta}, \hat{\rho}, \hat{\mathbf{z}}$.

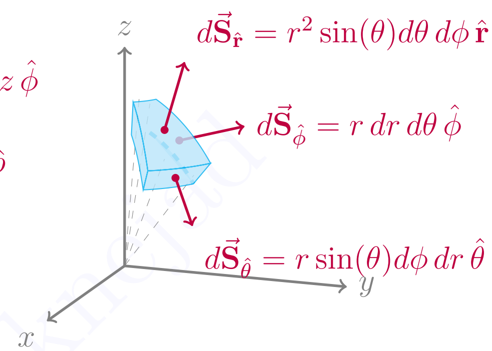
Cartesian:



Cylindrical:

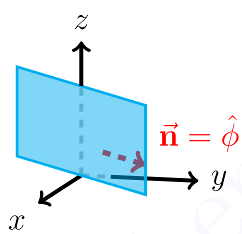


Spherical:



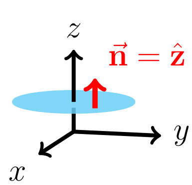
Example: Compute the flux of vector field $\vec{\mathbf{F}}(r, \phi, z) = \hat{\rho}$ through each of the following surfaces in direction $\vec{\mathbf{n}}$ shown in each figure.

(A) The surface inside the rectangle $\phi = \dfrac{\pi}{4}$, $0 \le z \le 2$ and $\rho = 2$.



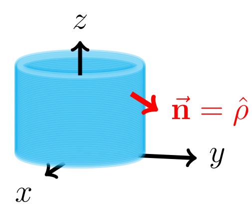
Solution:
$\hat{\rho}$ and $\hat{\phi}$ are perpendicular to each other so the flux is zero.

(B) The surface of the disk $z = 2$ and $\rho = 3$.



Solution:
$\hat{\mathbf{z}}$ and $\hat{\rho}$ are perpendicular to each other so the flux is zero.

(C) The surface of cylinder $\rho = 3$ and $0 \le z \le 4$.



Solution:
The flux is computed as

$$\iint_S \hat{\rho} \cdot d\vec{\mathbf{S}}$$

$$= \int_0^4 \int_0^{2\pi} \hat{\rho} \cdot \underset{\uparrow 4}{\rho}\,\hat{\rho}\,d\phi\,dz \qquad \rho \text{ is constant.}$$

$$= \int_0^3 dz \int_0^{2\pi} 4\,d\rho = 24\pi$$

15. **Stokes' Theorem** Let $\mathcal{S}$ be an oriented surface with smooth, simple closed boundary curves. Let $\vec{\mathbf{F}}$ be a vector field whose components have continuous partial derivatives. Then

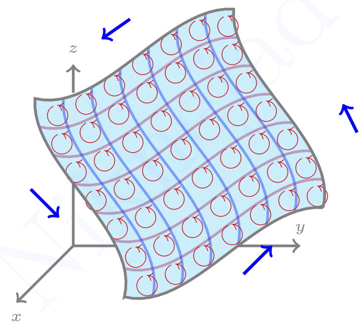
$$\oint_{\partial\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{\mathcal{S}} \text{Curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$$

where the components of $\partial\mathcal{S}$ are oriented using a right-hand-rule orientation.

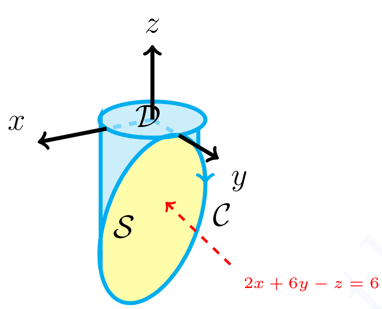
Curl and Circulation in Green and Stokes Green's and Stokes' Theorems both say that if a vector field pushes stuff (counter)clockwise around the boundary of a surface $\mathbb{R}^2$, then it rotates stuff (counter)clockwise in the surface itself.

Another way to define curl is using the Stokes' Theorem on an infinitesimal area:

$$\begin{aligned}
\lim_{\text{area}(\mathcal{S})\to 0} \oint_{\partial\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \lim_{\text{area}(\mathcal{S})\to 0} \iint_{\mathcal{S}} \text{Curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} \\
\text{Curl}(\vec{\mathbf{F}}) \cdot \vec{\mathbf{n}} &= \lim_{\text{area}(\mathcal{S})\to 0} \frac{\oint_{\partial\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}}{\mathcal{S}}
\end{aligned}$$



Example: Find the circulation of $\vec{F}(x,y,z) = \langle y^2, -y, 3z^2 \rangle$ through the ellipse formed from $2x + 6y - 3z = 6$ intersecting $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.



Solution:

Parametrize the surface of $z = \frac{2}{3}x + 2y - 2$ *inside the ellipse with an upwards orientation:*

$$\vec{\mathbf{G}}(r,\theta) = \left\langle r\cos(\theta), r\sin(\theta), \frac{2}{3}r\cos(\theta) + 2r\sin(\theta) - 2 \right\rangle \qquad \vec{\mathbf{G}}_r \times \vec{\mathbf{G}}_\theta = \left\langle \frac{-2r}{3}, -2r, r \right\rangle$$

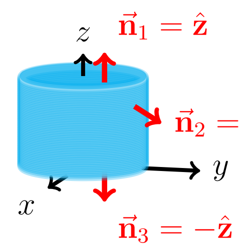
Using Stokes' Theorem,

$$\begin{aligned}
\int_{\partial\mathcal{S}} \vec{F} \cdot d\vec{r} &= \iint_{\mathcal{S}} curl\left(\vec{F}\right) \cdot d\vec{S} = \iint_{\mathcal{S}} \langle 0, 0, -2y \rangle \cdot d\vec{S} \\
&= \int_0^{2\pi} \int_0^1 -2r^2 \sin(\theta)\, dr\, d\theta \quad = 0
\end{aligned}$$

16. **The Divergence Theorem** Let $\mathcal{S}$ be a **closed surface** that encloses a solid $\mathcal{W}$ in $\mathbb{R}^3$. Assume that $\mathcal{S}$ is piecewise smooth and is oriented by normal vectors pointing outside $\mathcal{W}$. Let $\vec{\mathbf{F}}$ be a vector field whose domain contains $\mathcal{W}$. Then:

$$\oiint_{\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_{\mathcal{W}} \operatorname{Div}(\vec{\mathbf{F}})\, dV.$$

Example 1: Find the outward flux of $\vec{\mathbf{F}} = \hat{\rho}$ through the surface of closed cylindrical solid $\rho = 3$ and $0 \le z \le 4$.



Solution:

Method 1:

The **Divergence Theorem** can be find outward flux of $\vec{\mathbf{F}}$ through the surface of a solid:

$$\operatorname{Div}(\vec{\mathbf{F}}) = \frac{1}{\rho}\frac{\partial \rho(1)}{\partial \rho} = \frac{1}{\rho}$$

Compute the volume integral:

$$\iint_{S} \frac{1}{\rho} dV = \int_0^3 \int_0^{2\pi} \int_0^4 \frac{1}{\rho} \rho\, d\rho\, d\phi\, dz$$

$$= 24\pi$$

Method 2:

Comparing the result from Divergence Theorem and the direct computation:

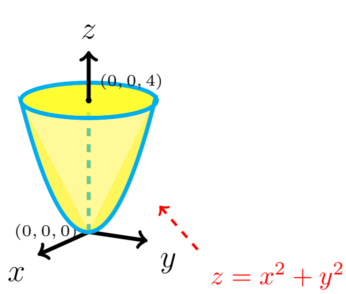
The flux can be computed **directly** as the sum of the outward flux thought the wall and the top and the bottom surfaces. Since the normal vector to the top and bottom surfaces are perpendicular to the vector field, the flux through those two surfaces is zero and the flux through the wall is:

$$= \iint_{S} \hat{\rho} \cdot d\vec{\mathbf{S}}$$

$$= \int_0^4 \int_0^{2\pi} \hat{\rho} \cdot \underset{\uparrow 4}{\rho} \hat{\rho}\, d\phi\, dz \qquad \rho \text{ is constant.}$$

$$= \int_0^3 dz \int_0^{2\pi} 4\, d\rho = 24\pi$$

Example 2: Find the flux of $\vec{F}(x,y,z) = \langle x^2y, xy^2, 2xyz \rangle$ outward through the surface of solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.



Solution:

Using the Divergence Theorem,

$$\iint_{S} \vec{F} \cdot d\vec{S} = \iiint_{\mathcal{T}} \operatorname{div}\left(\vec{F}\right) dV = \iiint_{\mathcal{T}} 6xy\, dV$$

$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 6r^2 \cos(\theta)\sin(\theta)\, r\, dz\, dr\, d\theta = 0$$