Problem (4)**.** *The Comparison Theorem For Injective Modules:*

Consider the following diagram

$$\begin{array}{rccccccccc}
\mathcal{B}_\bullet : & 0 & \longrightarrow & M & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{\varphi_2} & \cdots \\
& & & \big\downarrow{\scriptstyle \pi} & & & & & & \\
\mathcal{Q}_\bullet : & 0 & \longrightarrow & N & \xrightarrow{\psi_0} & Q_0 & \xrightarrow{\psi_1} & Q_1 & \xrightarrow{\psi_2} & \cdots
\end{array}$$

where $\mathcal{B}_\bullet$ is exact and each Q_i is injective. Prove that there exists a chain map $f : \mathcal{B}_\bullet \to \mathcal{Q}_\bullet$ lifting π. Further, if g is another chain map lifting π, then $f \sim g$.

Solution. By the injectiveness of Q_0, $\exists f_0$ such that the following diagram commutes.

$$\begin{array}{ccc}
M & \xrightarrow{\varphi_0} & B_0 \\
{\scriptstyle \psi_0\pi}\big\downarrow & \swarrow {\scriptstyle \exists f_0} & \\
Q_0 & &
\end{array}$$

That is, $\psi_0\pi = f_0\varphi_0$.

Now inductively, assume that we have defined $f_n : B_n \to Q_n$ such that $\psi_n f_{n-1} = f_n\varphi_n$ for some $n \geq 0$ (For convenience define $f_{-1} = \pi$).

By the injectiveness of Q_{n+1}, $\exists f_{n+1}$ such that the following diagram commutes.

$$\begin{array}{ccc}
B_n & \xrightarrow{\varphi_{n+1}} & B_{n+1} \\
{\scriptstyle \psi_{n+1}f_n}\big\downarrow & \swarrow {\scriptstyle \exists f_{n+1}} & \\
Q_{n+1} & &
\end{array}$$

That is, $\psi_{n+1}f_n = f_{n+1}\varphi_{n+1}$. By induction, $\exists f_n : B_n \to Q_n$ $\forall n \in \mathbb{N}$ that makes the diagram commute.

Now suppose g is a chain map lifting π.

We seek homomorphisms $s_n : B_n \to Q_{n-1}$ such that $f_n - g_n = \psi_n s_n + s_{n+1}\varphi_{n+1}$

Set $s_0 := 0$.

Note that

$$\psi_0\pi = f_0\varphi_0$$
$$\psi_0\pi = g_0\varphi_0$$

Then $(f_0 - g_0) \circ \varphi_0 = 0$. So $ker(\varphi_1) = im(\varphi_0) \subseteq ker(f_0 - g_0)$. Define the maps $\widetilde{\varphi_1}$ and $\widetilde{f_0 - g_0}$ by

$$\widetilde{\varphi_1} : \frac{B_0}{ker(\phi_1)} \to B_1$$
$$b + ker(\varphi_1) \mapsto \varphi_1(b)$$
$$\widetilde{f_0 - g_0} : \frac{B_0}{ker(\varphi_1)} \to Q_0$$
$$b + ker(\varphi_1) \mapsto (f_0 - g_0)(b)$$

These maps are well-defined since $ker(\phi_1)$ is contained in both $ker(\phi_1)$ and $\ker(f_0 - g_0)$. Note also that $\widetilde{\varphi_1}$ is injective.

Thus we have the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \frac{B_0}{ker(\varphi_1)} & \xrightarrow{\widetilde{\varphi_1}} & B_1 \\ & & \big\downarrow \widetilde{f_0 - g_0} & & \\ & & Q_0 & & \end{array}$$

By the injectivity of Q_0, $\exists s_1 : B_1 \to Q_0$ such that $\widetilde{f_0 - g_0} = s_1 \widetilde{\varphi_1}$.

Now let $b \in B_0$.

$$\begin{aligned} (f_0 - g_0)(b) &= (\widetilde{f_0 - g_0})(b + ker(\varphi_1)) \\ &= s_1 \widetilde{\varphi_1}(b + ker(\varphi_1)) \\ &= s_1(\varphi_1(b)) \\ &= s_1 \varphi_1(b). \end{aligned}$$

Thus $f_0 - g_0 = s_1 \varphi_1$.

Inductively assume that for all $1 \leq k \leq n$, we have defined $s_k : B_k \to Q_{k-1}$ such that

$$f_k - g_k = \psi_k s_k + s_{k+1} \varphi_{k+1}.$$

Note that

$$\begin{aligned} (f_n - g_n)\phi_n &= f_n \varphi_n - g_n \varphi_n \\ &= \psi_n f_{n-1} - \psi_n g_{n-1} \\ &= \psi_n (f_{n-1} - g_{n-1}) \\ &= \psi_n (\psi_{n-1} s_{n-1} + s_n \varphi_n) \\ &= \psi_n s_n \varphi_n \end{aligned}$$

Let $\gamma := f_n - g_n - \psi_n s_n$. Then $ker(\varphi_{n+1}) = im(\varphi_n) \subseteq ker(\gamma)$. Define functions $\widetilde{\varphi_{n+1}}$ and $\widetilde{\gamma}$ by

$$\begin{aligned} \widetilde{\varphi_{n+1}} : \frac{B_n}{ker(\varphi_{n+1})} &\to B_{n+1} \\ b + ker(\varphi_{n+1}) &\mapsto \varphi_{n+1}(b) \\ \widetilde{\gamma} : \frac{B_n}{ker(\varphi_{n+1})} &\to Q_n \\ b + ker(\varphi_{n+1}) &\mapsto \gamma b \end{aligned}$$

These maps are well-defined since $ker(\varphi_{n+1})$ is contained in both $ker(\varphi_{n+1})$ and $ker(\gamma)$. Note also that $\widetilde{\varphi_{n+1}}$ is injective.

So we have the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \frac{B_n}{ker(\varphi_{n+1})} & \xrightarrow{\widetilde{\varphi_{n+1}}} & B_{n+1} \\ & & \big\downarrow \widetilde{\gamma} & & \\ & & Q_n & & \end{array}$$

By the injectivity of Q_n, $\exists s_{n+1} : B_{n+1} \to Q_n$ such that $\widetilde{\gamma} = s_{n+1} \widetilde{\varphi_{n+1}}$.

Then for any $b \in B_n$,

$$\begin{aligned}(f_n - g_n)(b) - \psi_n s_n(b) &= \gamma(b) \\ &= \widetilde{\gamma}(b + ker(\varphi_{n+1}) \\ &= s_{n+1}\widetilde{\varphi_{n+1}}(b + ker(\varphi_{n+1}) \\ &= s_{n+1}\varphi_{n+1}(b).\end{aligned}$$

Thus $f_n - g_n = \psi_n s_n + s_{n+1}\varphi_{n+1}$.

By induction, there are maps s_n satisfying the desired equations. Thus $f \sim g$.

Problem (7)**.** *Let Γ be an oriented graph with vertices $\{v_1, \cdots, v_n\}$ and edges $\{e_1, \cdots, e_m\}$ and c connected components. Let C_0 be the free module on the vertices, C_1 be the free module on the edges, and ∂ be the incidence matrix of Γ. Consider the simplicial chain complex, $\Gamma_\bullet$, of Γ:*

$$\Gamma_\bullet : \; 0 \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow 0$$

Compute $H_0(\Gamma_\bullet)$ and $H_1(\Gamma_\bullet)$.

Solution. We seek to compute the rank and nullity of ∂. Suppose $\partial^T w = 0$. Note that ∂^T has exactly one 1 and one -1 in every row corresponding to an edge of Γ. Therefore if there is an edge from v_i to v_j, we must have $w_i - w_j = 0$. So $w_i = w_j$. Suppose now that two vertices v_i and v_j lie in the same connected component. Then there is a path $(v_i, v_{k_1}, \cdots, v_{k_s}, v_j)$. But then by the above observation, $w_i = w_{k_1} = \cdots = w_{k_s} = w_j$. Therefore $w_i = w_j$ if v_i and v_j lie in the same connected component. But then we see that when choosing w we had exactly c choices of values. So $nullity(\partial^T) = c$. And by Rank-Nullity (to be safe I will assume that R is a PID), $rank(\partial^T) = n - c$.

But $rank(\partial) = rank(\partial^T) = n - c$. So again by Rank-Nullity, $nullity(\partial) = m - (n - c) = m - n + c$.

Now we see that

$$H_0(\Gamma_\bullet) \cong \frac{C_0}{im(\partial)} \cong \frac{R^n}{R^{n-c}} \cong R^c$$
$$H_1(\Gamma_\bullet) \cong ker(\partial) \cong R^{m-n+c}$$

In particular, if Γ is connected, $c = 1$ and the original statement of the problem follows.

Problem (12). *Consider the following diagram of R-modules with exact rows*

$$\begin{array}{ccccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D & \xrightarrow{\delta} & E \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & D' & \xrightarrow{\delta'} & E' \end{array}$$

(a) If v and y are injective, and u is surjective, then w is injective.

(b) If v and y are surjective and z is injective, then w is surjective.

(c) If u, v, y, and z are isomorphisms, then w is an isormorphism.

Solution.

(a) Assume v, y are injective, and u is surjective. Suppose $w(c) = 0$. Then $y\gamma(c) = \gamma' w(c) = 0$. So $\gamma(c) \in ker(y) = \{0\}$. Thus $c \in ker(\gamma) = im(\beta)$. Therefore $\exists\ b \in B$ such that $\beta(b) = c$. Then

$$\begin{aligned} \beta' v(b) &= w\beta(b) \\ &= w(c) \\ &= 0. \end{aligned}$$

So $v(b) \in ker(\beta') = im(\alpha')$. And so $\exists\ a' \in A'$ with $\alpha'(a') = v(b)$. Since u is surjective, $\exists\ a \in A$ with $u(a) = a'$. Thus $v(b) = \alpha' u(a) = v\alpha(a)$. Since v is injective, $b = \alpha(a)$. And so $c = \beta(b) = \beta\alpha(a) = 0$. Thus w is injective.

(b) Assume v, y are surjective, and z is injective. Suppose $c' \in C'$. Since y is surjective, $\exists\ d \in D$ such that $y(d) = \gamma'(c')$. So $0 = \delta'\gamma'(c') = \delta' y(d) = z\delta(d)$. Since z is injective, $\delta(d) = 0$. That is $d \in ker(\delta) = im(\gamma)$. So $\exists\ c \in C$ with $\gamma(c) = d$. Then

$$\begin{aligned} \gamma'(c') &= y(d) \\ &= y\gamma(c) \\ &= \gamma' w(c) \end{aligned}$$

Therefore $c' - w(c) \in ker(\gamma') = im(\beta')$. So $\exists\ b' \in B'$ with $\beta'(b') = c' - w(c)$. Since v is surjective, $\exists\ b \in B$ with $v(b) = b'$. Then

$$\begin{aligned} c' - w(c) &= \beta'(b') \\ &= \beta' v(b) \\ &= w\beta(b) \end{aligned}$$

And so $c' = w(c + \beta(b))$, and w is surjective.

(c) Assume $u, v, y,$ and z are isomorphisms. Then by part (a), w is injective. By part (b) w is surjective. Thus w is an isomorphism.

Problem (13)**.** *For $\alpha \in \mathcal{A}$, let $\mathcal{C}_\alpha$ be the chain complex:*

$$\dots \xrightarrow{\partial_{\alpha,i+2}} C_{\alpha,i+1} \xrightarrow{\partial_{\alpha,i+1}} C_{\alpha,i} \xrightarrow{\partial_{\alpha,i}} \dots$$

Define the chain complex $\oplus_\alpha \mathcal{C}_\alpha$ and prove that $H_n(\oplus_\alpha \mathcal{C}_\alpha) \cong \oplus_\alpha H_n(\mathcal{C}_\alpha)$.

Solution. Define the chain complex $\oplus_\alpha \mathcal{C}_\alpha$ by:

$$\dots \xrightarrow{\oplus_\alpha \partial_{\alpha,i+2}} \oplus_\alpha C_{\alpha,i+1} \xrightarrow{\oplus_\alpha \partial_{\alpha,i+1}} \oplus_\alpha C_{\alpha,i} \xrightarrow{\oplus_\alpha \partial_{\alpha,i}} \dots$$

Where

$$(\oplus_\alpha \partial_{\alpha,i})\left((x_\alpha)_{\alpha\in\mathcal{A}}\right) = \left(\partial_{\alpha,i}(x_\alpha)\right)_{\alpha\in\mathcal{A}}.$$

$\oplus_\alpha \mathcal{C}_\alpha$ is a chain complex:

Let $(x_\alpha)_{\alpha\in\mathcal{A}} \in \oplus_\alpha C_{\alpha,i+1}$. Then

$$\begin{aligned}(\oplus_\alpha \partial_{\alpha,i} \circ \oplus_\alpha \partial_{\alpha,i+1})\left((x_\alpha)_{\alpha\in\mathcal{A}}\right) &= \oplus_\alpha \partial_{\alpha,i}\left((\partial_{\alpha,i+1}(x_\alpha))_{\alpha\in\mathcal{A}}\right) \\ &= \left((\partial_{\alpha,i}\partial_{\alpha,i+1})(x_\alpha)\right)_{\alpha\in\mathcal{A}} \\ &= (0)_{\alpha\in\mathcal{A}}.\end{aligned}$$

So $\oplus_\alpha \mathcal{C}_\alpha$ is a chain complex.

Now let $n \in \mathbb{N}$. Define $\varphi : H_n(\oplus_\alpha \mathcal{C}_\alpha) \to \oplus_\alpha H_n(\mathcal{C}_\alpha)$ by:

$$\varphi((x_\alpha)_{\alpha\in\mathcal{A}} + im(\oplus_\alpha \partial_{\alpha,n+1})) = \left(x_\alpha + im(\partial_{\alpha,n+1})\right)_{\alpha\in\mathcal{A}}$$

φ is well-defined: Suppose $(x_\alpha)_{\alpha\in\mathcal{A}} + im(\oplus_\alpha \partial_{\alpha,n+1}) = (y_\alpha)_{\alpha\in\mathcal{A}} + im(\oplus_\alpha \partial_{\alpha,n+1})$. Then

$$\begin{aligned}&(x_\alpha - y_\alpha)_{\alpha\in\mathcal{A}} + im(\oplus_\alpha \partial_{\alpha,n+1}) = im(\oplus_\alpha \partial_{\alpha,n+1}).\\ \implies &(x_\alpha - y_\alpha)_{\alpha\in\mathcal{A}} \in im(\oplus \partial_{\alpha,n+1}).\\ \implies &x_\alpha - y_\alpha \in im(\partial_{\alpha,n+1})\\ \implies &x_\alpha + im(\partial_{\alpha,n+1}) = y_\alpha + im(\partial_{\alpha,n+1})\\ \implies &\left(x_\alpha + im(\partial_{\alpha,n+1})\right)_{\alpha\in\mathcal{A}} = \left(y_\alpha + im(\partial_{\alpha,n+1})\right)_{\alpha\in\mathcal{A}}\\ \implies &\varphi((x_\alpha)_{\alpha\in\mathcal{A}} + im(\oplus_\alpha \partial_{\alpha,n+1})) = \varphi((y_\alpha)_{\alpha\in\mathcal{A}} + im(\oplus_\alpha \partial_{\alpha,n+1})).\end{aligned}$$

And so φ is well-defined.

φ is surjective: To see this let $\left(y_\alpha + im(\partial_{\alpha,n+1})\right)_{\alpha\in\mathcal{A}} \in \oplus_\alpha H_n(\mathcal{C}_\alpha)$. Then $y_\alpha \in ker(\partial_{\alpha,n})$ for all $\alpha \in \mathcal{A}$ and $y_\alpha \in im(\partial_{\alpha,n+1}$ for all but finitely many α. That is, $\exists B \subseteq \mathcal{A}$ with $|B| < \infty$ such that $y_\alpha \notin im(\partial_{\alpha,n+1})$ for all $\alpha \in B$ and $y_\alpha \in im(\partial_{\alpha,n+1})$ for all $\alpha \notin B$. Define

$$x_\alpha := \begin{cases} y_\alpha & \text{if } \alpha \in B \\ 0 & \text{if } \alpha \notin B. \end{cases}$$

Then $(x_\alpha)_{\alpha\in\mathcal{A}} \in \oplus_\alpha(C_{\alpha,n})$. Also, $(x_\alpha)_{\alpha\in\mathcal{A}} \in ker(\oplus_\alpha \partial_{\alpha,n})$. Thus, $(x_\alpha)_{\alpha\in\mathcal{A}} + im(\partial_{\alpha,n+1}) \in H_n(\oplus_\alpha C_\alpha)$. Now,

$$\begin{aligned}\varphi((x_\alpha)_{\alpha\in\mathcal{A}} + im(\partial_{\alpha,n+1})) &= \left(x_\alpha + im(\partial_{\alpha,n+1})\right)_{\alpha\in\mathcal{A}} \\ &= \left(y_\alpha + im(\partial_{\alpha,n+1})\right)_{\alpha\in\mathcal{A}}.\end{aligned}$$

Therefore φ is surjective.

φ *is injective:* Let $(x_\alpha)_{\alpha\in\mathcal{A}} + im(\partial_{\alpha,n+1}) \in H_n(\oplus_\alpha C_\alpha)$ and suppose that

$$(x_\alpha + im(\partial_{\alpha,n+1}))_{\alpha\in\mathcal{A}} = \varphi((x_\alpha)_{\alpha\in\mathcal{A}} + im(\partial_{\alpha,n+1})) = (0 + im(\partial_{\alpha,n+1}))_{\alpha\in\mathcal{A}}.$$

Then $x_\alpha \in im(\partial_{\alpha,n+1})$ for all α. So for each α, $\exists w_\alpha \in C_{\alpha,n+1}$ such that $\partial_{\alpha,n+1}(w_\alpha) = x_\alpha$. Note that since finitely many of the x_α are non-zero, finitely many of the w_α are non-zero. So $(w_\alpha)_{\alpha\in\mathcal{A}} \in \oplus(C_{\alpha,n+1})$, and

$$\begin{aligned}(\oplus_\alpha\partial_{\alpha,n+1})(w_\alpha) &= (\partial_{\alpha,n+1}(w_\alpha))_{\alpha\in\mathcal{A}} \\ &= (x_\alpha)_{\alpha\in\mathcal{A}}\end{aligned}$$

Therefore $(x_\alpha)_{\alpha\in\mathcal{A}} \in im(\partial\alpha, n+1)$ and φ is injective.

Since φ is an isomorphism, $H_n(\oplus_\alpha C_\alpha) \cong \oplus_\alpha H_n(C_\alpha)$.

Problem (15). *The simplicial homology of a tetrahedron, T is given by the homology of the following complex:*

$$0 \longrightarrow R^4 \xrightarrow{\partial_2} R^6 \xrightarrow{\partial_1} R^4 \longrightarrow 0$$

Where ∂_1 *and* ∂_2 *are given by the following matrices:*

$$\partial_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$\partial_2 = \begin{array}{c} \\ 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{array}{c} \begin{array}{cccc} 123 & 124 & 134 & 234 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Compute the simplicial homology of T.

Solution. The following computations were done using a computer algebra system.

Matrix	Rank	Nullity
∂_1	3	3
∂_2	3	1

Therefore:

$$H_0(T) \cong coker(\partial_1) \cong \frac{R^4}{R^3} \cong R$$

$$H_1(T) \cong \frac{ker(\partial_1)}{im(\partial_2)} \cong \frac{R^3}{R^3} \cong 0$$

$$H_2(T) \cong ker(\partial_2) \cong R$$