Math 601 Homework #4. Due Wed., Feb. 14, 2018

Read Chapter 3 of the textbook (Bierbrauer)

Most of the exercises in the text are quick and easy, and you should try them all, to make sure you get the basics. I will assign some of the less routine exercises.

Write up and hand in solutions to the following exercises. You must give complete justification for all answers.

Exercises (to hand in)

1. Let $C = \text{span}(\{11000, 01111, 11110, 01001\})$.
   (a) Find a generator matrix G for C.
   (b) Find the RREF (reduced row-echelon form) for G.
2. Encode the original message $u = 0101$ using the following generator matrix for a binary linear code C.
$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$
3. Prove: If C is a binary linear code, C is self-dual ($C = C^{\perp}$), and $v \in C$, then $\text{wt}(v)$ is even.
4. Find a 4×8 matrix G in standard form (with identity matrix in the first columns) such that G is the generator matrix for a self-dual $[8, 4, 4]_2$ code.
5. Find a parity-check matrix for the linear code with generator matrix
$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$
6. Find a parity-check matrix for the linear code $C = \text{span}\{110110, 101010, 111001\}$.
7. Find a generator matrix for the code with parity-check matrix
$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$
8. Suppose C is a binary linear code in F^n of dimension k. Give the following.
   (a) $\dim(C^{\perp})$
   (b) size of a generator matrix G (number of rows $\times$ number of columns)
   (c) size of a parity-check matrix H

(d) $|C|$

(e) $|C^{\perp}|$

9. Codes C_1 and C_2 are generated by generator matrices G_1 and G_2 below, respectively. Decide if C_1 and C_2 are equivalent. If they are equivalent, give a reordering of the positions that shows they are equivalent.

$$G_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

10. Prove: The codes C_1 and C_2, generated by generator matrices G_1 and G_2 below, respectively, are not equivalent.

$$G_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

11. Find the minimum distance of the linear code C with parity-check matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Extra Credit:

1. Prove: Let C be a linear code in $\mathbf{F}^n$. Let C^* be the length $n+1$ code obtained by adding a parity-bit: $C^* = \{v_1 v_2 \ldots v_n v_{n+1} : v_1 v_2 \ldots v_n \in C \text{ and } v_1 + v_2 + \cdots v_n + v_{n+1} = 0\}$. If H is a parity-check matrix for C, then H^* is a parity-check matrix for C^*, where

$$H^* = \begin{bmatrix} & & 0 \\ & & 0 \\ & H & \vdots \\ & & 0 \\ 1\,1 \cdots 1 & & 1 \end{bmatrix}.$$

2. Prove: Let $G_1 = [\mathrm{I}\ X_1]$ be a (standard) generator matrix for a binary linear code C. Let X_2 be obtained from X_1 by reordering the rows. Then $G_2 = [\mathrm{I}\ X_2]$ generates a code equivalent to C.