Problem. *The Snake Lemma: Consider the following commutative diagram, which is exact at B and B'.*

$$\begin{array}{ccccccccc} & & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & & \end{array}$$

Prove that there exists an exact sequence

$$Ker(u) \xrightarrow{\overline{\alpha}} Ker(v) \xrightarrow{\overline{\beta}} Ker(w) \xrightarrow{\delta} Coker(u) \xrightarrow{\overline{\alpha'}} Coker(v) \xrightarrow{\overline{\beta'}} Coker(w)$$

Where $\overline{\alpha} = \alpha\restriction_{ker(u)}$, $\overline{\beta} = \beta\restriction_{ker(v)}$, *and* $\overline{\alpha'}, \overline{\beta'}$ *are defined by*

$$\overline{\alpha'}(a' + im(u)) = \alpha'(a') + im(v) \quad \textit{for all } a' \in A'$$
$$\overline{\beta'}(b' + im(v)) = \beta'(b') + im(w) \quad \textit{for all } b' \in B'.$$

Solution. To define δ, suppose $x \in Ker(w)$. Since β is surjective, $\exists y \in B$ such that $\beta(y) = x$. By the commutativity of the diagram, we have that $\beta' v(y) = w\beta(y) = 0$. So $v(y) \in Ker(\beta')$. Since the lower sequence is exact at B', $\exists z \in A'$ with $\alpha'(z) = v(y)$. Note that α' is injective and so z is unique. Let $\delta(x) := z + \text{im}(u)$. It is not clear that this map is well-defined. That is, that the choice of the pre-image of x does not change our choice of coset of $\text{im}(u)$.

Suppose $y, \tilde{y} \in B$ such that $\beta(y) = x = \beta(\tilde{y})$. Let $z, \tilde{z} \in A'$ such that $\alpha'(z) = v(y)$ and $\alpha'(\tilde{z}) = v(\tilde{y})$. Since $\beta(y - \tilde{(y)}) = \beta(y) - \beta(\tilde{y}) = x - x = 0$, and the top sequence is exact at B, $\exists a \in A$ such that $\alpha(a) = y - \tilde{y}$. Therefore

$$\begin{aligned} \alpha'(z - \tilde{z}) &= v(y - \tilde{y}) \\ &= v\alpha(c) \\ &= \alpha' u(c) \end{aligned}$$

Since α' is injective, we conclude that $u(c) = z - \tilde{(z)}$. I.e. that $z - \tilde{z} \in \text{im}(u)$. So $z + \text{im}(u) = \tilde{z} + \text{im}(u)$, and we have that δ is well-defined.

We must also verify that $\overline{\alpha}$ and $\overline{\beta}$ have the promised codomains. Suppose $a \in Ker(u)$. Then $v\alpha(a) = \alpha' u(a) = \alpha'(0) = 0$. So $\overline{\alpha}(a) = \alpha(a) \in Ker(v)$. Now, given $b \in Ker(v)$, $w\beta(b) = \beta' v(b) = \beta'(0) = 0$. So $\overline{\beta}(b) = \beta(b) \in Ker(w)$.

Now for a procession of tedious verifications:

1. $\text{im}(\overline{\alpha}) = Ker(\overline{\beta})$: Let $b \in \text{im}(\overline{\alpha})$. Then $b = \overline{\alpha}(a)$ for some $a \in Ker(u)$. So $\overline{\beta}(b) = \overline{\beta}\overline{\alpha}(a) = \beta\alpha(a) = 0$. Thus $\text{im}(\overline{\alpha}) \subseteq Ker(\overline{\beta})$.
   Let $c \in Ker(\overline{\beta})$. Then $0 = \overline{\beta}(c) = \beta(c)$. So $c \in Ker(\beta) \cap Ker(v)$. By the exactness of the top sequence at B, $c \in \text{im}(\alpha)$. Let $a \in A$ be such that $\alpha(a) = c$. Then

$$\begin{aligned} \alpha' u(a) &= v\alpha(a) \\ &= v(c) \\ &= 0. \end{aligned}$$

But α' is injective, so $u(a) = 0$. Thus $a \in Ker(u)$. Then $c = \alpha(a) = \overline{\alpha}(a)$. I.e. $c \in \text{im}(\overline{\alpha})$. So $Ker(\overline{\beta}) \subseteq \text{im}(\overline{\alpha})$. Hence $\text{im}(\overline{\alpha}) = Ker(\overline{\beta})$.

2. $\text{im}(\overline{\beta}) = Ker(\delta)$: Let $c \in \text{im}(\overline{\beta})$. Then $c = \overline{\beta}(b) = \beta(b)$ for some $b \in Ker(v)$. As was noted in the construction of δ, $\exists z \in A'$ with $\alpha'(z) = v(b) = 0$. Since α' is injective, $z = 0$. So $\delta(c) = z + \text{im}(u) = 0 + \text{im}(u)$. I.e. $c \in Ker(\delta)$. So $\text{im}(\overline{\beta}) \subseteq Ker(\delta)$.
   Now suppose $c \in Ker(\delta)$. Then $\exists b \in B$ such that $\beta(b) = c$. As was noted in the constructions of δ, $\exists z \in A'$ such that $\alpha'(z) = v(b)$. Since $0 + \text{im}(u) = \delta(c) = z + \text{im}(u)$, we have $z \in \text{im}(u)$. Thus $\exists y \in A$ such that $u(y) = z$. Then

$$\begin{aligned} v(\alpha(y)) &= \alpha'(u(y)) \\ &= \alpha'(z) \\ &= v(b). \end{aligned}$$

   Therefore $b - \alpha(y) \in Ker(v)$, and

$$\begin{aligned} \overline{\beta}(b - \alpha(y)) &= \beta(b - \alpha(y)) \\ &= \beta(b) - \beta\alpha(y) \\ &= \beta(b) \\ &= c \end{aligned}$$

   So $c \in \text{im}(\overline{\beta})$ and $Ker(\delta) \subseteq \text{im}(\overline{\beta})$. Hence $\text{im}(\overline{\beta}) = Ker(\delta)$.

3. $\text{im}(\delta) = Ker(\overline{\alpha'})$: Let $a' + \text{im}(u) \in \text{im}(\delta)$. Then $\exists c \in Ker(w)$ and $b \in B$ such that $\beta(b) = c$ and $\alpha'(a') = v(b)$. Therefore

$$\begin{aligned} \overline{\alpha'}(a' + \text{im}(u)) &= \alpha'(a') + \text{im}(v) \\ &= v(b) + (im)(v) \\ &= 0 + \text{im}(v). \end{aligned}$$

   So $a' + \text{im}(u) \in Ker(\overline{\alpha'})$. Thus $\text{im}(u) \subseteq Ker(\overline{\alpha'})$.

   Now let $a' + \text{im}(u) \in Ker(\overline{\alpha'})$. Then $\alpha'(a') \in \text{im}(v)$. So $\exists b \in B$ such that $\alpha'(a') = v(b)$. Since $v(b) \in \text{im}(\alpha')$, and the lower sequence is exact at B', $\beta'(v(b)) = 0$. Then $w(\beta(b)) = \beta'(v(b)) = 0$. So $\beta(b) \in Ker(w)$. Therefore we have $\delta(\beta(b)) = a' + \text{im}(u)$. So $a' + \text{im}(u) \in \text{im}(\delta)$, and $Ker(\overline{\alpha'}) \subseteq \text{im}(\delta)$. Hence $\text{im}(\delta) = Ker(\overline{\alpha'})$.

4. $\text{im}(\overline{\alpha'}) = Ker(\overline{\beta'})$. Let $b' + \text{im}(v) \in \text{im}(\overline{\alpha'})$. Then $\exists a' + \text{im}(u) \in Coker(u)$ such that

$$\begin{aligned} b' + \text{im}(v) &= \overline{\alpha'}(a' + \text{im}(u)) \\ &= \alpha'(a') + \text{im}(v) \end{aligned}$$

   Thus $\exists b \in B$ such that $b' - \alpha'(a') = v(b)$.

$$\begin{aligned} \overline{\beta'}(b' + \text{im}(v)) &= \beta'(b') + \text{im}(w) \\ &= (\beta' v(b) + \beta'\alpha'(a')) + \text{im}(w) \\ &= (w\beta(b) + 0) + \text{im}(w) \\ &= 0 + \text{im}(w). \end{aligned}$$

Now let $b' + \mathrm{im}(v) \in Ker(\overline{\beta'})$. Then $\beta'(b') \in \mathrm{im}(w)$. So $\exists c \in C$ such that $w(c) = \beta'(b')$. Since β is surjective, $\exists b \in B$ such that $\beta(b) = c$. Therefore

$$\begin{aligned} \beta'(v(b)) &= w\beta(b) \\ &= w(c) \\ &= \beta'(b'). \end{aligned}$$

So $b' - v(b) \in Ker(\beta')$. By the exactness of the lower sequence at B', $\exists a' \in A'$ such that $\alpha'(a') = b' - v(b)$. So

$$\begin{aligned} \overline{\alpha'}(a' + \mathrm{im}(u)) &= \alpha'(a') + \mathrm{im}(v) \\ &= (b' - v(b)) + \mathrm{im}(v) \\ &= b' + \mathrm{im}(v). \end{aligned}$$

Thus $b' + \mathrm{im}(v) \in \mathrm{im}(\overline{\alpha'})$ and $Ker(\overline{\beta'}) \subseteq \mathrm{im}(\overline{\alpha'})$. Hence $\mathrm{im}(\overline{\alpha'}) = Ker(\overline{\beta'})$.

And thus we see that the sequence is exact.

Now further assume that α is injective. Suppose $\overline{\alpha}(a) = 0$ for $a \in Ker(u)$. Then $\alpha(a) = 0$. So $a = 0$. Thus $\overline{\alpha}$ is injective as well.

Assume now that β' is surjective. Let $c' + \mathrm{im}(w) \in Coker(w)$. Since β' is surjective, $\exists b' \in B'$ such that $\beta'(b') = c'$ Then

$$\begin{aligned} \overline{\beta'}(b' + \mathrm{im}(v)) &= \beta'(b') + \mathrm{im}(w) \\ &= c' + \mathrm{im}(w). \end{aligned}$$

Thus $\overline{\beta'}$ is surjective.

Problem. *Let* $R := \dfrac{\mathbb{C}[X,Y,Z]}{(X^2+Y^2+Z^2-1)}$, *and* x, y, z *denote the image of* X, Y, Z *respectively. Let* $\pi : R^3 \to R$ *be defined by* $\pi \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ax + by + cz$. *Prove that* $K := Ker(\pi)$ *is a free* $R-$*module.*

Solution. Define $j : R \to R^3$ by $j(r) := \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$. Then $\pi j(r) = rx^2 + ry^2 + rz^2 = r$. So j is a splitting map. Thus $R^3 \simeq K \oplus j(R) = K \oplus \mathrm{span} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. To show that K is free, it suffices to show that there are vectors u, v such that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, u, v$ form a basis form R^3. $\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ iz \\ x - iy \end{pmatrix}$ are such vectors, since

$$\det \begin{pmatrix} x & -i & 0 \\ y & 1 & iz \\ z & 0 & x - iy \end{pmatrix} = x^2 + y^2 + z^2 = 1.$$

We can construct an explicit basis for K as follows. Suppose α is such that

$$\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} - \alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in K.$$

Then

$$0 = (x\ y\ z)\left(\begin{pmatrix}-i\\1\\0\end{pmatrix} - \alpha\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = (y - ix) - \alpha.$$

So $\alpha = (y - ix)$.

Now suppose β is such that

$$\begin{pmatrix}0\\iz\\x-iy\end{pmatrix} - \beta\begin{pmatrix}x\\y\\z\end{pmatrix} \in K.$$

Then

$$0 = (x\ y\ z)\left(\begin{pmatrix}0\\iz\\x-iy\end{pmatrix} - \beta\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = xz - \beta.$$

So $\beta = xz$.

Substituting the values of α and β gives us an explicit basis of K.

$$\begin{pmatrix}ix^2 - xy - i\\ixy - y^2 + 1\\ixz - yz\end{pmatrix}, \begin{pmatrix}-x^2z\\iz - xyz\\x - iy - xz^2\end{pmatrix}.$$