

xtoaxaca - Extending the Kitagawa-Oaxaca-Blinder Decomposition Approach to Panel Data

January 8, 2021

Hannes Kröger¹, Jörg Hartmann²

Abstract

The Kitagawa-Oaxaca-Blinder decomposition approach has been widely used to attribute group-level differences in an outcome to differences in endowment, coefficients, and their interactions. The method has been implemented for Stata in the popular `oaxaca` program for cross-sectional analyses. In recent decades, however, research questions have been more often focused on the decomposition of group-based differences in change over time, e.g. diverging income trajectories, as well as decomposition of change in differences between groups, e.g. change in the gender pay gap over time. We review five existing methods for the decomposition of changes in group means and contribute an extension which takes an *interventionist* perspective suitable for applications with a clear before-after comparison. These decompositions of levels and changes over time can be implemented using the program `xtoaxaca`, which works as a postestimation command for different regression commands in Stata. It is built to maximize flexibility in modeling and implements all decomposition techniques presented in this paper.

This paper is accepted and currently in press at The Stata Journal

For the most current version of the ado please use (in Stata):

net install <http://xtoaxaca.uni-goettingen.de/xtoaxaca.pkg>, replace

WARNING: Versions downloaded before January 9th, 2021 contain mistakes in the bootstrapping option and resulting standard errors might be incorrectly calculated. Please rerun your analyses with the new version of the command.

¹Socio-economic Panel Study (SOEP) at the German Institute for Economic Research (DIW), Berlin, Germany, hkroeger@diw.de

²Georg-August-Universität Göttingen, Göttingen, Germany, joerg.hartmann@sowi.uni-goettingen.de

1 Introduction

The decomposition of group differences in means (Kitagawa, 1955; Oaxaca, 1973; Blinder, 1973) is a popular tool when researchers seek to attribute such differences to differences in the groups' characteristics and an unexplained part. As such, scholars have applied such decompositions to a variety of topics such as gender income inequality (Blau and Kahn, 2017), happiness (Arrosa and Gandelman, 2016), or obesity (Taber et al., 2016). This approach has also seen numerous extensions over the last decades to suit researchers' needs, such as its application to distributional parameters other than the mean (Freeman, 1980, 1984), to nonlinear models (Fairlie, 2005; Bauer and Sinning, 2008), to quantile regression (Machado and Mata, 2005), selection models (Neuman and Oaxaca, 2003), and other topics (for an overview, see Fortin et al. 2011). In large parts of the applied literature these kinds of decompositions are known as Oaxaca-Blinder decompositions, after two of the three scholars who pioneered these approaches (Oaxaca, 1973; Blinder, 1973). We refer to this way of decomposing group mean differences as the Kitagawa-Oaxaca-Blinder (KOB) approach to reference the earliest and often overlooked contribution to this literature as well (Kitagawa, 1955). As researchers became increasingly interested in research questions involving developments over time, further extensions were developed to decompose the changes in mean group differences between two points in time (Smith and Welch, 1989; Wellington, 1993; Makepeace et al., 1999; DeLeire, 2000; Kim, 2010). These decomposition techniques are based on principles similar to those in the original KOB decomposition and have been primarily used in repeated cross-sectional studies on income gaps. None of those approaches mentioned above have been coherently implemented in Stata. We therefore propose the `xtoaxaca` command. It implements extensions of the original KOB decomposition that focus on the decomposition of changes between groups across time and makes decompositions available for pooled cross-section and panel data.

The command enables a user-friendly implementation of five existing decomposition methods for change (Smith and Welch, 1989; Wellington, 1993; Makepeace et al., 1999; DeLeire, 2000; Kim, 2010) and retains the possibility of applying it to panel instead of only repeated cross-sectional regression models. It provides a generalization of the existing `oaxaca` command (Jann, 2008) to longitudinal and panel data. It also includes a modified version of an existing decomposition approach (Wellington, 1993), which is suitable for easy interpretation of the results under an *interventionist* perspective. This approach is aimed at before-after comparisons such as settings of interventions, policy changes, or (natural) experiments with a post-treatment follow-up. We believe that, in many instances, this perspective is applicable to numerous applied research settings in social sciences and has a more natural interpretation than the existing five decompositions of group differences in change over time.

The purpose of this paper is to introduce both the `xtoaxaca` command for Stata and the new decomposition approach for change. In the next section, we introduce the concepts of longitudinal decompositions of **levels** and **change** of mean group differences over time. Both decompositions are implemented in the `xtoaxaca` command. We then elaborate on cross-sectional decomposition of **levels** – recapitulating the original KOB decomposition – and apply these principles to the decomposition of **levels** over

time using longitudinal and panel data. The fourth section presents a discussion of different approaches to the decomposition of **change**. Thereafter, in the fifth section, we argue that an interventionist perspective on the decomposition of **change** has a crucial advantage in terms of interpretation for applied research. We then describe in the sixth section the syntax and options of the new `xtoaxaca` command. We present three empirical examples in the seventh section. The eighth section summarizes the main limitations of the `xtoaxaca` command. The last section concludes and provides an outlook for future research.

2 Decomposition of Levels and Change

Generally speaking, there are two ways in which we can use mean decomposition techniques with longitudinal data.

The first way examines the contribution of past changes or events to **levels** of outcomes differences between two groups A and B at a single time t_u . The second way of exploiting longitudinal data is the decomposition of group differences in **change** in an outcome between two groups A and B between times s and t . We address both approaches in turn, but highlight the importance of the latter.

Using longitudinal data to determine the contribution of past changes, a typical research question is

1. How much of the difference in an outcome Y between groups A and B at time t is due to the differences in the incidence of a past event X , its different effects, or its different cumulative effects over the last n years?

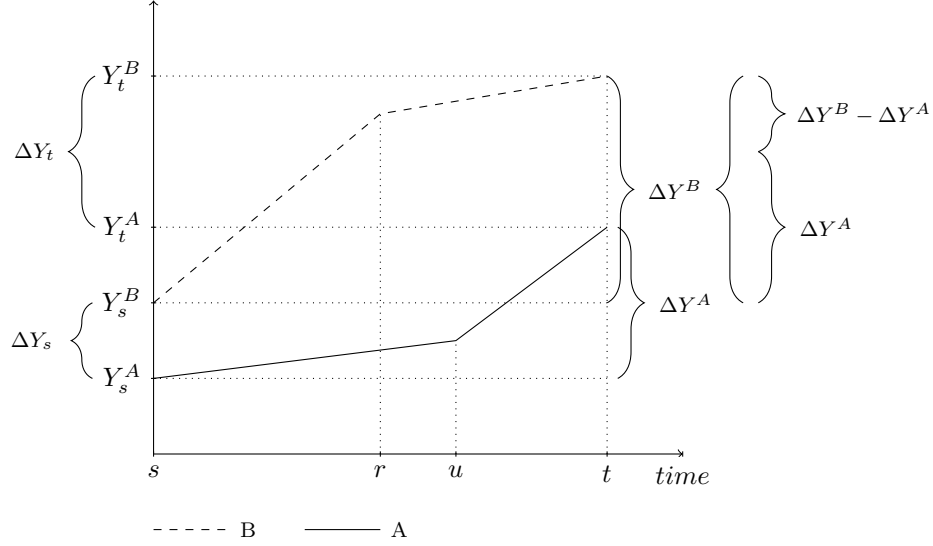
This type of question utilizes longitudinal data by taking individuals' past experiences into account. For instance, we may ask to what extent differences in past unemployment spells and their cumulative impact affect the gender wage gap at time t . In Figure 1, this would translate into the decomposition of the outcome difference at t_u , ΔY_t . It would take the different incidences and effects of events at r and u into account to explain the **level** difference ΔY_t . Analytically, this type of question is still cross-sectional in nature and we can examine it using the traditional KOB decomposition³.

In the following, we denote the repeated decomposition of group differences over time as the longitudinal decomposition of **levels** over time. Decomposing levels is a distinct approach from the second way of using longitudinal data for decomposition, the decomposition of **change**.

If we seek to decompose the **change** in mean group differences over time, we compare the mean group differences between groups A and B between two points in time, s and t , and ask what factors narrowed or widened the outcome difference over time. For example, the wage gap between men and women may have decreased over the last ten years. A researcher might ask whether this occurred due to compositional changes (differential changes in endowments of the groups) or changes in the contribution of

³Possibly taking into account panel attrition (Oaxaca and Choe, 2016).

Figure 1: Decomposition of changes over time.



coefficients of the two groups. Thus, the second kind of longitudinal question can be expressed as follows:

2. How much of the change in differences in an outcome Y between groups A and B and between times t and s is due to changes in the groups' composition or the effects of the explanatory variables?

In Figure 1, this amounts to decomposing the change in group differences between time s and t , $\Delta Y^B - \Delta Y^A$.

As has been shown (Smith and Welch, 1989; Wellington, 1993; Makepeace et al., 1999; DeLeire, 2000; Kim, 2010), this type of question can be answered with repeated cross-sectional data. In this paper, we demonstrate that these existing approaches to the decomposition of **change** can be easily generalized to the use of panel regression models as well. We argue, however, that the existing approaches are not always easy to interpret when asking a set of research questions that falls under what we call *interventionist* perspective. Therefore, we argue that they can be usefully complemented by a new approach to the decomposition of change, which we lay out in detail in section 5.

3 Decomposition of levels

3.1 The KOB decomposition for cross-sectional data

Before we review the existing approaches to the decomposition of **change**, it is useful to recapitulate what the original KOB decomposition does using cross-sectional data (Kitagawa, 1955; Oaxaca, 1973; Blinder, 1973). We will also introduce the notation we

will use throughout the paper. We start with a basic linear regression model for an outcome Y and two groups A and B :

$$Y_t^l = X_t^{l'} \beta_t^l + \epsilon_t^l, \quad E(\epsilon^l) = 0, COV(X, \epsilon) = 0 \quad l \in [A, B] \quad (1)$$

where X represents the matrix of covariates, including the unity vector, while β contains the $k - 1$ coefficients and the constant, t denotes the time, and ϵ is the error-term. The basic KOB decomposition applies to data with one point time and divides the mean outcome difference between the two groups into a part that is explained by differences in the groups' characteristics and an unexplained part (twofold approach). Given that the decompositions of change we introduce in section 4 are often based on the assumption that there are groups differences in the coefficients (threefold decomposition), we illustrate our notation on this approach. Given the outcome difference

$$\begin{aligned} \Delta Y_t &= E(Y_t^A) - E(Y_t^B) \\ &= E(X_t^A)' \beta_t^A - E(X_t^B)' \beta_t^B \end{aligned} \quad (2)$$

and given that $E(X_t^l \beta_t^l + \epsilon_t^l) = E(X_t^l \beta_t^l)$, the outcome difference can be decomposed into

$$\begin{aligned} \Delta Y_t &= E_t + C_t + I_t \\ E_t &= [E(X_t^A) - E(X_t^B)]' \beta_t^B \\ C_t &= E(X_t^B)' (\beta_t^A - \beta_t^B) \\ I_t &= [E(X_t^A) - E(X_t^B)]' (\beta_t^A - \beta_t^B) \end{aligned} \quad (3)$$

E_t is defined as the part of the difference that is due to differences in the groups' characteristics at time t (endowments effect). C_t is the part of the difference that is due to differences in the coefficients at time t . I_t , finally, is the part of the difference at time t that is due to the interaction of the groups' different characteristics and coefficients.

The presented decomposition in equation (10) is a threefold decomposition from the viewpoint of group B , meaning that E_t is weighted by B 's coefficients and that C_t is weighted by B 's characteristics. While this suffices to represent the basic principle of the KOB decomposition, other decompositions, such as a twofold decomposition or a decomposition from the viewpoint of group A , are possible (cp. Jann, 2008; Fortin et al., 2011).

From the viewpoint of group A , we would weight the differences in characteristics and coefficients with the characteristics and coefficients of group A instead of group B :

$$\Delta Y_t = [E(X_t^A) - E(X_t^B)]' \beta_t^A + E(X_t^A)' (\beta_t^A - \beta_t^B) + [E(X_t^B) - E(X_t^A)]' (\beta_t^B - \beta_t^A) \quad (4)$$

For the twofold decomposition, which is the one originally devised by Oaxaca and Blinder, the outcome difference (with group A as the reference) is decomposed by

$$\Delta Y_t = [E(X_t^B) - E(X_t^A)]' \beta_t^A + E(X_t^B)' (\beta_t^B - \beta_t^A) \quad (5)$$

This is also implemented in `xtoaxaca`, but not discussed in detail in this article. Normalization of explanatory variables (Yun, 2005) has also been implemented as is explained in the next sub-section (3.2).

3.2 Normalization of categorical variables

As has been noted in the literature (Jann, 2008; Kim, 2010; Yun, 2005), there is an identification problem when categorical variables are used for decomposition. A widely used solution is to normalize the coefficients of categorical variables by subtracting the variable-specific mean of the coefficient from each category of the variable-specific coefficients and adding all subtracted means to the intercept for decomposition purposes. This yields a new set of coefficients for the decomposition defined as:

$$\tilde{\beta}_j^l = \beta_j^l - \frac{\sum_{k=1}^k \beta_k^l}{k} \quad (6)$$

In this notation, j indicates the j th categorical variable and k the k th category within the j th categorical variable, with β_1 constrained to zero for identification in the original model. The intercept is then defined as:

$$\tilde{\beta}_0^l = \beta_0 + \sum_{i=1}^j \frac{\sum_{k=1}^k \beta_k^l}{k} \quad (7)$$

As can be seen, the basic principle of the original KOB decomposition is to get counterfactual estimates for the outcome for, e.g., group B assuming it had the same endowments or coefficients as group A . This reasoning is retained in the decomposition of **levels** and **change** over time as well.

3.3 Longitudinal data using non-panel regression models

The use of the KOB decomposition with non-panel regression models is unproblematic with longitudinal data as long as time is measured discretely. Discretely in this context means that observations are categorized together to have been observed at the same, undifferentiated time point, for example in waves of a cohort of panel study. In this case, the analysis is identical to a repeated cross-sectional approach. However, the assumption can also be that the time variable is (quasi-)continuous. In this case, we have to define a bandwidth (b) around the time points to allow for an estimation of the endowment part. The wider the bandwidth is, the more reliable the estimate is, because more data points fall into the bandwidth around the time point (and the smaller the standard errors become). Broadening the bandwidth comes at the price of losing sensitivity to time-dependent changes in the endowment. From a nonparametric modeling this is similar to the trade-off which has to be made between bias and variance (Härdle et al., 2012, 28). We estimate:

$$E(X_{t-b;t+b}^l) \quad (8)$$

In the case of the continuous-time variable, the decomposition components become a function of the chosen bandwidth b .

This raises the question of whether the coefficients, like the endowment, can also become dependent on the chosen bandwidth. This has to be decided in line with the choice of the functional form of the time variable in the regression models used for the decomposition. Even when a bandwidth of some kind is used for the endowments, a parametric form can be chosen for the coefficients over time. However, the time variable can be constructed to reflect the bandwidth around pre-specified points in time. In such a case, the coefficients can be estimated nonparametrically for each of these time intervals separately, and the decomposition can be done for each of these time intervals. Under these circumstances coefficients and endowments would be treated in an analogous fashion.

For simplicity's sake, we leave out the bandwidth in the index for the remainder of the paper, but note that it is theoretically necessary and practically possible to set the bandwidth in cases in which time is assumed to be continuous.

3.4 Longitudinal data using panel regression models

Using panel data, we can also estimate β from a panel regression model. Since panel regressions model time-constant individual error terms, a decomposition using panel regression models must take empirical group differences in these time-constant, unobserved variables into account. Thereby, the time constant individual error terms u^l become part of the decomposition of group-level differences.

$$Y_t^l = X_t^{l'} \beta_t^l + u^l + \epsilon_t^l, \quad E(\epsilon_t^l) = 0, COV(X_t, \epsilon_t) = 0 \quad l \in [A, B] \quad (9)$$

Taking the time-constant error-terms into account adds the differences in the expectation of u^l as a fourth component U to the decomposition. This component is not time-dependent. It only comprises differences between groups in the time-constant error terms.⁴

$$\begin{aligned} \Delta Y_t &= E_t + C_t + I_t + U \\ E_t &= [E(X_t^A) - E(X_t^B)]' \beta_t^B \\ C_t &= E(X_t^B)' (\beta_t^A - \beta_t^B) \\ I_t &= [E(X_t^A) - E(X_t^B)]' (\beta_t^A - \beta_t^B) \\ U &= [E(u^A) - E(u^B)] \end{aligned} \quad (10)$$

Accordingly, a decomposition using panel regression models attributes parts of the differences between groups to unobserved factors that do not change within the period of observation.

⁴Note that in applied research an unbalanced panel might mean that taking the expectation of the time-constant error term at different time points t can lead to time varying results if the drop-out is related to the time-constant error terms.

3.5 Model assumptions

3.5.1 Selection and causal identification

It is important to note that any results produced by decompositions of levels or change rely on the assumptions made in the original regression models. This pertains especially to the interpretation of the results in a causal manner. Following a counterfactual interpretations of causality in social sciences (Morgan and Winship, 2014), the estimators for the explanatory variables would need to be unbiased. Only then could the results of any of the decomposition approaches presented here (including the original KOB decomposition) be given interpretations like “how much would the gap between group A and group B be reduced, if group A had the same endowments as group B”. Panel regression models offer some advantages when it comes to arguing that the assumptions for causal interpretation are fulfilled, but still rest on certain assumptions which are often not realistic in applied research (Firebaugh et al., 2013).

For example, from a more technical perspectives it should be noted that in a standard random-effects model (which includes the grouping variable l as a covariate), the assumption is that $cov(l, u) = 0$. This implies that $E[E(u^A) - E(u^B)] = 0$. If our model conforms to this assumption, we should, therefore, expect that the time-constant error terms cannot contribute to the explanation of differences in group levels of the outcome. If we see strong deviations in an empirical estimation of $E[E(u^A) - E(u^B)] = 0$, this might indicate a misspecified model leading to biased coefficient estimates, which in turn could lead to a biased decomposition.

3.5.2 Panel drop-out

With longitudinal data, panel drop-out is a serious issue that may affect the estimation of coefficients (Oaxaca and Choe, 2016) as well as the estimation of the time-specific endowment component. If the results are to be interpreted causally, endogeneity problems resulting from panel drop-out have to be solved when designing the panel regression models before the decomposition is applied.

It is possible for drop-out rates to differ between the groups under study. This can also affect the estimation of the endowments in equation 8. To avoid biased endowments estimators, weights that account for the effects of differential panel-drop out must be constructed and applied in the estimation of the endowments. How these are constructed is beyond the scope of this article; however, we recommend standard procedures from the literature on survey research (Kalton and Flores-Cervantes, 2003; Deming and Stephan, 1940; Kim and Kim, 2007) which can be implemented in Stata using `survwgt` (Winter, 2018).

3.5.3 Functional form of the time variable

In the modeling process, it will be possible to either specify time nonparametrically, estimating an interaction of each time point with the group variable and each decomposition variable, or assume a certain functional form like linear growth. The decomposition will rely on these assumptions made in the modeling process. If the functional form is chosen

incorrectly, this will also affect the decomposition and the results will consequently be biased. This is not only important for the overall growth of the dependent variable, but also for the change in the effect of the decomposition variables over time. A nonparametric approach is less statistically efficient, but has much weaker assumptions than any parametric function and might therefore be preferred if researchers are uncertain in this regard.

3.6 Decomposition in a multi-level framework

Since all panel models can be understood as a special case of multi-level models (with time points nested within units), we believe that `xtoaxaca` can also be used to decompose **levels** and **differences**⁵ between clusters or higher-level units. For this purpose, the time variable needs to reflect the cluster variable (e.g. countries) and should be used in categorical interaction with the desired variables in the model. Differences between units in a multi-level framework and differences between time points can both be seen as a form of difference-in-differences decomposition.

The interpretation of the decomposition of levels over different clusters does not deviate from repeated cross-sectional Kitagawa-Oaxaca-Blinder decompositions, and for this purpose using `xtoaxaca` would not contribute much extra benefit over using `oaxaca`. However, decomposition of **difference in differences** between groups over clusters or higher-level units might be of interest (Blau and Kahn, 1992). The interpretation then refers to whether differences between clusters or higher-level units in the group differences in the outcome can be attributed to cluster differences in the group differences in endowments, coefficients, or their interactions.

4 Decomposition of change

Regardless of whether we have repeated cross-sectional or panel data, given two groups A and B for which we have data for at least two points in time, t and s with $t > s$, the change in the outcome difference between the two groups and between the two points in time is given by

$$\Delta Y = \Delta Y_t - \Delta Y_s \quad (11)$$

Alternatively, changes in outcome differences between two groups and two points in time can be expressed as the difference of group differences over time

$$\begin{aligned} \Delta Y &= \Delta Y_t - \Delta Y_s \\ &= (Y_t^A - Y_t^B) - (Y_s^A - Y_s^B) \\ &= Y_t^A - Y_t^B - Y_s^A + Y_s^B \\ &= Y_t^A - Y_s^A - Y_t^B + Y_s^B \\ &= Y_t^A - Y_s^A - (Y_t^B - Y_s^B) \\ &= \Delta Y^A - \Delta Y^B \end{aligned} \quad (12)$$

⁵The multilevel analogue to **change**

Essentially, changes over time can therefore be expressed as the difference between two KOB-decompositions at different time points.

Several approaches for the decomposition of **change** in group differences over time exist. We cover the five most prominent examples.⁶ These decompositions of **change** have been applied to both points using repeated cross-sectional data. The generalization of the decomposition of **levels** to continuous-time and panel data introduced in the previous section applies in the same way to the decomposition of **change** as it does to the decomposition of **levels**.

4.1 Simple subtraction method (SSM)

The simplest decomposition of **change** is a simple subtraction of the decomposition components of the original KOB-decomposition at time s from the components at time t and is defined in our notation as SSM:

$$\begin{aligned}
\Delta Y &= \Delta Y_t - \Delta Y_s \\
&= E_t + C_t + I_t - (E_s + C_s + I_s) \\
&= (E_t - E_s) + (C_t - C_s) + (I_t - I_s) \\
&= \underbrace{[E(X_t^A) - E(X_t^B)]' \beta_t^B}_{E_t} - \underbrace{[E(X_s^A) - E(X_s^B)]' \beta_s^B}_{E_s} \\
&\quad + \underbrace{E(X_t^B)' (\beta_t^A - \beta_t^B)}_{C_t} - \underbrace{E(X_s^B)' (\beta_s^A - \beta_s^B)}_{C_s} \\
&\quad + \underbrace{[E(X_t^A) - E(X_t^B)]' (\beta_t^A - \beta_t^B)}_{I_t} - \underbrace{[E(X_s^A) - E(X_s^B)]' (\beta_s^A - \beta_s^B)}_{I_s}
\end{aligned}$$

This method is straightforward to calculate, applied for example in DeLeire (2000). The endowment part can be interpreted as the part in the change in the gap that is due to changes in the endowments given changes in the evaluation (coefficient) in the reference group over time. The coefficient part is the part in the change in the gap that is due to changes in the coefficients given changes in the evaluation (endowment) in the reference group over time. The interaction part is difference in the interactions of group differences in coefficients and endowments, and similarly to the original KOB, this component is difficult to interpret and might often be treated as the substantively unexplained part.

The approach has also attracted criticism because it does not estimate the unique contribution of coefficient changes and changes in the variable distributions over time (Kim, 2010). As Kim (2010) shows, the coefficient differences at each time point are weighted by the mean distribution of the endowments at their respective time and, as the endowments likely change over time, the coefficient effect captures these changes.

⁶There are $2 * 2 * 2 = 8$ (endowments and coefficients per two groups per two time points) estimates which make up equation 17. Decompositions can take any number of possible differences and products as part of the decomposition and combine them to form a new kind of decomposition.

Similarly, the endowment effect contains interactions between the coefficient and endowment changes. This kind of criticism applies in a different way for all the decompositions presented here, except for the one by Kim (2010).

4.2 Smith and Welch (1989) (SW)

Smith and Welch (1989) propose a fourfold decomposition of change which is defined in our notation as SW (Smith and Welch, 1989, p. 529):

$$\begin{aligned} \text{i} &= \left\{ [(E(X_t^A) - E(X_t^B)) - (E(X_s^A) - E(X_s^B))] \right\}' \beta_s^B \\ \text{ii} &= [E(X_t^A) - E(X_s^A)]' (\beta_s^A - \beta_s^B) \\ \text{iii} &= [E(X_t^A) - E(X_t^B)]' (\beta_t^B - \beta_s^B) \\ \text{iv} &= E(X_t^A)' [(\beta_t^A - \beta_t^B) - (\beta_s^A - \beta_s^B)] \end{aligned}$$

The components can be given the following interpretation:⁷

- (i) *Main Effect*: The component estimates the predicted change in the outcome between the two groups that can be attributed to the two groups are changing in the endowments (valued at base time s) between time t and s .
- (ii) *Group interaction*: The second component describes the part of change in the endowment of group that is valued differently at time s . Therefore, a secular rise in endowments gives a higher benefit to the group with the higher return to this endowment at time s .
- (iii) *Time interaction*: This component takes the endowment differences at the second time point and attributes change to the change in the coefficient of group B. This would mean that higher returns to an endowment benefit the group with higher endowments at time point t .
- (iv) *Group-time interaction*: The last component attributes change to a change in the differences in the coefficients (returns to endowments) given the initial level of group A. If group A were the disadvantaged group, reduction in the differences to the return to their endowments would close the overall gap between the groups.

4.3 Wellington (1993) (WL)

Wellington (1993) proposes a simple twofold decomposition of change in differences in labor market returns. Her decomposition is defined in our notation as WL (Wellington, 1993, p. 393):

$$\begin{aligned} \text{WL1} &= [(E(X_t^A) - E(X_s^A))]' \beta_t^A - [E(X_t^B) - E(X_s^B)]' \beta_t^B \\ \text{WL2} &= (\beta_t^A - \beta_s^A) E(X_s^A)' - (\beta_t^B - \beta_s^B) E(X_s^B)' \end{aligned}$$

⁷See (Smith and Welch, 1989, p. 529-530) for their interpretation of the change in the race gap in wages.

Wellington (1993, p. 393-394) gives the following description of the two components:

- (WL1) The portion of the change in the gap which can be accounted for by changes in the means if the returns to the independent variables were constant at t (not at baseline s).
- (WL2) The portion of the change in the gap that can be explained by changes in the coefficients (including the constant term) over the period, evaluated at the groups' baseline (s) means.

This approach is the one that is closest to our own addition (see next sub-section) to the set of possible decomposition approaches, but there is a slight but significant difference, as we discuss in section 5.

4.4 A threefold extension of WL (Interventionist)

There is another useful way in which the change in gaps can be decomposed. This is an extension of the WL decomposition which takes the form of a threefold decomposition.

$$\Delta Y^A - \Delta Y^B = \Delta Y = \underbrace{\Delta E + \Delta I}_{\text{WL (1)}} + \underbrace{\Delta C}_{\text{WL (2)}} \quad (13)$$

The three components are named in analogue to the original KOB decomposition. To obtain the endowments effect, we allow only the groups' composition to vary over time and hold the coefficients constant at their initial group-specific levels at time s .

$$\begin{aligned} \Delta E &= E(X_t^A)' \beta_s^A - E(X_s^A)' \beta_s^A - E(X_t^B)' \beta_s^B + E(X_s^B)' \beta_s^B \\ &= [E(X_t^A) - E(X_s^A)]' \beta_s^A - [E(X_t^B) - E(X_s^B)]' \beta_s^B \end{aligned} \quad (14)$$

As can be seen in equation (14), we obtain the endowments component by subtracting the groups' compositional changes over time weighted by their initial coefficients. The endowment component then answers the question: *Given the initial differences in coefficients, how much does the gap between groups change due to the changes in the endowments between the both points (if the coefficients do not change)?*

Similar to the endowments effect, the component attributable to a change in coefficients is obtained by fixing the groups' endowments so that $E(X_t^I) = E(X_s^I)$

$$\begin{aligned} \Delta C &= E(X_s^A)' \beta_t^A - E(X_s^A)' \beta_s^A - E(X_s^B)' \beta_t^B + E(X_s^B)' \beta_s^B \\ &= E(X_s^A)' (\beta_t^A - \beta_s^A) - E(X_s^B)' (\beta_t^B - \beta_s^B) \end{aligned} \quad (15)$$

which denotes the change of the difference due to a change in coefficients (including the constant) over time between the groups given the groups' initial differences in endowments at s . The coefficient component answers the question: *Given the initial differences in endowments, how much does the gap between groups change due to changes in the coefficients (if the endowments do not change)?*

The interaction between the change in endowments and coefficients is the last component of the decomposition:

$$\Delta I = [E(X_t^A) - E(X_s^A)]' (\beta_t^A - \beta_s^A) - [E(X_t^B) - E(X_s^B)]' (\beta_t^B - \beta_s^B)$$

Similar to the original KOB-decomposition, it is difficult to give this component a straightforward interpretation on its own. Additionally, it should be noted that the sub-component of ΔC which is attributable to a change in the intercept is usually also a kind of residual, unexplained by the (change in) X variables in the model.

We can show that our suggested approach is a direct extension of Wellington (1993). First, our ΔC component is exactly the same as WL (2) of the Wellington decomposition.

$$\begin{aligned} WL(2) &= (\beta_t^A - \beta_s^A) E(X_s^A)' - (\beta_t^B - \beta_s^B) E(X_s^B)' \\ &= E(X_s^A)' (\beta_t^A - \beta_s^A) - E(X_s^B)' (\beta_t^B - \beta_s^B) \\ &= \Delta C \end{aligned}$$

In addition, if we add up the endowment and interaction term of the interventionist decomposition, we get the first part of the Wellington decomposition.

$$\begin{aligned} \Delta C + \Delta I &= [E(X_t^A) - E(X_s^A)]' \beta_s^A - [E(X_t^B) - E(X_s^B)]' \beta_s^B + \\ &\quad [E(X_t^A) - E(X_s^A)]' (\beta_t^A - \beta_s^A) - [E(X_t^B) - E(X_s^B)]' (\beta_t^B - \beta_s^B) \\ &= [E(X_t^A) - E(X_s^A)]' \beta_t^A - [E(X_t^B) - E(X_s^B)]' \beta_t^B \\ &= WL(1) \end{aligned}$$

4.5 Makepeace et al. (1999) (MPJD)

Another well-known approach aims at partially mirroring the twofold cross-sectional KOB-decomposition into explained and unexplained in the decomposition of **change**. The authors further divide the explained and unexplained components into a component related to change in endowments (pure) and one aspect related to change in coefficients (price) (Makepeace et al., 1999, p. 539). Their decomposition is defined in our notation as MPJD:

$$\begin{aligned} \Delta E_{pure} &= \left\{ [E(X_t^A) - E(X_s^A)] - [E(X_t^B) - E(X_s^B)] \right\}' \beta_t^A \\ \Delta E_{price} &= [E(X_s^A) - E(X_s^B)]' (\beta_t^A - \beta_s^A) \\ \Delta U_{pure} &= E(X_t^B)' [(\beta_t^A - \beta_s^A) - (\beta_t^B - \beta_s^B)] \\ \Delta U_{price} &= [E(X_t^B) - E(X_s^B)]' (\beta_s^A - \beta_s^B) \end{aligned} \tag{16}$$

4.6 Kim (2010) (KIM)

While many decomposition approaches were developed with particular research questions in mind, Kim (2010) develops the most analytical approach. It yields five components of which two can be attributed purely to the change in endowments and coefficients. He argues that all methods discussed so far in this paper confuse or at least conflate the pure change in endowment and the pure change in the coefficients with interactions of such changes with initial (or current) level differences in coefficients and endowments. We agree with the analysis, but argue in section 5 that for the sake of interpretability this might be a desirable property of a decomposition.

His decomposition (Kim, 2010, p. 629) has five parts and is defined in our notation as:

$$\begin{aligned}
 D1 &= [(\beta_{0,t}^A - \beta_{0,s}^A) - (\beta_{0,t}^B - \beta_{0,s}^B)] + [(\bar{\beta}_t^{*A} - \bar{\beta}_s^{*A}) - (\bar{\beta}_t^{*B} - \bar{\beta}_s^{*B})] \\
 D2 &= \left[\frac{E(X_t^{*A}) + E(X_s^{*A}) + E(X_t^{*B}) + E(X_s^{*B})}{4} \right]' \left[(\tilde{\beta}_t^A - \tilde{\beta}_s^A) - (\tilde{\beta}_t^B - \tilde{\beta}_s^B) \right] \\
 D3 &= \left\{ \frac{[E(X_t^{*A}) - E(X_s^{*A})] + [E(X_t^{*B}) - E(X_s^{*B})]}{2} \right\}' \left[\frac{(\tilde{\beta}_t^A + \tilde{\beta}_s^A)}{2} - \frac{(\tilde{\beta}_t^B + \tilde{\beta}_s^B)}{2} \right] \\
 D4 &= \left\{ [E(X_t^{*A}) - E(X_s^{*A})] - [E(X_t^{*B}) - E(X_s^{*B})] \right\}' \left[\frac{\tilde{\beta}_t^A + \tilde{\beta}_s^A + \tilde{\beta}_t^B + \tilde{\beta}_s^B}{4} \right] \\
 D5 &= \left\{ \frac{[E(X_t^{*A}) + E(X_s^{*A})]}{2} - \frac{[E(X_t^{*B}) + E(X_s^{*B})]}{2} \right\}' \left[\frac{(\tilde{\beta}_t^A - \tilde{\beta}_s^A) + (\tilde{\beta}_t^B - \tilde{\beta}_s^B)}{2} \right]
 \end{aligned}$$

with β^{*l} denoting the coefficient vector in group $l \in \{A, B\}$ without the intercept and $\tilde{\beta}_t^l$ denoting the demeaned coefficient at time t for group $l \in \{A, B\}$.

Following Kim (2010), we can give the following descriptions of the five components:

- D1** *Intercept Effect*: This is purely the difference in differences between group and overall intercepts.
- D2** *Pure Coefficient Effect*: This component measures how much the gap between groups changes due to changes in the coefficients if there were no differences in the endowments at all, neither between groups nor over time.
- D3** *Coefficient Interaction Effect*: This component measures how much the gap between groups changes due to the average change in endowment combined with the difference in the averaged coefficient. It is supposed to capture the aspect of initial level differences in coefficients, which affect the change in the gap in interaction with changes in endowments.
- D4** *Pure Endowment Effect*: This component is the analog to D2 for endowments. It measures how much the gap between groups changes due to changes in the

endowments if there were no differences in the coefficients at all, neither between groups nor over time.

D5 *Endowment Interaction Effect*: This component is the analog to D3, but reverses the role of endowment and coefficients. It measures how much the gap between groups changes due to the average change in coefficients combined with the difference in the averaged endowments. It is supposed to capture the aspect of initial level differences in endowments, which affect the gap in interaction with changes in coefficients.

In order to better understand how the KIM decomposition decomposes group differences over time, we need to take a closer look at the intercepts in D1. So far, decompositions have used the standard matrix notation of multiple regression. This means that the intercept (or constant) β_0^l is part of the coefficient vector:

$$\beta^l = [\beta_0^l, \beta_1^l, \dots, \beta_k^l] = [\beta_0^l, \beta^{*l}]$$

with k being the number of decomposition variables used in the model. Accordingly, the means matrices so far have contained a unity vector which is multiplied with the intercept:

$$E(X^l) = [1, E(x_1^l), \dots, E(x_k^l)] = [1, E(X^{*l})]$$

Kim (2010) uses a different approach by explicitly distinguishing between intercepts and covariates. Therefore, the notation for the coefficients uses $\tilde{\beta}^{*l}$ to represent the coefficient vector without the intercept and X^{l*} as the means vector without the 1. Additionally, the decomposition uses the normalization proposed by Yun (2005):

$$\tilde{\beta}^l = [\tilde{\beta}_0^l, \tilde{\beta}_1^l, \dots, \tilde{\beta}_k^l] = [\beta_0^l + \bar{\beta}^{*l}, \beta^{*l} - \bar{\beta}^{*l}]$$

with $\tilde{\beta}_0^l$ being the intercept for the demeaned coefficients (see equation 7), $\tilde{\beta}_1^l, \dots, \tilde{\beta}_k^l$ being the demeaned coefficient vectors, and $\bar{\beta}^{*l}$ being a vector of variable specific means of the coefficients (see section 3.2).

With $\tilde{\beta}^{*l}$ being the demeaned coefficient vector without the intercept, it follows that:

$$\begin{aligned} E(X^{*l})' \tilde{\beta}^{*l} + \beta_0^l + \bar{\beta}^{*l} &= [1, E(X^{*l})]' [\beta_0^l, \tilde{\beta}^{*l} + \bar{\beta}^{*l}] \\ &= E(X^l)' \beta^l \end{aligned}$$

4.7 Panel models and time-constant error terms

As mentioned in section 3.4, decompositions can also attribute parts of group differences in levels of the outcome to factors that are time constant within the period of observation. The same can be done for change over time (ΔU). However, this only makes sense if we have an unbalanced panel. If the panel is balanced, the expectations of the time-constant error terms cannot change over time and cannot contribute anything to the decomposition of change between groups.

If we see substantial contributions of the time-constant error terms, the data suffers from group-specific panel attrition, which contributes to a change in the group differences in the outcome. In this case, we can add a component ΔU to all of the previous five decompositions as well as to the interventionist decomposition method introduced in the next section.

$$\Delta U = [E(u_t^A) - E(u_s^A)] - [E(u_t^B) - E(u_s^B)]$$

4.8 The relationship between the different types of decompositions

Since all decomposition approaches decompose the same differences in change over time, each decomposition can be expanded and transformed into any other of the existing decompositions. Nevertheless, there are some direct relationships which are worth mentioning and which are also depicted in figure 4.8.⁸

The six decomposition methods can be divided into a heuristic based on a combination of two characteristics. The first one is simply the number of components which is used. Here we see between two and five components. The second characteristic divides the methods into those which conduct decompositions group-wise across time and those which conduct decompositions time-wise across groups. Time-wise across groups means that the differences between groups at one time point are subtracted from another the differences between groups from another time point. In contrast, group-wise across time means that the differences between time-points within one group are subtracted from the differences between time-points within the other group.

WL, KIM, and the interventionist approach fall clearly into the group-wise over time category, while SSM is a time-wise over group approach. For MPJD and SW not all components of their decomposition follow this logic. For MPJD the pure components are group-wise over time and for SW the components (i) and (iv) are time-wise over groups.

In section 4.4, we have already shown that WL can be further divided to yield a threefold decomposition which we label *interventionist* and which we argue has a certain desirable property in contrast to all other approaches, which we elaborate on in section 5.

The simple subtraction method does time-wise subtractions across groups of the components of endowments, coefficients, and interactions. If we were to exchange time points with groups we would end up with a group-wise subtraction across time. This is exactly what is done in the interventionist perspective. So if we were to substitute $t = A$ and $s = A$, the equations would be:

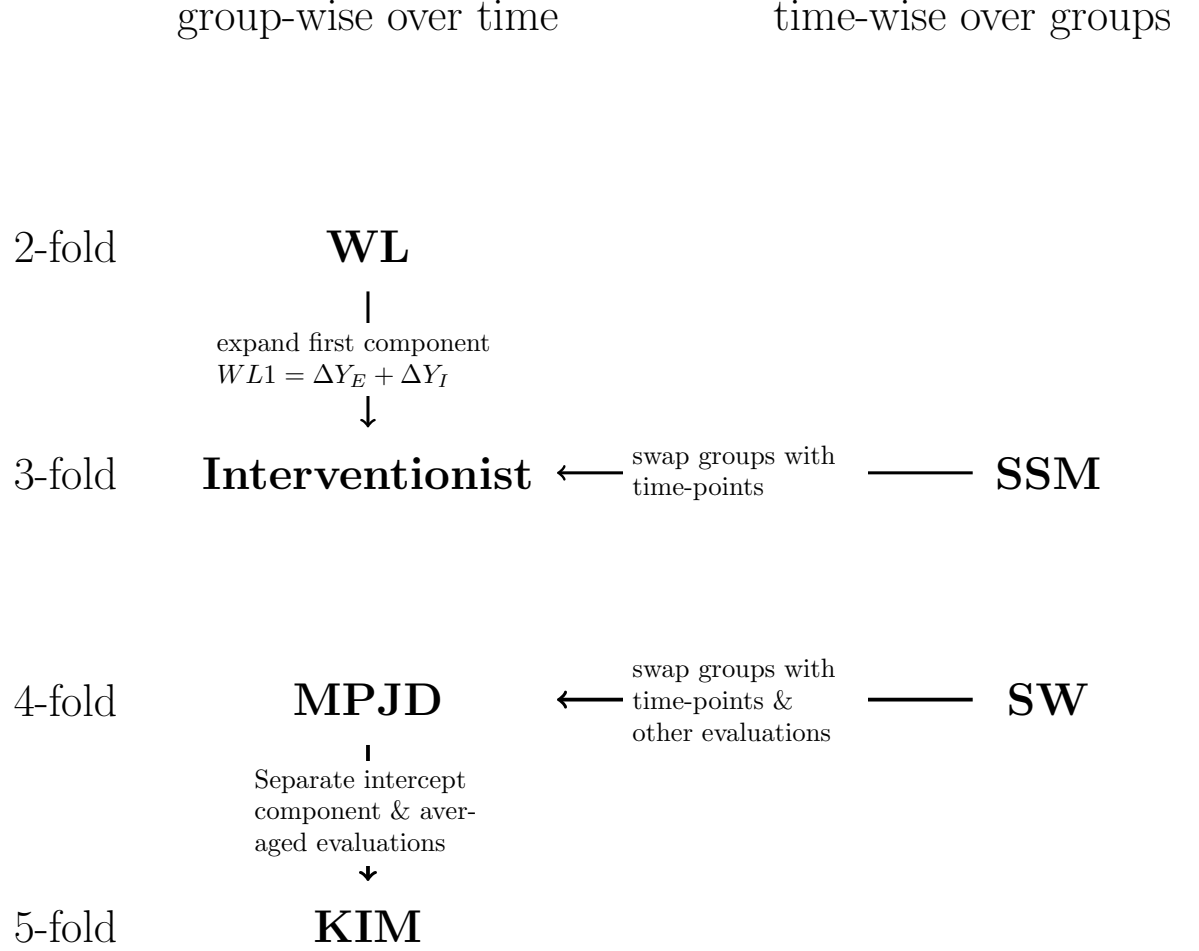
⁸We are very grateful to an anonymous reviewer for pointing out some of these relationships.

$$\begin{aligned}
\Delta E &= E_A - E_B = [E(X_A^t) - E(X_A^s)]' \beta_A^s - [E(X_B^t) - E(X_B^s)]' \beta_A^s \\
\Delta C &= C_A - C_B = E(X_A^s)' (\beta_A^t - \beta_A^s) - E(X_B^s)' (\beta_B^t - \beta_B^s) \\
\Delta I &= I_A - I_B = [E(X_A^t) - E(X_A^s)]' (\beta_A^t - \beta_A^s) - [E(X_B^t) - E(X_B^s)]' (\beta_B^t - \beta_B^s)
\end{aligned} \tag{17}$$

These are the same components as in the interventionist perspective and after changing groups with time-points, the SSM could also be reduced to the twofold decomposition of Wellington (1993).

Note that each decomposition retains its substantively different interpretation even if they can each be transformed into a different decomposition. When interpreting the results they should therefore not be treated as 1:1 substitutes for each other.

Figure 2: Relationship among decomposition approaches



Note: SSM=Simple subtraction method (Section 4.1), WL=(Wellington, 1993) (Section 4.3), MPJD=(Makepeace et al., 1999), (Section 4.5), SW=(Smith and Welch, 1989) (Section 4.2), KIM=(Kim, 2010) (Section 4.6)

5 An interventionist perspective on the decomposition of change

While all of the decomposition approaches that we discussed in the previous section have their uses, we argue that the *interventionist* approach is best suited to address a certain kind of research question which regularly arises in applied social science research and similar fields like epidemiology or public health (see section 4.4). The premise of this approach is that we take the initial differences in **levels** between the groups at the reference time point s as given. We then ask how the difference between the groups could have changed if either the **change** in endowments or the **change** in coefficients had been different. This reflects real-world applications in which either an intervention is designed, or a (natural) experiment or policy change occurs between s and t and is evaluated at time point t . The initial differences in both coefficients and endowments at time point s are seen as inextricably linked to an explanation of **change**, as any change is built on the existing **levels** at s . These initial **levels** are assumed to be beyond intervention and are therefore not subject to counterfactual predictions within the decomposition approach.

There are two combinable types of counterfactual predictions about endowments and coefficients at time s : (1) across groups and (2) across time (and a combination of both). The former would make statements such as “if group A had the same endowment as group B at time point s ”, while the latter would make statements such as “if group A already had the coefficients of time t at time point s ”.

From these two types of counterfactual statements we can derive two formal requirements for a decomposition approach to conform to the assumptions of our *interventionist* perspective. First, no component should contain a term that takes group differences at s (which constitutes a counterfactual prediction at time s across groups). Instead, only differences of within-group **change**⁹ should be used for the decomposition. Second, changes within groups should only be multiplied (valued) at the initial levels (s) or at change ($t - s$), but not at the levels at t (or any function of the levels, endowments, or coefficients at t). If we value at levels of t we make a counterfactual prediction at time s across time.

Except for the *interventionist* approach, all other decompositions described in section 4 violate these assumptions. They are therefore not applicable under an *interventionist* perspective¹⁰. In such a research scenario, it is therefore desirable to use a decomposition that can attribute **changes** in the gaps to **changes** in endowments and coefficients *given the initial differences in levels in the outcome between groups*. We designed the *interventionist* approach to fill exactly this lack of a decomposition approach to the mean based decompositions of **change** in linear models.

Thus, using this decomposition, we seek to answer questions such as how group differences in an outcome would have developed over time had both groups’ characteristics or coefficients changed in the same way. These are counterfactual statements about

⁹Because we assume that we could manipulate the change through an intervention.

¹⁰For SSM it is all three components, for SW, it is components (i), (ii) and (iv), for WL, it is component (1), for MPJD, it is all components, and for KIM it is components (D2) – (D5) that violate the assumptions.

changes that might have been the result of an intervention, policy change, or natural experiment, or any other processes or events that occur between two time points s and t . To this end, we need to find a decomposition that does not violate our two *interventionist* assumptions. This can be achieved by setting the endowments and coefficients at which within-group changes are valued to their groups' initial values.¹¹

6 The `xtoaxaca` command

The `xtoaxca` postestimation command provides decomposition techniques after any kind of regression based growth curve analysis. For the programming of `xtoaxca` Stata version 14.1 was used. Currently `xtoaxaca` supports the analysis of stored models estimated using `reg`, `xtreg`, or `mixed`.

`xtoaxaca` relies heavily on the use of factor-variables. Users are therefore actively encouraged to specify all variables in their regression command explicitly as factor variables (including non-interacted, continuous variables). All variables which are not specified as fv-variables are treated as continuous and the use of dummy variables is **not** supported.

The exception to this rule are interactions among decomposition variables. These must be created as hand-made interaction terms (possibly using dummy variables) and interacted using fv-syntax with time and grouping variables. The example in section 7.5 shows how this can be achieved.

The maximum length of variable names for decomposition variables that is supported by `xtoaxaca` is 20.

A longitudinal decomposition using `xtoaxaca` works in two steps.

1. Estimate a (growth curve) model. This model should condition on the variables that are used as decomposition variables to explain the gap over time between the groups.
2. `xtoaxaca` takes the model and the data set as input to decompose the gaps over time using Stata's `margins` command in the background.

¹¹However, by definition and as mentioned above, it does depend on the differences in the initial *levels* of coefficients, and would be regarded as conflating change in endowments with levels of coefficients in Kim's (2010) analytical perspective.

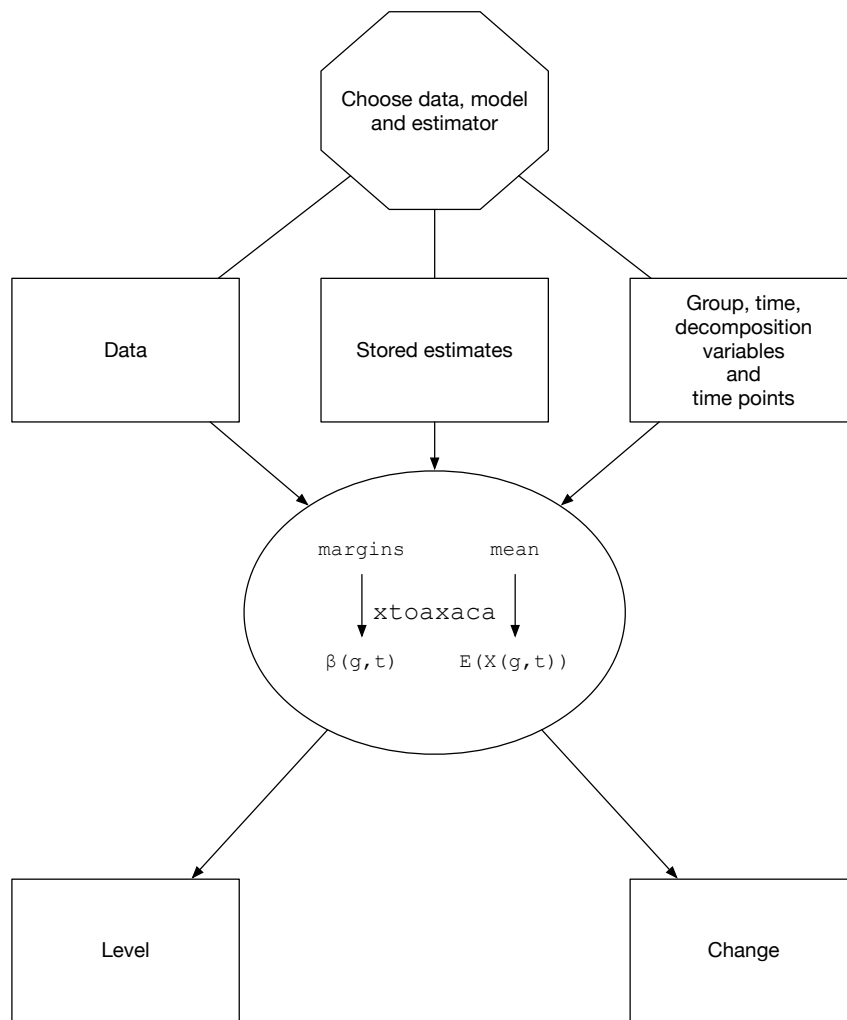


Figure 3: xtoaxaca work flow

6.1 Syntax

The general syntax is:

```

xtoaxaca varlist, groupvar(varname) groupcat(##) timevar(varname)
times(#[#]) timeref(#) timebandwidth(#) model(name)
[basemodel(name)] [weights(varname)] [change(changetype)]
[normalize(varlist)] [noisily] [detail] [forcesample]

```

```
[twofold(weight|pooled|off)] [resultsdata(string [, replace])]
[blocks(blockname1 = (varlist1)[,blockname2 = (varlist2)...])]
[tfweight(#)] [fmt(#)] [nolevels|nochange] [seed(#)]
[bootstrap(#)]
```

varlist contains all decomposition variables and should include all variables interacted with the variable specified in *timevar()*. Otherwise, the decomposition will be incomplete.

groupvar(*varname*) specifies the group variable for decomposition.

groupcat(*##*) identifies the group categories in **groupvar** between which differences across time will be decomposed. Only two codes are allowed.

timevar(*varname*) specifies the time variable in the growth curve model.

times(*numlist*) defines the values of **timevar** at which group differences will be decomposed.

timeref(*#*) defines the reference time point for which change decomposition will be calculated. Is required if the option **change** is specified. It must be one of the time points specified in **times**.

timebandwidth(*#*) defines the time span around the time points in **times**(*numlist*) that is used to estimate the time-specific means of the decomposition variables. This option is only required if **timevar** is specified as a continuous (factor) variable in the models. Default is 0.1.

model(*name*) is the name under which the (growth curve) model is stored. Please ensure that the basic functional form of **timevar** and possible interactions with the **groupvar** variable are correctly specified. Please use variable notation for all variables in the model. Therefore, do not use dummy variables to represent categorical variables (both **timevar**, **groupvar**, and decomposition variables), but use the factor variable prefix "i." instead. All variables without a fv-prefix are treated as continuous by **xtoaxaca**.

basemodel(*name*) is the name under which an optional baseline model is stored. This is not necessary for the decomposition. It can be used to ensure that the functional form and interactions of the **timevar** with the grouping variable are correctly specified. This might be helpful if the code of model contains many complicated interactions which might be prone to errors or typos. Please use factor variable notation for all variables in the baseline model. Therefore, do not use dummy variables to represent categorical variables (both **timevar**, **groupvar** and decomposition variables). Instead, use the factor variable prefix "i.". All variables without a fv-prefix are treated as continuous by **xtoaxaca**.

weights(*varname*) specifies the variable containing (longitudinal) weights for the estimation of the endowments (means).

change(*changetype*) *changetype* may be **ssm**, **interventionist**, **interventionist_twofold**, **smithwelch**, **wellington**, **mpjd**, **none**, or **kim**. **interventionist** yields the decomposition using the *interventionist* perspective presented in this paper (see section 5). **ssm** gives the simple subtraction method described in 4.1; **smithwelch** gives the decomposition by Smith and Welch (1989) described

in 4.2; **wellington** gives the decomposition by Wellington (1993) described in 4.3; **mpjd** gives the Makepeace et al. (1999) decomposition described in section 4.5; **kim** gives the decomposition presented in Kim (2010) described in 4.6. **twofold** gives the decomposition presented in section A.1 in the appendix, which is akin to the original twofold KOB decomposition. The default is **none** which only shows the decomposition of levels.

normalize(*varlist*) *varlist* may be any categorical variables from the decomposition *varlist*. The categorical variables will be normalized according to the method by Yun (2005), as described in the section 3.2

noisily yields more output from the in-between estimation steps of e.g. matrices of means and coefficients over time. If you specify **noisily** with **bootstrap()**, you also need to specify **resultsdata()**.

detail same as **noisily**.

forcesample forces the command to accept differences in the current sample in the dataset and the samples used to estimate the models. If a **basemodel** is specified, it also forces **xtoaxaca** to accept differences in sample size between **model** and **basemodel**. In normal circumstances, it is not recommended to use this option and it should only be used if the output is interpreted in accordance with the differences between the samples.

twofold(weight|pooled|off) gives the twofold decomposition of the level over time. **weight** allows the user to specify a weight for the coefficients. **pooled** uses the method proposed by Oaxaca and Ransom (1994), which is equivalent to a pooled regression model and takes into account the relative amount of variance in the decomposition variables between the two groups found in the data. Default is **off**, which conducts the threefold decomposition.

resultsdata(string [, replace]) saves the main decomposition results in a results data set that can be used for further results-presentation in tables or graphs.

blocks(blockname1 = (varlist1)[,blockname2 = (varlist2)...]) allows the calculation of decomposition in blocks of variables. This is especially useful together with the **bootstrap()** option as it will generate block-specific standard errors. If you specify **blocks()**, you also need to specify **resultsdata()**.

tfweight(#) specifies the weight which is to be given to the first of the two groups specified in **groupcat(##)**. Only allowed if **twofold(weight)** is specified.

fmt() specifies the decimal points that are to be used in the results presentation.

nolevels skips the output of the decomposition of levels.

nochange skips the output of the decomposition of change.

seed() specifies the seed for the bootstrapping option. The default is the seed currently set in the Stata session.

bootstrap() Estimates standard errors via bootstrapping with `iterations` iterations. In addition to the normal results, it returns an `e(dec_b)` and `e(dec_V)` matrix for further processing. If bootstrapped clustered standard errors are to be estimated, the clustering has to be specified in the original regression command using Stata's **cluster(*varlist*)** option.

Table 6.1: Overview of stored results after <code>xtoaxaca</code>		
Method	Stored results	
	absolute values	percentages
Levels	$e(E/C/CE)$	$e(pE/pC/pCE)$
SSM	$e(dE/dC/dCE)$	$e(pdE/pdC/pdCE)$
Interventionist	$e(dE/dC/dCE)$	$e(pdE/pdC/pdCE)$
Smith and Welch (1989)	$e(sw_X)$	$e(psw_X)$
Wellington (1993)	$e(wl_X)$	$e(pwl_X)$
Makepeace et al. (1999)	$e(mpjd_X)$	$e(pmpjd_X)$
Kim (2010)	$e(kim_DX)$	$e(pkim_DX)$
Note: X stands for the different components within one method.		

Table 6.2: Overview of additional results stored after <code>xtoaxaca</code>	
Name	Content
$catX_{coef_{mean}}$	coefficients for category X of group variable
$catX_{endow_{mean}}$	means of decomposition variables for category X of group variable
$change_{model}$	change in the gap predicted by the model
$change_{observed}$	change in the gap as observed in the dataset
$means_{model}$	group means predicted by the model
$means_{observed}$	group means as observed in the dataset
$(p)(d)refmat$	contribution of RE/FE to the decomposition (p for percentage, d for change)
$summary_{levels}$	summary matrix of level decomposition results
$summary_{change}$	summary matrix of change decomposition results

6.2 Storing results

When using the `xtoaxaca` command, the results of the original regression command are retained in Stata's `e()` format. Additionally, the user will find matrices specific to the `xtoaxaca` command which are named after the decomposition method chosen (see table 6.1).

The `resultsdata` option allows users to save the results matrices as a dataset as well, which can be useful if they are to be presented in graphical form. The `resultsdata` option will also store all draws from the bootstrapping procedure if this has been chosen as an option. There are also several additional results which are saved by `xtoaxaca` as described in table 6.2.

7 Example: Increasing household income inequality and composition effects

7.1 Example 1 - Decomposition of changes in household income between East and West Germany

To demonstrate the capabilities of the `xtoaxaca` command, we take an example from research on income inequality and examine the extent to which the decreasing gap in household income between households in the new and old German federal states can be traced back to their changing household composition. To this end, we use data from the German Socioeconomic Panel (SOEP) v.34.1 (Goebel et al., 2019), liebig.2018, and compute the logarithmized monthly net equivalent and inflation-adjusted household income. In addition to income data, we use information on the households' composition and employment situation (`hh_emp`). The variable we build from this information captures whether the household consists of a full-time working person, a full-time part-time working person, a not working person, a full-time and part-time working couple, a couple both working full-time, and other constellations.

In a first step, we estimate the panel regression model and store its results. Because we are interested in the effects of changing household compositions over time, the model includes a threefold interaction term which includes time, the group variable `hh_east`, and the decomposition variable of interest `hh_emp`. In this way, the model predicts income changes for each group and every household composition at every year. Note that we model time as a categorical variable in this example. However, the `xtoaxaca` command does not assume any functional form of time and that we can also specify time as a continuous variable.

After having estimated the model, we run the `xtoaxaca` command. In our example, we specify `hh_emp` as the decomposition variable we are interested in, the group variable (`hh_east`), and the two values of the group variable which we want to compare (0 and 1). Further, we need to specify the time variable (`year`) and the specific time values (2006 2010 2014), the reference time (2006), and the name of the stored estimated model. Finally, we tell `xtoaxaca` that we wish to use the interventionist approach, denoted by `change(interventionist)`.

```
. qui: eststo est1: xtreg hh_inc i.year##i.hh_east##i.hh_emp
.
. xtoaxaca hh_emp, groupvar(hh_east) groupcat(0 1) timevar(year) ///
>          times(2006 2010 2014) timeref(2006) ///
>          model(est1) change(interventionist)
WARNING: This is a beta version. Please check the results carefully
         and report bugs and suggestions to hkroeger@diw.de
```

Decomposition of Levels

Summary of level decomposition

	year		
	2006	2010	2014

Level			
non-parametric	0.199	0.182	0.176
Decomp			
Endowments	0.011	0.010	0.018
Coefficients	0.170	0.151	0.136
Interaction	0.002	-0.002	-0.002
RE	0.017	0.023	0.025
Total	0.199	0.182	0.176
Decomp %			
Endowments	5.404	5.647	9.998
Coefficients	85.200	82.685	77.310
Interaction	0.764	-1.063	-1.324
RE	8.631	12.730	14.017
Total	100.000	100.000	100.000

RE = random effects.

Decomposition of Change

Summary of changes in the outcome

	year		
	2006	2010	2014
Change			
non-parametric	0.000	-0.017	-0.023
Decomp			
Endowments	0.000	-0.007	-0.004
Coefficients	0.000	-0.015	-0.024
Interactions	0.000	-0.001	-0.002
RE	0.000	0.006	0.008
Total	0.000	-0.017	-0.023
Decomp %			
Endowments	.	40.920	16.559
Coefficients	.	89.446	105.874
Interactions	.	5.111	10.334
RE	.	-35.478	-32.767
Total	.	100.000	100.000

RE = random effects.

For an explanation of this change decomposition, please see:

Kröger, H., & Hartmann, J. (2020). xtoaxaca - Extending the Kitagawa-Oaxaca-Blinder Decomposition Approach to P
> Data. <https://doi.org/10.31235/osf.io/egj79>

.

By default, `xtoaxaca` displays a table of the decomposition of levels and another table showing the decomposition of change. In the first table, the first row (**Observed**) denotes the mean group differences in log household incomes as estimated non-parametrically from the observed data. The rows in the **Decomp** section show the results of the income gaps' decomposition into an endowments part, a coefficient part, an interactions part, and a part that is due to the time-constant error term (RE). In our example, household compositions hardly contribute to the gap in household income, and their contribution grows only in 2018. As the RE component is close to 0, our model is also reasonably

well specified. However, the RE component is non-negligible compared to the size of the other components. Further, we see that the decomposed parts sum up to the difference predicted by the base model for all years. The lower part of the table displays the relative contribution of the four decomposition effects to the overall gap.

The second table displays the results of the decomposition of change. We see the change in the income gap in comparison to the reference year 2006 in the second and third row. For the observed data, the gap decreased between 2006 and 2014 by 0.032 log incomes. We can now examine the role of changing endowments and coefficients over time. As we can see, the changing household composition decreased the gap by 0.004 log incomes and the changing coefficients contributed 0.024 log incomes to the narrowing gap between 2006 and 2014. The part that is due to differences between groups in the time-constant error term increased the gap by 0.008 log incomes. While this part is rather small, it still indicates that group-specific panel dropout has a small effect on the results.

7.2 Bootstrapping for standard errors

So far, we have decomposed the changes in household incomes between 2006 and 2014 in Germany and have the estimates, but no standard errors. The `xtoaxaca` command provides a `bootstrap` option to estimate the standard errors. As there have been no attempts to analytically derive standard errors for all decomposition models of change over time, we believe that bootstrapping is a viable alternative. By default, `xtoaxaca` does not bootstrap the standard errors because it is a potentially time-consuming endeavor.

Below, we see the same example as above with bootstrapped standard errors. The point estimates of the decomposition components are by design identical to the previous example. However, we now get a standard error below the point estimate and can now be more confident that the households' composition contributes about 5.4 percent to the income gap in 2006 and almost 10 percent in 2014, as the components are statistically significant. The size of the component of households' changing composition is small in size, however, and is not statistically significant.

```
. qui: eststo est2: xtreg hh_inc i.year##i.hh_east i.year##i.hh_east##i.hh_emp
.
.
. xtoaxaca hh_emp, groupvar(hh_east) groupcat(0 1) timevar(year) ///
> times(2006 2010 2014) timeref(2006) ///
> model(est2) change(interventionist) bootstrap(100)
WARNING: This is a beta version. Please check the results carefully
and report bugs and suggestions to hkroeger@diw.de
WARNING: Bootstrapping requires the original data set for estimation and the regressions need to be repeated 100
> . This may take some time.
(running xtoaxaca_bootstrap_wrapper on estimation sample)
Bootstrap replications (100)
-----| 1 |-----| 2 |-----| 3 |-----| 4 |-----| 5
..... 50
..... 100
```

Decomposition of Levels	
year	
2006	0.032
2010	0.032
2014	0.032
2006	0.004
2010	0.004
2014	0.004
2006	0.024
2010	0.024
2014	0.024
2006	0.008
2010	0.008
2014	0.008

	2006 b/se	2010 b/se	2014 b/se
Outcome			
Observed	0.199*** (0.011)	0.182*** (0.013)	0.176*** (0.013)
Decomp			
Endowments	0.011*** (0.003)	0.010* (0.004)	0.018** (0.006)
Coefficients	0.170*** (0.010)	0.151*** (0.010)	0.136*** (0.011)
Interactions	0.002 (0.002)	-0.002 (0.002)	-0.002 (0.002)
RE	0.017 (0.010)	0.023* (0.011)	0.025* (0.012)
Total	0.199*** (0.011)	0.182*** (0.013)	0.176*** (0.013)
Decomp%			
Endowments	5.404*** (1.503)	5.647* (2.336)	9.998*** (2.948)
Coefficients	85.200*** (4.994)	82.685*** (5.672)	77.310*** (6.486)
Interactions	0.764 (1.060)	-1.063 (0.954)	-1.324 (1.332)
RE	8.631 (4.905)	12.730* (5.517)	14.017* (6.309)
Total	100.000 (.)	100.000 (.)	100.000 (.)

Decomposition of Change

	year 2006 b/se	2010 b/se	2014 b/se
Outcome			
Observed	0.000 (.)	-0.017 (0.012)	-0.023 (0.014)
Decomp			
Endowments	0.000 (.)	-0.007* (0.003)	-0.004 (0.004)
Coefficients	0.000 (.)	-0.015 (0.008)	-0.024** (0.009)
Interactions	0.000 (.)	-0.001 (0.001)	-0.002 (0.002)
RE	0.000 (.)	0.006 (0.008)	0.008 (0.010)
Total	0.000 (.)	-0.017 (0.012)	-0.023 (0.014)
Decomp%			
Endowments	0.000 (.)	40.920 (237.566)	16.559 (102.940)
Coefficients	0.000 (.)	89.446 (283.015)	105.874 (458.214)
Interactions	0.000 (.)	5.111 (43.023)	10.334 (42.913)
RE	0.000 (.)	-35.478 (397.911)	-32.767 (466.494)
Total	0.000	100.000	100.000

(.) (.) (.)

For an explanation of this change decomposition, please see:

Kröger, H., & Hartmann, J. (2020). xtoaxaca - Extending the Kitagawa-Oaxaca-Blinder Decomposition Approach to P
> Data. <https://doi.org/10.31235/osf.io/egj79>

7.3 Blocks of variables

It is also possible to combine two or more variables to blocks and get standard error via bootstrapping for their combined contribution to the decomposition of both levels and change. The relevant option is `blocks` and the `resultsdata` option needs to be specified as well. Below is an example of code and output for variable blocks. The results for the variables are shown after all other output is given. Here, the contribution of all categories of the variables `hh_emp` and `hh_edu` are combined into the block `socio`. The variable `hh_married` is treated separately. It is possible to specify more than one block of variables.

```
. qui: eststo est3: xtreg hh_inc i.year##i.hh_east i.year##i.hh_east##i.(hh_emp hh_married hh_edu) , fe
.
.
.
. xtoaxaca hh_emp hh_married hh_edu, groupvar(hh_east) groupcat(0 1) timevar(year) ///
> times(2006 2010 2014) timeref(2006) normalize(hh_emp hh_married hh_edu) ///
> model(est3) change(interventionist) bootstrap(10) detail ///
> blocks(socio = (hh_emp hh_edu)) resultsdata(${path}Example/results_example1.
[output omitted]
```

Detailed Decomposition of Levels - Variable Blocks

Level Component: C

	2006 b/se	2010 b/se	2014 b/se
hh_married-n	-0.009 (0.005)	-0.004 (0.007)	0.003 (0.005)
hh_married_1	0.010 (0.006)	0.005 (0.008)	-0.004 (0.005)
socio	-0.119* (0.051)	-0.109** (0.039)	-0.075 (0.070)
Intercept	0.244*** (0.070)	0.216** (0.067)	0.167* (0.074)
Total	0.126*** (0.036)	0.108** (0.041)	0.092* (0.038)

Level Component: CE

	2006 b/se	2010 b/se	2014 b/se
hh_married-n	0.001 (0.001)	0.000 (0.001)	-0.000 (0.000)
hh_married_1	0.001 (0.001)	0.000 (0.001)	-0.000 (0.000)

socio	0.035*** (0.010)	0.030** (0.010)	0.021 (0.011)
Intercept	0.000 (.)	0.000 (.)	0.000 (.)
Total	0.037*** (0.011)	0.031** (0.011)	0.020 (0.011)

Level Component: E

	2006 b/se	2010 b/se	2014 b/se
hh_married-n	-0.000 (0.000)	0.000 (0.001)	0.000 (0.000)
hh_married_1	-0.000 (0.000)	0.000 (0.001)	0.000 (0.000)
socio	-0.057*** (0.016)	-0.060*** (0.012)	-0.037** (0.014)
Intercept	0.000 (.)	0.000 (.)	0.000 (.)
Total	-0.058*** (0.016)	-0.060*** (0.012)	-0.036** (0.014)

Percentages of Level Component: C

	2006 b/se	2010 b/se	2014 b/se
hh_married-n	-4.304 (2.681)	-2.344 (3.883)	1.971 (2.623)
hh_married_1	5.018 (3.089)	2.694 (4.299)	-2.363 (3.109)
socio	-59.852* (23.887)	-59.522** (20.890)	-42.358 (40.748)
Intercept	122.507*** (34.509)	118.227*** (35.711)	94.820* (44.731)
Total	63.368*** (19.209)	59.055** (21.464)	52.069* (21.899)

Percentages of Level Component: CE

	2006 b/se	2010 b/se	2014 b/se
hh_married-n	0.491 (0.335)	0.211 (0.480)	-0.175 (0.248)
hh_married_1	0.491 (0.335)	0.211 (0.480)	-0.175 (0.248)
socio	17.445** (5.487)	16.477** (6.298)	11.835 (7.070)
Intercept	0.000 (.)	0.000 (.)	0.000 (.)
Total	18.427** (5.962)	16.898* (6.843)	11.486 (6.963)

Percentages of Level Component: E

	2006 b/se	2010 b/se	2014 b/se
--	--------------	--------------	--------------

hh_married-n	-0.064 (0.231)	0.000 (0.277)	0.128 (0.182)
hh_married_1	-0.064 (0.231)	0.000 (0.277)	0.128 (0.182)
socio	-28.714*** (8.371)	-32.669*** (7.227)	-20.913* (8.747)
Intercept	0.000 (.)	0.000 (.)	0.000 (.)
Total	-28.842*** (8.651)	-32.669*** (7.341)	-20.657* (8.659)

7.4 Example 2 - An intervention

As a further example, we simulate a dataset with a group variable (`group`), two time points, and two binary intervening variables, of which the first is exogenous (`int1`) and the second is endogenous to the first one (`int2`). The purpose of this example is to demonstrate how `xtoaxaca` can decompose change in group differences over time using multiple intervening and exogeneous decomposition variables similar to experimental settings. For instance, an exogenous treatment effect, such as a policy change, could lead to increases in one group's endowments, and we can then ask whether the treatment has any effect on the changing group differences.

Similar to the previous example, the first step involves estimating the model which now includes two interaction terms: the interaction of the group variable with time and the first decomposition variable, and the interaction of the group variable with time and the second decomposition variable. These two interaction terms are included so that the `xtoaxaca` command can estimate the groups' counterfactual trajectories. In a second step, we call the `xtoaxaca` command followed by the two decomposition variables and specify that we choose the interventionist decomposition method (`interventionist`).

```
.
. qui: eststo m: xtreg dep i.time##i.group##i.int1 i.time##i.group##i.int2, i(id)
.
. xtoaxaca int1 int2, groupvar(group) groupcat(0 1) timevar(time) times(1 2) ///
>      timeref(1) model(m) change(interventionist)
WARNING: This is a beta version. Please check the results carefully
        and report bugs and suggestions to hkroeger@diw.de
```

Decomposition of Levels

Summary of level decomposition

	time	
	1	2
Level		
non-parame-c	1.623	3.770
Decomp		
Endowments	0.034	-0.005
Coefficients	1.711	3.824
Interaction	-0.122	-0.049

RE	-0.000	-0.000
Total	1.623	3.770

Decomp %		
Endowments	2.086	-0.132
Coefficients	105.445	101.424
Interaction	-7.531	-1.292
RE	-0.000	-0.000
Total	100.000	100.000

RE = random effects.

Decomposition of Change

Summary of changes in the outcome

	time	
	1	2

Change		
non-parametric	0.000	2.147

Decomp		
Endowments	0.000	-0.097
Coefficients	0.000	2.265
Interactions	0.000	-0.021
RE	0.000	0.000
Total	0.000	2.147

Decomp %		
Endowments	.	-4.510
Coefficients	.	105.482
Interactions	.	-0.972
RE	.	0.000
Total	.	100.000

RE = random effects.

For an explanation of this change decomposition, please see:

Kröger, H., & Hartmann, J. (2020). xtoaxaca - Extending the Kitagawa-Oaxaca-Blinder Decomposition Approach to P

> Data. <https://doi.org/10.31235/osf.io/egj79>

.

As can be seen from the output, changing endowments cause the gap to decrease by 4.5% over time, while changing coefficients increase the gap by 105%. Thus, we conclude that the increasing gap between the groups over time is not caused by their changing endowments. Quite the contrary, they decrease the gap over time, while all of the increase in the gap can be attributed to changing coefficients. If the role of the changing return to the decomposition variables is to be investigated, results using the `detail` option should be used (as illustrated in the last example in section 7.5), because the intercept is part of the coefficient block. A change in the intercept does not usually have a meaningful substantive interpretation with respect to the intervening variables and therefore might be considered an unexplained part within the coefficient component. It should also be noted that because the example uses simulated data, the contribution of the time-constant error term is now exactly zero as expected by model assumptions.

7.5 Example 3 - Interaction of decomposition variables

In the last example, we show how a regression model has to be set up if interactions of decomposition variables are to be used in `xtoaxaca`. For this purpose we distinguish between three types of interactions: Categorical-categorical, Continuous-continuous, and categorical-continuous interactions.

7.5.1 Categorical-categorical interactions

For these kinds of interactions we recommend generating a new variable which contains all combinations of the two categorical variables.

```
egen newvar = group(var1 var2)
```

Then, we can use this new variable (`newvar`) as a decomposition variable in both the regression and the `xtoaxaca` command.

7.5.2 Continuous-continuous interactions

For these kinds of interactions the key is to create interaction terms by hand and then combine them with the standard Stata-`fv.` notation of the group and time variable.

```
.  
.   
.   
. *** Create interaction term for a squared term  
.      replace exp = exp-10  
(10,000 real changes made)  
.      gen exp2 = exp*exp  
.   
.   
.   
. *** for interactions among decomposition variables (squared experience)  
. *** a hand-made interaction-term is used  
. qui: eststo est3: xtreg inc i.group##i.time##c.(exp exp2) i.group##i.time##i.edu  
.   
. xtoaxaca edu exp exp2, groupvar(group) groupcat(0 1) timevar(time) nolevels ///  
>      times(4 2 1) model(est3) timeref(2) change(interventionist) detail normalize(edu)  
WARNING: This is a beta version. Please check the results carefully  
and report bugs and suggestions to hkroeger@diw.de
```

```
Group variable:      group (0,1)  
Decomposition variables:  edu exp exp2  
Times:              time (4 2 1)
```

```
[ Progress ] Calculating endowment means for each time ...
```

```
[ Progress ] Calculating mean outcomes ...  
[ Estimating random effects ] Estimating mean of random effects for both groups ...  
[ Progress ] Calculating coefficients ...  
[ Progress ] Running level decomposition ...  
[ Progress ] Running Interventionist decomposition ...  
[ Progress ] Display output ...
```

Outcome

Mean predicted outcome differences between the groups (empirical values)

	Time		
	4	2	1
Group A	3663.107	2983.06	2715.14
Group B	6524.738	4533.923	3951.633
Diff	-2861.632	-1550.863	-1236.493

Endowments

Endowment means for each time (Group (group) 0, variables: edu exp exp2)

	Time		
	4	2	1
0.edu	.0914697	.1438849	.1819116
1.edu	.3329908	.4110997	.4635149
2.edu	.3103803	.307297	.270298
3.edu	.2651593	.1377184	.0842754
exp	-4.367934	-4.75334	-4.971223
exp2	24.46249	27.72148	29.62076
Intercept	1	1	1

Endowment means for each time (Group (group) 1, variables: edu exp exp2)

	Time		
	4	2	1
0.edu	.0681597	.2755599	.5423564
1.edu	.2375852	.4333009	.3417722
2.edu	.3446933	.2259007	.1051607
3.edu	.3495618	.0652386	.0107108
exp	2.057449	.661149	-.0467381
exp2	15.46933	11.35054	10.73418
Intercept	1	1	1

Coefficients

Group A coefficients

	Time		
	r1	r2	r3
0.edu	-235.8521	83.33642	238.1277
1.edu	-78.87413	22.97487	67.34441
2.edu	76.95988	-28.07654	-98.34574
3.edu	237.7663	-78.23475	-207.1264
exp	28.87513	-4.020149	-14.08591
exp2	.9153148	.5256663	.278332
Intercept	3727.745	2947.345	2606.376

Group B coefficients

	Time		
	r1	r2	r3
0.edu	-312.1004	93.12871	297.7939
1.edu	-98.5876	55.99646	72.83483
2.edu	108.9784	-35.06983	-137.1094

3.edu	301.7096	-114.0553	-233.5193
exp	79.22486	-32.17301	-65.14026
exp2	.2232951	-.2076859	-.0671643
Intercept	6259.948	4522.989	3779.826

Decomposition of Change

Summary of changes in the outcome

	time 4	2	1
Change			
non-parame-c	-1,310.769	0.000	314.370
Decomp			
Endowments	93.165	0.000	-41.610
Coefficients	-1,073.681	0.000	431.704
Interactions	-330.253	0.000	-75.724
RE	0.000	0.000	0.000
Total	-1,310.769	0.000	314.370
Decomp %			
Endowments	-7.108	.	-13.236
Coefficients	81.912	.	137.324
Interactions	25.195	.	-24.088
RE	0.000	.	0.000
Total	100.000	.	100.000

RE = random effects.

Total change: Endowments

	time 4	2	1
0.edu	14.947	0.000	-21.677
1.edu	9.165	0.000	6.330
2.edu	4.079	0.000	-3.196
3.edu	22.458	0.000	-2.038
exp	43.374	0.000	-21.899
exp2	-0.858	0.000	0.870
Intercept	0.000	0.000	0.000
Total	93.165	0.000	-41.610

Note: Change in group differences over time if only the groups' endowments had changed.

Total change: Coefficients

	time 4	2	1
0.edu	65.738	0.000	-34.125
1.edu	25.111	0.000	10.944
2.edu	-0.263	0.000	1.457
3.edu	16.395	0.000	-9.957
exp	-230.013	0.000	69.642
exp2	5.910	0.000	-8.451
Intercept	-956.559	0.000	402.194
Total	-1,073.681	0.000	431.704

Note: Change in group differences over time if only the groups' coefficients had changed.

Total change: Interactions

	time		
	4	2	1
0.edu	-67.314	0.000	-48.718
1.edu	-22.299	0.000	3.867
2.edu	-16.788	0.000	-9.720
3.edu	-77.940	0.000	0.374
exp	-142.867	0.000	-21.144
exp2	-3.045	0.000	-0.383
Intercept	0.000	0.000	0.000
Total	-330.253	0.000	-75.724

Note: Change in group differences over time attributable to the interaction of change in endowments and change > coefficients.

Total Percentages of change: Endowments

	time		
	4	2	1
0.edu	-1.140	.	-6.896
1.edu	-0.699	.	2.013
2.edu	-0.311	.	-1.016
3.edu	-1.713	.	-0.648
exp	-3.309	.	-6.966
exp2	0.065	.	0.277
Intercept	0.000	.	0.000
Total	-7.108	.	-13.236

Total Percentages of change: Coefficients

	time		
	4	2	1
0.edu	-5.015	.	-10.855
1.edu	-1.916	.	3.481
2.edu	0.020	.	0.464
3.edu	-1.251	.	-3.167
exp	17.548	.	22.153
exp2	-0.451	.	-2.688
Intercept	72.977	.	127.937
Total	81.912	.	137.324

Total Percentages of change: Interactions

	time		
	4	2	1
0.edu	5.135	.	-15.497
1.edu	1.701	.	1.230
2.edu	1.281	.	-3.092
3.edu	5.946	.	0.119
exp	10.899	.	-6.726
exp2	0.232	.	-0.122
Intercept	0.000	.	0.000
Total	25.195	.	-24.088

For an explanation of this change decomposition, please see:

Kröger, H., & Hartmann, J. (2020). *xtoaxaca* - Extending the Kitagawa-Oaxaca-Blinder Decomposition Approach to P
> 1 Data. <https://doi.org/10.31235/osf.io/egj79>

.

The first difference to the previous examples is the additional output that is generated by using the `detail` option. It provides not only the components of the decomposition but also the estimates for the endowments and coefficients for each variable and group. Note that for the interpretation of each variable and its contribution separate assumptions about the reference category or value have to be made (see section 3.2 or Yun (2005); Jann (2008)).

The second difference is that this example shows interactions of two continuous decomposition variables. If interactions of decomposition variables are used, it is important to note that the individual contribution of each interacted variable is now conditional on the value of the other variable. This means that in our example that we interacted experience with itself to get a squared term in the regression equation. The contribution of labor market experience (`exp`) now varies along the values of labor market experience.

For this particular example, we could state that the difference in change between groups A and B and time points 4 and 2 is explained about 20 percent by the difference in change in coefficients in the experience variable at the 15 years of experience in the sample (value of zero). If labor market experience is not centered at a meaningful value this might not be a particular useful result.

7.5.3 Categorical-continuous interactions

The strategy described in 7.5.2 for continuous-continuous interactions technically also works for categorical-continuous interactions. There are two limitations, however. First, normalization (3.2) for the categorical variable is not possible. This also implies that we cannot interpret the results for the decomposition by (Kim, 2010) as originally intended, because this decomposition relies on normalization. Second, we believe that it is overall very difficult the detailed output for such a decomposition. users might consider performing the decomposition separately for the categorical variable they wish to interact with the continuous variable to ease interpretation.

8 Limitations

We focus on continuous outcomes and linear models but believe that the general approach can be generalized to non-linear models as well, as it has been for the cross-sectional KOB decomposition Bauer and Sinning (2008); Jann (2008). Furthermore, applying regression with recentered influence functions (RIF) in the modeling step might also be a way to circumvent the current restrictions to linear models of *xtoaxaca* (Essama-Nssah and Lambert, 2012; Firpo et al., 2018) as exemplified in the user-written Stata command `oaxaca_rif` (?).

Our decomposition approach is further limited to mean decompositions. Longitudinal decompositions or using other distributional statistics (e.g. percentiles, variances) might also be useful as user-friendly programs (Fortin et al., 2011; Blau and Kahn, 1992; Juhn et al., 1993).

9 Conclusion

We provide a systematical extension of the KOB-decomposition to longitudinal (and multilevel) data. We reviewed five central approaches to the decomposition of change. We noted that none of them is directly useful for the evaluation of an intervention, policy changes, or natural experiments. We proposed an extension of the Wellington (1993) decomposition of change over time from an *interventionist* perspective. We introduced the `xtoaxaca` command, which implements the decomposition of levels and change (all six variants) over time in a user-friendly postestimation command for Stata. We are open to suggestions from users of `xtoaxaca` concerning new functions and other improvements so that we can update and improve it regularly.

10 Acknowledgments

We would like to thank Benita Combet and Christoph Halbmeier for helpful feedback on earlier versions of this paper. We would also like to extend our special thanks to Benita Combet for extensive testing of earlier versions of `xtoaxaca`. All remaining mistakes remain, of course, the sole responsibility of the authors.

Hannes Kröger was funded by the Deutsche Forschungsgemeinschaft (DFG, German Science Foundation) - 415809395;427279591;40965412 and the German Federal Ministry of Education and Research (BMBF, grants: 01UJ1911BY;01NV1601B).

References

- Arrosa, M. L., and N. Gandelman. 2016. Happiness Decomposition: Female Optimism. *Journal of Happiness Studies* 17(2): 731–756.
- Bauer, T. K., and M. Sinning. 2008. An extension of the Blinder–Oaxaca decomposition to nonlinear models. *AStA Advances in Statistical Analysis* 92(2): 197–206.
- Blau, F. D., and L. M. Kahn. 1992. The gender earnings gap: learning from international comparisons. *The American Economic Review* 82(2): 533–538.
- . 2017. The Gender Wage Gap: Extent, Trends, and Explanations. *Journal of Economic Literature* 55(3): 789–865.
- Blinder, A. S. 1973. Wage discrimination: Reduced form and structural estimates. *The Journal of Human Resources* 8: 436–455.
- DeLeire, T. 2000. The Wage and Employment Effects of the Americans with Disabilities Act. Technical Report 4, Harris School of Public Policy Studies, University of Chicago.
- Deming, W. E., and F. F. Stephan. 1940. On a Least Squares Adjustment of a Sampled Frequency Table When the Expected Marginal Totals are Known. *The Annals of Mathematical Statistics* 11(4): 427–444.
- Essama-Nssah, B., and P. J. Lambert. 2012. Chapter 6 Influence Functions for Policy Impact Analysis. In *Inequality, mobility and segregation: Essays in honor of Jacques Silber*, 135–159. Emerald Group Publishing Limited.
- Fairlie, R. W. 2005. An extension of the Blinder–Oaxaca decomposition technique to logit and probit models. *Journal of Economic and Social Measurement* 30(4): 305–316.
- Firebaugh, G., C. Warner, and M. Massoglia. 2013. Fixed Effects, Random Effects, and Hybrid Models for Causal Analysis. In *Handbook of Causal Analysis for Social Research*, ed. S. L. Morgan, 113–132. Springer, Dordrecht. URL http://link.springer.com/10.1007/978-94-007-6094-3_7.
- Firpo, S., N. Fortin, and T. Lemieux. 2018. Decomposing Wage Distributions Using Recentered Influence Function Regressions. *Econometrics* 6(2): 28. URL <http://www.mdpi.com/2225-1146/6/2/28>.
- Fortin, N., T. Lemieux, and S. Firpo. 2011. Decomposition methods in economics. In *Handbook of labor economics*, vol. 4, 1–102. Elsevier.
- Freeman, R. B. 1980. Unionism and the Dispersion of Wages. *ILR Review* 34(1): 3–23.
- . 1984. Longitudinal Analyses of the Effects of Trade Unions. *Journal of Labor Economics* 2(1): 1–26.

- Goebel, J., M. M. Grabka, S. Liebig, M. Kroh, D. Richter, C. Schröder, and J. Schupp. 2019. The German Socio-Economic Panel (SOEP). *Jahrbucher fur Nationalokonomie und Statistik* 239(2): 345–360. URL <https://doi.org/10.1515/jbnst-2018-0022>.
- Härdle, W. K., M. Müller, S. Sperlich, and A. Werwatz. 2012. *Nonparametric and semiparametric models*. Springer Science & Business Media.
- Jann, B. 2008. The Blinder-Oaxaca decomposition for linear regression models. *The Stata Journal* 8(4): 453–479.
- Juhn, C., Murphy, Kevin, M., and B. Pierce. 1993. Wage Inequality and the Rise in Returns to Skill. *The Journal of Political Economy* 101(3): 410–442.
- Kalton, G., and I. Flores-Cervantes. 2003. Weighting methods. *Journal of official statistics* 19(2): 81.
- Kim, C. 2010. Decomposing the Change in the Wage Gap Between White and Black Men Over Time, 1980-2005: An Extension of the Blinder-Oaxaca Decomposition Method. *Sociological Methods & Research* 38(4): 619–651.
- Kim, J. K., and J. J. Kim. 2007. Nonresponse weighting adjustment using estimated response probability. *Canadian Journal of Statistics* 35(4): 501–514.
- Kitagawa, E. M. 1955. Components of a Difference Between Two Rates. *Journal of the American Statistical Association* 50(272): 1168–1194. URL <http://www.tandfonline.com/doi/abs/10.1080/01621459.1955.10501299>.
- Machado, J. A. F., and J. Mata. 2005. Counterfactual decomposition of changes in wage distributions using quantile regression. *Journal of Applied Econometrics* 20(4): 445–465.
- Makepeace, G., P. Paci, H. Joshi, and P. Dolton. 1999. How Unequally Has Equal Pay Progressed Since the 1970s? *The Journal of Human Resources* 34(3): 534–556.
- Morgan, S. L. S. L., and C. Winship. 2014. *Counterfactuals and causal inference: methods and principles for social research*. 2nd ed. New York: Cambridge University Press.
- Neuman, S., and R. L. Oaxaca. 2003. Gender versus Ethnic Wage Differentials among Professionals: Evidence from Israel. *Annales d'Économie et de Statistique* (71/72): 267.
- Oaxaca, R. L. 1973. Male–female wage differentials in urban labor markets. *International Economic Review* 14: 693–709.
- Oaxaca, R. L., and C. Choe. 2016. Wage Decompositions Using Panel Data Sample Selection Correction. *IZA Discussion Paper* (10157).
- Oaxaca, R. L., and M. R. Ransom. 1994. On discrimination and the decomposition of wage differentials. *Journal of Econometrics* 61(1): 5–21. URL <https://www.sciencedirect.com/science/article/pii/0304407694900744>.

- Smith, J. P., and F. R. Welch. 1989. Black economic progress after Myrdal. *Journal of economic literature* 27(2): 519–564.
- Taber, D. R., W. R. Robinson, S. N. Bleich, and Y. C. Wang. 2016. Deconstructing race and gender differences in adolescent obesity: Oaxaca-blinder decomposition. *Obesity (Silver Spring, Md.)* 24(3): 719–726.
- Wellington, A. J. 1993. Changes in the Male/Female Wage Gap, 1976-85. *The Journal of Human Resources* 28(2): 383. URL <https://www.jstor.org/stable/146209?origin=crossref>.
- Winter, N. 2018. SURVWGT: Stata module to create and manipulate survey weights. *EconPapers*. URL <https://econpapers.repec.org/software/bocbocode/S427503.htm>.
- Yun, M.-S. 2005. A Simple Solution to the Identification Problem in Detailed Wage Decompositions. *Economic Inquiry* 43(4): 766–772. URL <http://doi.wiley.com/10.1093/ei/cbi053>.

11 About the authors

Hannes Kröger is the leader of the Junior Research Group “Social and Psychological Determinants of Mental Health in the Life Course” (SocPsych-MH) at the Socio-Economic Panel Study (SOEP) at the German Institute for Economic Research (DIW) in Berlin. His research interests include health inequalities and applied statistical methods for social science research.

Jörg Hartmann is a senior researcher at the University of Göttingen. His main research interests comprise the causes and consequences of social inequality, right-wing populism, immigrant integration, as well as quantitative methods.

A Appendix

A.1 Twofold decomposition of change

We can also derive an analog to the twofold KOB within the interventionist perspective. This based on a model that assumes a valuation of the endowments at a non-discriminatory coefficient vector.

$$\begin{aligned}\Delta Y_2^l &= Y_t^l - Y_s^l \\ &= E(X_t^l)' \beta_t - E(X_s^l)' \beta_s\end{aligned}$$

$$\beta_t = W \beta_t^A + (I - W) \beta_t^B$$

The common coefficient is defined as a weighted average of the two group-specific coefficients. Weights can be defined similarly to the implementation of the original KOB twofold decomposition, as described by (Jann, 2008, 455-457).

In our adaption of the twofold decomposition for change, we have two components. The first relates to the differences in change in endowments given the same coefficients at time s ΔY_{E2} . The second component is the remaining unexplained part ΔY_U .

$$\begin{aligned}\Delta Y_{E2} &= \{ [E(X_t^A) - E(X_s^A)] - [E(X_t^B) - E(X_s^B)] \}' \beta_s \\ &= E(X_t^A)' \beta_s - E(X_s^A)' \beta_s - E(X_t^B)' \beta_s + E(X_s^B)' \beta_s \\ &= E(X_t^A)' W \beta_s^A + E(X_t^A)' (I - W) \beta_s^B - E(X_s^A)' W \beta_s^A - E(X_s^A)' (I - W) \beta_s^B \\ &\quad - E(X_t^B)' W \beta_s^A - E(X_t^B)' (I - W) \beta_s^B + E(X_s^B)' W \beta_s^A + E(X_s^B)' (I - W) \beta_s^B\end{aligned}\tag{18}$$

$$\begin{aligned}\Delta Y_U &= E(X_t^A)' \beta_t^A - E(X_t^B)' \beta_t^B + E(X_s^B)' W \beta_s^B - E(X_s^A)' (I - W) \beta_s^A \\ &\quad + [E(X_t^B) - E(X_s^B) - E(X_s^A)]' W \beta_s^A + [E(X_s^A) + E(X_t^B) - E(X_t^A)]' (I - W) \beta_t^B\end{aligned}\tag{19}$$

The first component mirrors the cross-sectional endowment component of the twofold decomposition. We immediately see, however, that it violates the formal condition that no term is to be restricted across groups (coefficient β_s). It therefore imagines a situation in which there would not have been differences in the coefficients at time s , something that is beyond a potential intervention occurring after s . Still, for specific purposes, research might find this decomposition useful as well if **change** is to be decomposed twofold, similar to the twofold **level** decomposition in the original KOB approach. We can also show that together the two components add to the total change over time.

Proof.

$$\begin{aligned}
\Delta Y_{E2} + \Delta Y_U &= E(X_t^A)'W\beta_s^A + E(X_t^A)'(I-W)\beta_s^B - E(X_s^A)'W\beta_s^A - E(X_s^A)'(I-W)\beta_s^B \\
&\quad - E(X_t^B)'W\beta_s^A - E(X_t^B)'(I-W)\beta_s^B + E(X_s^B)'W\beta_s^A + E(X_s^B)'(I-W)\beta_s^B \\
&\quad + E(X_t^A)'\beta_t^A - E(X_t^B)'\beta_t^B + E(X_s^B)'W\beta_s^B - E(X_s^A)'(I-W)\beta_s^A \\
&\quad + [E(X_t^B) - E(X_s^B) - E(X_s^A)]'W\beta_s^A + [E(X_s^A) + E(X_t^B) - E(X_t^A)]'(I-W)\beta_t^B \\
&= E(X_t^A)'\beta_t^A - E(X_t^B)'\beta_t^B - E(X_s^A)'W\beta_s^A + E(X_s^B)'(I-W)\beta_t^B \\
&\quad - E(X_s^A)'(I-W)\beta_s^A + E(X_s^B)'W\beta_s^B \\
&= [E(X_t^A)'\beta_t^A - E(X_s^A)'\beta_s^A] - [E(X_t^B)'\beta_t^B - E(X_s^B)'\beta_s^B] \\
&= \Delta Y^A - \Delta Y^B \\
&= \Delta Y
\end{aligned}$$

□

B Relations between change decompositions and KOB

In this section are the proofs that all decompositions of change presented in section 4 are derivatives of the difference between two KOB decomposition at two time points¹².

B.1 Smith and Welch (1989)

Together, the four components fully decompose changes in group differences over time.

¹²For the SSM this is directly shown in section 4.1

Proof.

$$\begin{aligned}
\Delta Y &= \text{i} + \text{ii} + \text{iii} + \text{iv} \\
&= \left\{ [E(X_t^A) - E(X_t^B)] - [E(X_s^A) - E(X_s^B)] \right\}' \beta_s^B \\
&\quad + [E(X_t^A) - E(X_s^A)]' (\beta_s^A - \beta_s^B) \\
&\quad + [E(X_t^A) - E(X_t^B)]' (\beta_t^B - \beta_s^B) \\
&\quad + E(X_t^A)' [(\beta_t^A - \beta_t^B) - (\beta_s^A - \beta_s^B)] \\
&= E(X_t^A)' \beta_s^B - E(X_t^B)' \beta_s^B - E(X_s^A)' \beta_s^B + E(X_s^B)' \beta_s^B \\
&\quad + E(X_t^A)' \beta_s^A - E(X_t^A)' \beta_s^B - E(X_s^A)' \beta_s^A + E(X_s^A)' \beta_s^B \\
&\quad + E(X_t^A)' \beta_t^B - E(X_t^A)' \beta_s^B - E(X_t^B)' \beta_t^B + E(X_t^B)' \beta_s^B \\
&\quad + E(X_t^A)' \beta_t^A - E(X_t^A)' \beta_t^B - E(X_t^A)' \beta_s^A + E(X_t^A)' \beta_s^B \\
&= E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A - E(X_t^B)' \beta_t^B + E(X_s^B)' \beta_s^B \\
&\quad + E(X_t^A)' \beta_s^B - E(X_t^A)' \beta_s^B + E(X_t^B)' \beta_s^B - E(X_t^B)' \beta_s^B \\
&\quad + E(X_s^A)' \beta_s^B - E(X_s^A)' \beta_s^B + E(X_t^A)' \beta_s^A - E(X_t^A)' \beta_s^A \\
&\quad + E(X_t^A)' \beta_t^B - E(X_t^A)' \beta_t^B + E(X_t^A)' \beta_s^B - E(X_t^A)' \beta_s^B \\
&= [E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A] - [E(X_t^B)' \beta_t^B - E(X_s^B)' \beta_s^B] \\
&= E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A - E(X_t^B)' \beta_t^B + E(X_s^B)' \beta_s^B \\
&= \Delta Y^A + \Delta Y^B
\end{aligned}$$

□

B.2 Wellington (1993)

As can be shown, WL1 and WL2 fully decompose changes in group differences over time.

Proof.

$$\begin{aligned}
\Delta Y &= \text{WL1} + \text{WL2} \\
&= [(E(X_t^A) - E(X_s^A))] \beta_t^A [E(X_t^B) - E(X_s^B)]' \beta_t^B \\
&\quad + (\beta_t^A - \beta_s^A) E(X_s^A)' - (\beta_t^B - \beta_s^B) E(X_s^B)' \\
&= E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A - E(X_t^B)' \beta_t^B + E(X_s^B)' \beta_s^B \\
&\quad + E(X_s^A)' \beta_t^A - E(X_s^A)' \beta_t^A + E(X_s^B)' \beta_s^B - E(X_s^B)' \beta_s^B \\
&= E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A - E(X_t^B)' \beta_t^B + E(X_s^B)' \beta_s^B \\
&= [E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A] - [E(X_t^B)' \beta_t^B - E(X_s^B)' \beta_s^B] \\
&= \Delta Y^A - \Delta Y^B
\end{aligned}$$

□

B.3 Interventionist

Finally, we can show that the three components added up to give the total difference in change between the two groups:

$$\Delta Y = \Delta E + \Delta C + \Delta I \quad (20)$$

Proof.

$$\begin{aligned} \Delta E + \Delta C + \Delta I &= [E(X_t^A) - E(X_s^A)]' \beta_s^A - [E(X_t^B) - E(X_s^B)]' \beta_s^B \\ &\quad + E(X_s^A)' (\beta_t^A - \beta_s^A) - E(X_s^B)' (\beta_t^B - \beta_s^B) \\ &\quad + [E(X_t^A) - E(X_s^A)]' (\beta_t^A - \beta_s^A) - [E(X_t^B) - E(X_s^B)]' (\beta_t^B - \beta_s^B) \\ &= E(X_t^A)' \beta_s^A - E(X_s^A)' \beta_s^A - E(X_t^B)' \beta_s^B + E(X_s^B)' \beta_s^B \\ &\quad + E(X_s^A)' \beta_t^A - E(X_s^A)' \beta_s^A - E(X_s^B)' \beta_t^B + E(X_s^B)' \beta_s^B \\ &\quad + E(X_t^A)' \beta_t^A - E(X_t^A)' \beta_s^A - E(X_s^A)' \beta_t^A + E(X_s^A)' \beta_s^A \\ &\quad - E(X_t^B)' \beta_t^B + E(X_t^B)' \beta_s^B + E(X_s^B)' \beta_t^B - E(X_s^B)' \beta_s^B \\ &= E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A - E(X_t^B)' \beta_t^B + E(X_s^B)' \beta_s^B \\ &= [E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A] - [E(X_t^B)' \beta_t^B - E(X_s^B)' \beta_s^B] \\ &= \Delta Y^A - \Delta Y^B \\ &= \Delta Y \end{aligned}$$

□

B.4 Makepeace et al. (1999)

Together, the components fully decompose the change in group differences over time.

Proof.

$$\begin{aligned}
\Delta Y &= \Delta E_{pure} + \Delta E_{price} + \Delta U_{pure} + \Delta U_{price} \\
&= \left\{ [E(X_t^A) - E(X_s^A)] - [E(X_t^B) - E(X_s^B)] \right\}' \beta_t^A \\
&\quad + [E(X_s^A) - E(X_s^B)]' (\beta_t^A - \beta_s^A) \\
&\quad + E(X_t^B)' \left[(\beta_t^A - \beta_s^A) - (\beta_t^B - \beta_s^B) \right] \\
&\quad + [E(X_t^B) - E(X_s^B)]' (\beta_s^A - \beta_s^B) \\
&= E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A - E(X_t^B)' \beta_t^B + E(X_s^B)' \beta_s^B \\
&\quad + E(X_s^A)' \beta_s^A - E(X_s^A)' \beta_s^A + E(X_t^B)' \beta_t^A - E(X_t^B)' \beta_t^A \\
&\quad + E(X_s^B)' \beta_t^A - E(X_s^A)' \beta_t^A + E(X_s^B)' \beta_s^A - E(X_s^B)' \beta_s^A \\
&\quad + E(X_t^B)' \beta_s^A - E(X_t^B)' \beta_s^A + E(X_t^B)' \beta_s^B - E(X_t^B)' \beta_s^B \\
&= E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A - E(X_t^B)' \beta_t^B + E(X_s^B)' \beta_s^B \\
&= [E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A] - [E(X_t^B)' \beta_t^B - E(X_s^B)' \beta_s^B] \\
&= \Delta Y^A - \Delta Y^B
\end{aligned} \tag{21}$$

□

B.5 Kim (2010)

We can show that the KIM decomposition fully decomposes changes in group differences over time.

Proof.

$$\begin{aligned}
\Delta Y &= D1 + D2 + D3 + D4 + D5 \\
&= [(\beta_{0,t}^A - \beta_{0,s}^A) - (\beta_{0,t}^B - \beta_{0,s}^B)] + [(\bar{\beta}_t^{*A} - \bar{\beta}_s^{*A}) - (\bar{\beta}_t^{*B} - \bar{\beta}_s^{*B})] \\
&+ \left[\frac{E(X_t^{*A}) + E(X_s^{*A}) + E(X_t^{*B}) + E(X_s^{*B})}{4} \right]' \left[(\tilde{\beta}_t^A - \tilde{\beta}_s^A) - (\tilde{\beta}_t^B - \tilde{\beta}_s^B) \right] \\
&+ \left\{ \frac{[E(X_t^{*A}) - E(X_s^{*A})] + [E(X_t^{*B}) - E(X_s^{*B})]}{2} \right\}' \left[\frac{(\tilde{\beta}_t^A + \tilde{\beta}_s^A)}{2} - \frac{(\tilde{\beta}_t^B + \tilde{\beta}_s^B)}{2} \right] \\
&+ \left\{ [E(X_t^{*A}) - E(X_s^{*A})] - [E(X_t^{*B}) - E(X_s^{*B})] \right\}' \left[\frac{\tilde{\beta}_t^A + \tilde{\beta}_s^A + \tilde{\beta}_t^B + \tilde{\beta}_s^B}{4} \right] \\
&+ \left\{ \frac{[E(X_t^{*A}) + E(X_s^{*A})]}{2} - \frac{[E(X_t^{*B}) + E(X_s^{*B})]}{2} \right\}' \left[\frac{(\tilde{\beta}_t^A - \tilde{\beta}_s^A) + (\tilde{\beta}_t^B - \tilde{\beta}_s^B)}{2} \right] \\
&= [(\beta_{0,t}^A - \beta_{0,s}^A) - (\beta_{0,t}^B - \beta_{0,s}^B)] + [(\bar{\beta}_t^{*A} - \bar{\beta}_s^{*A}) - (\bar{\beta}_t^{*B} - \bar{\beta}_s^{*B})] \\
&+ [E(X_t^{*A})' \tilde{\beta}_t^{*A} - E(X_t^{*A})' \tilde{\beta}_s^{*A} - E(X_t^{*A})' \tilde{\beta}_t^{*B} + E(X_t^{*A})' \tilde{\beta}_s^{*B}] / 4 \\
&+ [E(X_s^{*A})' \tilde{\beta}_t^{*A} - E(X_s^{*A})' \tilde{\beta}_s^{*A} - E(X_s^{*A})' \tilde{\beta}_t^{*B} + E(X_s^{*A})' \tilde{\beta}_s^{*B}] / 4 \\
&+ [E(X_t^{*B})' \tilde{\beta}_t^{*A} - E(X_t^{*B})' \tilde{\beta}_s^{*A} - E(X_t^{*B})' \tilde{\beta}_t^{*B} + E(X_t^{*B})' \tilde{\beta}_s^{*B}] / 4 \\
&+ [E(X_s^{*B})' \tilde{\beta}_t^{*A} - E(X_s^{*B})' \tilde{\beta}_s^{*A} - E(X_s^{*B})' \tilde{\beta}_t^{*B} + E(X_s^{*B})' \tilde{\beta}_s^{*B}] / 4 \\
&+ [E(X_t^{*A})' \tilde{\beta}_t^{*A} + E(X_t^{*A})' \tilde{\beta}_s^{*A} - E(X_t^{*A})' \tilde{\beta}_t^{*B} - E(X_t^{*A})' \tilde{\beta}_s^{*B}] / 4 \\
&- [E(X_s^{*A})' \tilde{\beta}_t^{*A} - E(X_s^{*A})' \tilde{\beta}_s^{*A} + E(X_s^{*A})' \tilde{\beta}_t^{*B} + E(X_s^{*A})' \tilde{\beta}_s^{*B}] / 4 \\
&+ [E(X_t^{*B})' \tilde{\beta}_t^{*A} + E(X_t^{*B})' \tilde{\beta}_s^{*A} - E(X_t^{*B})' \tilde{\beta}_t^{*B} - E(X_t^{*B})' \tilde{\beta}_s^{*B}] / 4 \\
&- [E(X_s^{*B})' \tilde{\beta}_t^{*A} - E(X_s^{*B})' \tilde{\beta}_s^{*A} + E(X_s^{*B})' \tilde{\beta}_t^{*B} + E(X_s^{*B})' \tilde{\beta}_s^{*B}] / 4 \\
&+ [E(X_t^{*A})' \tilde{\beta}_t^{*A} + E(X_t^{*A})' \tilde{\beta}_s^{*A} + E(X_t^{*A})' \tilde{\beta}_t^{*B} + E(X_t^{*A})' \tilde{\beta}_s^{*B}] / 4 \\
&- [E(X_s^{*A})' \tilde{\beta}_t^{*A} - E(X_s^{*A})' \tilde{\beta}_s^{*A} - E(X_s^{*A})' \tilde{\beta}_t^{*B} - E(X_s^{*A})' \tilde{\beta}_s^{*B}] / 4 \\
&- [E(X_t^{*B})' \tilde{\beta}_t^{*A} - E(X_t^{*B})' \tilde{\beta}_s^{*A} - E(X_t^{*B})' \tilde{\beta}_t^{*B} - E(X_t^{*B})' \tilde{\beta}_s^{*B}] / 4 \\
&+ [E(X_s^{*B})' \tilde{\beta}_t^{*A} + E(X_s^{*B})' \tilde{\beta}_s^{*A} + E(X_s^{*B})' \tilde{\beta}_t^{*B} + E(X_s^{*B})' \tilde{\beta}_s^{*B}] / 4 \\
&+ [E(X_t^{*A})' \tilde{\beta}_t^{*A} - E(X_t^{*A})' \tilde{\beta}_s^{*A} + E(X_t^{*A})' \tilde{\beta}_t^{*B} - E(X_t^{*A})' \tilde{\beta}_s^{*B}] / 4 \\
&+ [E(X_s^{*A})' \tilde{\beta}_t^{*A} - E(X_s^{*A})' \tilde{\beta}_s^{*A} + E(X_s^{*A})' \tilde{\beta}_t^{*B} - E(X_s^{*A})' \tilde{\beta}_s^{*B}] / 4 \\
&- [E(X_t^{*B})' \tilde{\beta}_t^{*A} + E(X_t^{*B})' \tilde{\beta}_s^{*A} - E(X_t^{*B})' \tilde{\beta}_t^{*B} + E(X_t^{*B})' \tilde{\beta}_s^{*B}] / 4 \\
&- [E(X_s^{*B})' \tilde{\beta}_t^{*A} + E(X_s^{*B})' \tilde{\beta}_s^{*A} - E(X_s^{*B})' \tilde{\beta}_t^{*B} + E(X_s^{*B})' \tilde{\beta}_s^{*B}] / 4 \\
&= [(\beta_{0,t}^A - \beta_{0,s}^A) - (\beta_{0,t}^B - \beta_{0,s}^B)] + [(\bar{\beta}_t^{*A} - \bar{\beta}_s^{*A}) - (\bar{\beta}_t^{*B} - \bar{\beta}_s^{*B})] \\
&+ E(X_t^{*A})' \tilde{\beta}_t^{*A} - E(X_s^{*A})' \tilde{\beta}_s^{*A} - E(X_t^{*B})' \tilde{\beta}_t^{*B} + E(X_s^{*B})' \tilde{\beta}_s^{*B} \\
&= [1, E(X_t^{*A})]' [\beta_{0,t}^A, \tilde{\beta}_t^A + \tilde{\beta}_t^A] - [1, E(X_s^{*A})]' [\beta_{0,s}^A, \tilde{\beta}_s^A + \tilde{\beta}_s^A] \\
&- [1, E(X_t^{*B})]' [\beta_{0,t}^B, \tilde{\beta}_t^B + \tilde{\beta}_t^B] + [1, E(X_s^{*B})]' [\beta_{0,s}^B, \tilde{\beta}_s^B + \tilde{\beta}_s^B] \\
&= [E(X_t^A)' \beta_t^A - E(X_s^A)' \beta_s^A] - [E(X_t^B)' \beta_t^B - E(X_s^B)' \beta_s^B] \\
&= \Delta Y^A - \Delta Y^B
\end{aligned}$$

□