Exercise Set 1.1 (Page 11)

1. There are various answers depending on the algorithm used.

(a) In six-digit decimal rounded arithmetic \( A = 1.33333, B = 0.33333, C = 0.099999, \) and \( U = 0.00001 \times 10^{-5} \),
(b) Same as those in part (a).
(c) The exact \( a = 0.5 \times 10^{-5} \) in rounded arithmetic and \( 10^{-5} \) in chopped.
(d) There are various answers depending on the computer/calculator used.

1.3 (a) With \( n = 3 \) in six-digit decimal chopped arithmetic \( A = 123.456, B = 123.579, y = 0.123000 \), and

\( x - y = 0.000456 \).

(b) With \( n = 5, A = 12345.6, B = 12345.7, y = 0.100000 \), and \( x - y = 0.023456 \).

1.4 \( |F(x) - F(0)|/|F(0)| = |\cos x - 1| \leq 0.5 \times 10^{-2} \cos x \leq 0.582 \)

1.5 \( (1 + 2\epsilon + \epsilon^2)x^2 + 2(1 + \epsilon)x + 1 - x^2 + 2x - 1)/(x - 1)^2 = \epsilon^3 x^2/(x - 1)^2 \). The last quantity becomes infinite as \( x \) approaches 1.

1.6 (a) \( S_{15} \) should be negative on most computers and will have large magnitude if calculations are performed in single precision. For example, in IEEE single precision \( S_{15} = -4.3902E1 + 98 \) and in IEEE double precision \( S_{15} = -4.36726 \) using the recursion.

(b) \( S_n - \tilde{S}_n = \left[ 1 - 2n(2n - 1) \right] \bar{x} \bar{x} \nabla - \left[ 1 - 2n(2n - 1) \right] \tilde{S}_{n-1} \nabla / \pi^2 \)\n
\[ S_n - \tilde{S}_n = -2n(2n - 1) \tilde{S}_{n-1} / \pi^2 \]

For \( n = 15 \), the final perturbation is \(-3.2297E+17\mu\).

(c) This recursion should be stable since the \((2n)\) is in the denominator rather than in the numerator. Starting with \( 0 \), the backwards recursion yielded a very accurate \( S_0 = 2.0000000000 \) so any initial error was damped out.

Exercise Set 1.2 (Page 23)

1.7 With \( s = \sqrt{1 - e^2} = F(c), \delta s \approx F'(c) \delta c = -(c/s) \delta c \). For \( \pi/4 \leq \theta \leq \pi/2 \), the factor \(|c/s| = |\cot(\theta)| \leq 1 \). In this range, the value of \( s \) is as accurate as the value of \( c \), and it will be of comparable accuracy for \( b \) not greatly smaller than \( \pi/4 \). However, for \( \theta \ll 1 \), the factor \( \cot(\theta) \approx 1/\theta \) is very large and \( s \) computed in this manner is much less accurate than \( c \). The relative error

\[ \frac{\delta s}{s} \approx -\frac{1}{s} \left( \frac{c}{s} \right) \delta c = - \left( \frac{c}{s} \right)^2 \frac{\delta c}{c}. \]

Because the relative error in \( s \) is related to the relative error in \( c \) by a factor that is the square of that arising for absolute errors, conclusions about the usefulness of the scheme for the various \( b \) are essentially the same as for absolute errors.
The double precision result is exact as expected, the single precision value is the size of the unit roundoff (0.01) but not very good.

1.9 In 4 digit decimal chopped \((0.8717 + 0.8719)/2 = 1.743/2 = 0.8715\), in 4 digit decimal rounded the midpoint is \(1.744/2 = 0.8720\); neither of these is even inside the desired interval. A good alternative is to compute \(h = b - a\) and then the midpoint is \(a + h/2\). For this example, we get \(h = 0.000200\) and the midpoint is 0.8718, the exact value.

1.10 To construct a case for which there is a severe cancellation error, let \(a = 1/3\), \(b = 1/7\), \(c = 301/100\), and \(d = 701/100\). Then \(ac - bd = 1/525\). In three-digit decimal rounded arithmetic \(ac - bd\) is 0.00190; but \(f(l(ac)) = 1.01\) and \(f(l(bd)) = 1.00\) are accurate, while \(f(l(ac - bd)) = 0.0100\) has a large relative error.

1.11 (a) In IEEE single precision the final \(k = 46\) with \(S = 0.00008138\) has relative error is -0.793 or 79%.
(b) For \(x = +10\), the final \(k = 32\) with \(S = 22026.5\) compared to the exact answer of 22026.45. The relative error is an acceptable -0.22E-5.

(c) The series is a better algorithm for positive \(x\) because there is no cancellation error.

1.12 For small values of \(q\) there will be loss of significance followed by a division by a small number. Since

\[
(1 - 2q)^{-3/2} = 1 + (-3/2)(-2q) + (-3/2)(-5/2)(-2q)^2/2! + (\cdots)
\]

\[
= 1 + 3q + \frac{3 \cdot 5}{2!} q^2 + \frac{3 \cdot 7}{3!} q^3 + \cdots
\]

\[f(q) = -3 - 3 \cdot 5q/2! - 3 \cdot 5 \cdot 7q^2/3! - \cdots
\]

1.13 Some information is lost in the subtraction of \(N^2/4\) from 1 because \(N \to 0\) as \(N \to \infty\), \(1 - L_N^2/4 \approx 1 - L_N^2/8\), showing that there is severe cancellation when this quantity is subtracted from 1. The information lost is small in magnitude, but it becomes important due to cancellation, especially after it is multiplied by the large value of \(N\) to form the approximation \(N\sqrt{L_{2N}/2}\) to \(\pi\). The rearranged form avoids the cancellation in the original form of the recurrence.

1.14 For small \(r^2/\pi\) there will be loss of significance in the \((1 - \exp)\) part. Rewrite

\[
\mu_0 \frac{r_0}{\pi} \exp \left( -\frac{r^2}{8\pi} \right) \sinh \frac{r^2}{8\pi}.
\]

Miscellaneous Exercises for Chapter 1 (Page 29)

1.15 (a) Let \(x = 8.01\), \(y = 1.25\), \(z = 80.8\); then \((x \otimes y) \otimes z = 808\). While \(x \otimes (y \otimes z) = 809\).
(b) Let \(x = 113\), \(y = 111\), \(z = 7.51\); then \((x \otimes y) \otimes z = 9.51\) while \(x \otimes (y \otimes z) = 10.0\).
(c) Let \(x = 200\), \(y = -60.0\), \(z = 6.03\); then \((x \otimes y \otimes z) = -10700\). While \((x \otimes y) \otimes (x \otimes z) = -10800\).
(d) Let \(x = 1.00\), \(y = 0.00500\), \(z = -1.00\); then \((x \otimes y) \otimes z = 0.00\) while the exact answer is 0.005.

1.16 \[(N - 1)y^2 = \sum_{i=1}^{N} (x_i - \bar{x})^2 \leq \sum_{i=1}^{N} \left[ x_i^2 - 2x_i \sum_{j=1}^{N} x_j / N + \left( \sum_{j=1}^{N} x_j \right)^2 / N^2 \right] \leq \frac{N}{N^2} \sum_{i=1}^{N} x_i^2 - \left( \sum_{j=1}^{N} x_j \right)^2 / N\]
If $(N - 1)s^2$ is small relative to $\sum x_i^2$ then loss of significance must occur. This will affect the second formula the most.

2. (17) Use $\sin nx = \sin(1 + n - 1)x = \sin x \cos(n - 1)x + \cos x \sin(n - 1)x$ and $\cos nx = \cos(1 + n - 1)x = \cos x \cos(n - 1)x - \sin x \sin(n - 1)x$.

Let $e_n = s_n - \hat{e}_n = s_1(c_{n-1} - \hat{c}_{n-1}) + c_1(e_{n-1} - \hat{e}_{n-1}) = s_1 \tau_{n-1} + c_1 e_{n-1}$ and $\tau_n = c_n - \hat{c}_n = c_1(e_{n-1} - \hat{e}_{n-1}) - s_1(s_{n-1} - \hat{s}_{n-1}) = c_1 \tau_{n-1} - s_1 e_{n-1}$.

Hence, $e_n^2 + \tau_n^2 = s_1^2 \tau_{n-1}^2 + 2c_1 s_1 e_{n-1} \tau_{n-1} + c_1^2 e_{n-1}^2 + c_1^2 \tau_{n-1}^2 - 2c_1 s_1 e_{n-1} \tau_{n-1} + s_1^2 e_{n-1}^2 = e_{n-1}^2 + \tau_{n-1}^2 = \cdots = e_2^2 + e_2^2$.

Exercise Set 2.1 (Page 42)

2. (21) (a) nonsingular; $x_1 = -1, x_2 = 1/2, x_3 = 3$
(b) nonsingular; $x_1 = 1/2, x_2 = -1/2, x_3 = 1$
(c) singular (but consistent)
(d) singular (and inconsistent)
(e) singular (but consistent)
(f) nonsingular; $x_1 = -5, x_2 = 6, x_3 = -2, x_4 = 7$

2.2 $R_1 = 51.67, R_2 = 26.66, R_3 = 31.67$.

2.3 $R_G = 223, R_A = 177, C_h = 56.7607, D_h = -56.7607, C_v = 29.3397, D_v = -252.340, B_v = -147.660, B_h = 56.7607$

Exercise Set 2.2 (Page 48)

2. (24) (a) $x_1 = 9, x_2 = -36, x_3 = 30$
(b) $\begin{bmatrix} 1.0 & 0.50 & 0.33 \\ 0.50 & 0.33 & 0.25 \\ 0.33 & 0.25 & 0.20 \end{bmatrix} x = \begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix}$

(c) $U = \begin{bmatrix} 1.0 & 0.50 & 0.33 \\ 0 & 0.080 & 0.090 \\ 0 & 0 & 0.001 \end{bmatrix}$ $L^{-1}b = \begin{bmatrix} 1.0 \\ -0.50 \\ 0.22 \end{bmatrix}$

so $x_3 = 220, x_2 = -230, and x_1 = -37$.

(d) $U = \begin{bmatrix} 1.0 & 0.50 & 0.33 \\ 0 & 0.090 & 0.10 \\ 0 & 0 & 0.002 \end{bmatrix}$ $L^{-1}b = \begin{bmatrix} 1.0 \\ -0.33 \\ -0.21 \end{bmatrix}$

so $x_3 = -100, x_2 = 110, and x_1 = -21$.

(e) $x_1 = 500/9, x_2 = -2500/9, x_3 = 2300/9$
2.5 (a) 
\[ L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2/3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 6 & -2 \\ 0 & 0 & -7/3 \end{bmatrix} \]

(b) 
\[ L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1.5 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -4 & 0 \\ 0 & 0 & 7 \end{bmatrix} \]

(c) 
\[ L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 1.5 & 13 & 1 & 0 \\ 0.5 & 5 & 4 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & -3 & 2 & 5 \\ 0 & 0.5 & 0 & -0.5 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \]

2.6 The answer is unique in spite of the singularity.

\[ L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \]

Exercise Set 2.3 (Page 61)

2.7 \( r = (0.000772, 0.000350) \) and \( s = (0.000001, -0.000003) \), so the more accurate answer has the larger residual.

2.8 (a) \( 10 + 10^{18} \) becomes \( 10^{18} \), so after elimination the second equation is \( -10^{17}x_2 = -10^{17} \), which yields \( x_2 = 1 \) and \( x_1 = 0 \) after the back substitution.

(b) The scaled system is
\[
\begin{align*}
x_1 + 10^{17}x_2 &= 2 \\
10^{-17}x_1 + x_2 &= 1
\end{align*}
\]

After elimination the second equation becomes \( x_2 = 1 \), and back substitution yields \( x_1 = 1 \).

(c) \( x_1 = x_2 = 1 \)

(d) \( r_1 = 1 \) and \( r_2 = 10 \) in exact arithmetic; for the scaled version \( r_1 = r_2 = 0 \), which indicates an "exact" answer.

2.9 (a) 
\[ A^{-1} = \begin{bmatrix} \frac{10^{18}}{10^{18} - 10} & \frac{-1}{10^{18} - 10} \\ \frac{-1}{10^{18} - 10} & \frac{10^{18}}{10^{18} - 10} \end{bmatrix} \]

so \( \text{cond}(A) = (10^{18} + 10)(10^{18} + 1)/(10^{18} - 10) \approx 10^{18} \).

(b) The uncertainty in each \( x_i \) is \( \pm ||A|| \) times the right side of the condition number inequality, that is, \( \pm 0.00075386 \).

2.10 \( \text{cond}(A) \approx 10^5 \).
Exercise Set 2.4 (Page 63)

2.11 Factor/Solve produces the exact answers \( R_1 = 51.67, R_2 = 26.66, \) and \( R_3 = 31.67 \) with minor perturbations at the roundoff level. The value of COND is 1.50.

2.12 \( R_k = 223, R_A = 1177, C_h = 56.7607, D_h = -56.7607, C_v = -29.3397, D_v = -252.340, B_s = -147.660, \) \( B_h = 56.7607, \) COND = 31.26; the computed residual is at the roundoff level.

2.13 \( \text{COND} = 1.438E+4 \)

\[
A^{-1} = \begin{bmatrix}
98.331 & -199.329 & 99.998 \\
-198.662 & 399.658 & -199.995 \\
99.998 & -199.995 & 99.998 \\
\end{bmatrix}
\]

2.14 (a) \( \text{cond}(A) = 748, \) since

\[
A^{-1} = \begin{bmatrix}
-9 & -36 & 30 \\
-36 & 192 & -180 \\
30 & -180 & 180 \\
\end{bmatrix}
\]

(b) \( \text{COND} = 2659, x_1 = 55.5556, x_2 = -277.7778, \) and \( x_3 = 255.5556. \) The condition number inequality has the form \( 1 - \text{cond}(A)||\Delta A||/||A|| \) which indicates so large a perturbation that the new matrix \( A + \Delta A \) could be singular. In any case, the analysis does not apply here and all digits of the computed \( x \) are suspect.

2.15 COND = 112.9; for \( V = 50 \) we have \( v = (35, 26, 20, 15.5, 11, 5) \) with minor perturbations at the roundoff level.

(a) Output pivots (IEEE double precision) are 0.731, -0.782, and -0.100E-16; since none of these are exactly zero, \( \text{FLAG} = 0 \) on output. However, the third pivot is roughly the size of the unit roundoff (but nowhere near underflow). \( \text{COND} = 0.461E+17 \) is large.

(b) \( x = (-0.005, -0.752, 0.250) \) (answers may vary because of the near singularity). This seems harmless enough.

(c) \( x = (-0.214E+14, -0.880E+14, -0.703E+14) \). The large size of the entries makes one suspicious.

(d) The actual values will vary but the residual for (b) should have small entries while those for (c) should be large.

2.17 (a) \( \text{COND} = 6.44 \)

\[
X = \begin{bmatrix}
3.14 & 22.38 & 0.21 & 7.05 \\
-0.014 & 0.19 & 0.62 & 0.83 \\
6.25 & 3.46 & 8.85 & 18.93 \\
\end{bmatrix}
\]

For exact data the answers are very reliable; however, if the entries in \( A \) are known only to \( \pm 0.0005 \) and those in \( B \) to \( \pm 0.005 \), then from the condition number inequality

\[
||\Delta X_k||/||X_k|| \leq 6.44(0.0011 + 0.005/||B_k||)
\]

where \( X_k \) is the \( k \)th column of \( X \). For example, when \( k = 1 \) we have \( ||\Delta X_k||/||X_k|| \leq 0.018 \). In particular, \( x_{21} \) is probably incorrect; it certainly makes no sense physically.

(b) \( \text{cond}(A) = 13.65 \)

\[
A^{-1} = \begin{bmatrix}
2.78 & 2.24 & -3.06 \\
-3.05 & 1.37 & 0.21 \\
3.90 & -2.80 & 3.64 \\
\end{bmatrix}
\]

(c) The analog of (2.27) here is \( \Delta x_{ij} \approx a_{ij} \Delta b_{jj}. \) For this problem, \( a_{23} \) is much smaller than the other entries, but the rest are about the same. The \( x_3 \) values are most sensitive to changes in \( b \) since the third row of \( A^{-1} \) contains the
\[ f_2^T = (-0.1979, -0.8628E-1, -0.3957, 0.1187, 0.2699, 0.2374, 0.2015, \\
-0.5406E-1, -0.2519, -0.4819) \]
\[ f_3^T = (-0.6409, -0.2958, -0.3176E-1, 0.3845, 0.1077, 0.1906E-1, \\
0.6309, -0.1478, 0.4613, -0.3436) \]

2.22 Let \( \alpha = v^T y / (1 + v^T z) \); then
\[
(A + n v^T)(y - \alpha z) = Ay + n v^T y - \alpha Az - \alpha n v^T z = b
\]
(a) Let \( e_j \) be the vector of all zeros except the entry in row \( j \) is one. To replace column \( j \) of \( A \) by column \( j \) plus \( u \),
choose \( v = e_j \).
(b) To replace \( a_{ij} \) by \( a_{ij} + \delta \) let \( n = e_i \) and \( v = e_j \).
(c) Singularity of \( A + n v^T \) is revealed when \( 1 + v^T z = 0 \), i.e., when the Sherman-Morrison formula breaks down because of division by zero.

2.23 (a) \( \det A = 1,500,000 \)
(b) \( \det A = -0.0300 \)

Exercise Set 3.1 (Page 89)

3.1 No, coefficients of higher degree terms may be zero. For example, for the data \( x_i = i, y_i = 2i, 1 \leq i \leq 4 \), the interpolating polynomial is clearly \( P_4(x) = 2x \) which has degree only one.

3.2 If \( f(x) \) is a polynomial of degree \( N - 1 \) or less then it certainly interpolates itself at any \( N \) different points; hence, the uniqueness theorem says \( f(x) \equiv P_N(x) \).

3.3 \( P_3(x) = 2 \frac{(x-2)(x-3)}{(1)(-2)} + d \frac{(x-1)(x-3)}{(1)(-1)} + c \frac{(x-1)(x-2)}{(2)(1)} \) interpolates (1,2) (2,4) and (3,c) for any \( c \). The formula for \( P_3(x) \) simplifies to
\[
\frac{1}{2}(c-6)x^2 + (11-3c/2)x + (c-6)
\]
so that we get exact degree two as long as \( c \neq 6 \). This does not contradict Theorem 1 since the degree of \( P_2 \) is too large for that Theorem.

3.4 (a) \( a_{ij} = x_j^{i-1} \) for \( 1 \leq j \leq N \); \( b_i = f_i \) for \( 1 \leq i \leq N \).
(b) \( \text{COND} = 0.14E+9 \), yet output at \{5, 15, 30, 50, 70, 90\} agrees with that in Example 3.4.

3.5 The standard algorithm requires \( 1 + 2 + \cdots + N - 1 \) multiplications for a total of \( N(N-1)/2 \). The nested version requires only \( N - 1 \).

3.6 The coefficient matrix has \( \text{COND} = 0.66E+24 \) indicating severe ill-conditioning, yet the output \( P_{12} \) is not too bad (except for a "bulge" in [90, 100]). Note: it is important to use a very stable algorithm to evaluate \( P_{12} \) (for example, the one suggested in Exercise 3.5).
3.7 \( w_2(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.0</th>
<th>1.5</th>
<th>2.5</th>
<th>3.5</th>
<th>4.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_2(x) )</td>
<td>-362,880</td>
<td>3959.03</td>
<td>-791.81</td>
<td>304.54</td>
<td>-193.80</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>5.5</th>
<th>6.5</th>
<th>7.5</th>
<th>8.5</th>
<th>10.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_2(x) )</td>
<td>193.80</td>
<td>-304.54</td>
<td>791.81</td>
<td>-3959.03</td>
<td>362,880</td>
</tr>
</tbody>
</table>

Clearly \( w_2(x) \) has smaller values near \( x = 5 \) and becomes quite large in magnitude near the ends.

3.8 (a) From the Lagrange form \( P_N^{(N-1)}(x) = \sum f_i L_i^{(N-1)}(x) = \sum f_i \cdot (N-1)! \cdot \left( \frac{1}{\prod(x_k-x_j)} \right) \).

(b) \( f''(x) \approx f_1/(x_1-x_2)+f_2/(x_1-x_2) = (f_2-f_1)/(x_2-x_1) \).

3.9 \( H(x_n) = a \), \( H'(x_n) = b = f_n' \).

\[
H(x_{n+1}) = a + bh + ch^2 + dh^3 = f_n + hf_n' + [3(f_n-1-f_n) - 2hf_n' - hf_{n+1}'] + [hf_n' + hf_{n+1}' - 2(f_{n+1} - f_n)] = f_{n+1}.
\]

Similarly, \( H'(x_{n+1}) = f_{n+1}' \).

3.10 \( f_n = a_n, f_{n+1} = a_n + b_n h + c_n h^2 + d_n h^3, f_n' = b_n, \) and \( f_{n+1}' = b_n + 2c_n h + 3d_n h^2 \), so \( a_n = f_n \) and \( b_n = f_n' \) immediately. The other two equations say \( e_n + hd_n = (f_{n+1} - f_n - hf_n')/h^2 \) and \( 2e_n + 3hd_n = (f_{n+1}' - f_n')/h \), which can be solved by elimination to get the \( c_n \) and \( d_n \) formulas in the book.

Exercise Set 3.2 (Page 93)

3.11

<table>
<thead>
<tr>
<th>( m )</th>
<th>( N )</th>
<th>error on ([-5,5])</th>
<th>error on ([-1,1])</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>15</td>
<td>7.19</td>
<td>0.019716</td>
</tr>
<tr>
<td>10</td>
<td>21</td>
<td>59.77</td>
<td>0.005280</td>
</tr>
<tr>
<td>13</td>
<td>27</td>
<td>538.17</td>
<td>0.000748</td>
</tr>
</tbody>
</table>

The errors on \([-5,5]\) are clearly diverging, but they appear to be converging in \([-1,1]\). This data came from a sample of 1001 points.
Clearly there is rapid convergence.

Error increases as $N$ does.

After a good start the error appears to increase as $N$ does.

Exercise Set 3.3 (Page 98)

(a) $P_4(x) = \frac{x(x-1)(x-2)}{6} + 2 \frac{(x+1)(x-1)(x-2)}{2} + 2 \frac{(x+1)x(x-2)}{2} + 5 \frac{(x+1)x(x-1)}{6}$

(b)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

Hence $P_4(x) = 2 + \frac{1}{2}(x+1)x(x-1)$.

(c)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

Hence $P_4(x) = 2 + \frac{1}{2}(x+1)x(x-1)$. 
Exercises for Section 3.5 (Page 115)

3.16 (a) 
\[ P_3(x) = -4\frac{(x+1)(x-1)(x-2)}{24} + \frac{(x+2)x(x-1)(x-2)}{6} + \frac{(x+2)(x+1)(x-1)(x-2)}{4} + 2\frac{(x+2)(x+1)x(x-2)}{-6} + 10\frac{(x+2)(x+1)x(x-1)}{24} \]

(b)

\[
\begin{array}{cccc}
-2 & -4 & \ \\
-1 & 1 & 5 & \ \\
0 & 1 & 0 & -2.5 & \ \\
1 & 2 & 1 & 0.5 & 1 & \ \\
2 & 10 & 8 & 3.5 & 1 & 0
\end{array}
\]

Hence \( P_3(x) = -4 + 5(x+2) - \frac{5}{2}(x+2)(x+1) + (x+2)(x+1)x \).

(c) Let \( P_3(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 \). Then

\[
\begin{align*}
c_1 - 2c_2 + 4c_3 - 8c_4 + 16c_5 &= -4 \\
c_1 - c_2 + c_3 - c_4 + c_5 &= 1 \\
c_1 + c_2 + c_3 + c_4 + c_5 &= 2 \\
c_1 + 2c_2 + 4c_3 + 8c_4 + 16c_5 &= 10
\end{align*}
\]

so \( c_1 = 1, c_2 = -0.5, c_3 = 0.5, c_4 = 1.0, \) and \( c_5 = 0 \).

3.17

\[
\begin{array}{ccc}
0 & 0.0 & \ \\
\pi/2 & 1.0 & 2/\pi & \ \\
\pi & 0.0 & -2/\pi & -4/\pi^2 & \ \\
\end{array}
\]

Hence \( P_3(x) = 2\pi/\pi - (4/\pi^2)x(x-\pi/2) = -4x^2/\pi^2 + 4x/\pi \) as before.

3.18 A total of \( N(N-1)/2 \) divisions and \( N(N-1) \) additions or subtractions are needed for the computation of the divided differences. For each evaluation only \( N-1 \) multiplications and \( 2N-2 \) additions are needed. In contrast, for the Lagrange form it is not possible to easily separate evaluation from any other work. The total for each evaluation without being tricky is \( 2N^2 - 2N \) additions/subtractions, \( 2N^2 - 2N \) multiplications, and \( N \) divisions.

3.20

\[ f'(x_N) = S'(x_N) = b_{N-1} + 2h_{N-1}c_{N-1} + 3h_{N-1}^2d_{N-1} \]

\[ = (f_N - f_{N-1})/h_{N-1} + h_{N-1}c_{N-1} + 2h_{N-1}^2d_{N-1} \quad \text{from (3.32)} \]

\[ = (f_N - f_{N-1})/(h_{N-1} + h_{N-1}c_{N-1}/3 + 2h_{N-1}^2/3) \quad \text{from (3.33)} \]

* which is a rearrangement of (3.36).

3.21 For \( S''(x_1) = f''(x_1) \) use \( c_1 = f''(x_1); \) for \( S''(x_N) = f''(x_N) \) use \( c_N = f''(x_N) \).

3.22 \( S(5) = 0.008762, S(45) = 0.095840, S(95) = 0.84540 \), which match the known values reasonably well. Note: if a polynomial interpolant is used with all the data then \( P_{11}(5) = 0.008701, P_{11}(45) = 0.095845, \) and \( P_{11}(95) = 0.84525 \), which is pretty good considering that the condition number is on the order of \( 10^{20} \).
The results for $S(x)$ are the same (to the displayed digits) for the three different sets of $(x_i)$.

The values of $P_9(x)$ hardly change for (a)-(c), but the condition numbers drop from $10^{10}$ to $10^{13}$ to 2700. The accuracy is surprisingly good, but it does deteriorate near the ends. The cubic spline on #3.15 is a little better. Note: because of the large condition number, results may vary greatly from one machine to another.

The results from (b) are better than those from (a) but neither is really accurate (especially for large $f$).

Using all the data but that at \{765, 825, 855, 875, 915, 935\} produces an $S(T)$ for which

This is not all that accurate; a graph would show some undesirable oscillations.

Using all the data but that at \{21, 22.6, 22.8, 23.0, 23.2, 23.4\} produces an $S(x)$ for which

This is good for small $x$, but deteriorates eventually. For this choice of interpolating points there are 10 sign changes in the $(x_i)$ sequence indicating ten inflection points, not one. Hence, there must be a lot of undesired oscillation; however, a graph of $S(x)$ would show that, for the most part, the amplitudes of the oscillations are small enough to be visible.

Using all the data except at \{5.1, 5.3, 5.5, 5.7, 5.9, 6.5, 7.5\} produces an $S(r)$ for which

Note: the bottom line of the table is for a polynomial interpolant over the same data. There is oscillation and poor
There are two ways to approach this problem: (a) interpolate the given data with \( S(x) \), then evaluate \( |xS'(x)|^2 \) at the data points; (b) interpolate \( \{x_i, y_i\} \) then evaluate \( S'(x_i) \) for each \( i \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>5.0</th>
<th>7.5</th>
<th>9.9</th>
<th>12.9</th>
<th>15.1</th>
<th>16.3</th>
<th>16.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.0561</td>
<td>0.1482</td>
<td>0.1480</td>
<td>1.449</td>
<td>1.129</td>
<td>5.053</td>
<td>6.724</td>
</tr>
<tr>
<td>(b)</td>
<td>0.0733</td>
<td>0.1652</td>
<td>0.0979</td>
<td>1.491</td>
<td>1.062</td>
<td>5.270</td>
<td>6.590</td>
</tr>
</tbody>
</table>

The large discrepancy between the two methods suggests that neither may be all that accurate. Using the shape-preserving spline \( H(x) \) is probably a better idea. The two approaches for \( H(x) \) produced:

<table>
<thead>
<tr>
<th>( x )</th>
<th>5.0</th>
<th>7.5</th>
<th>9.9</th>
<th>12.9</th>
<th>15.1</th>
<th>16.3</th>
<th>16.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.0452</td>
<td>0.1214</td>
<td>0.2762</td>
<td>0.858</td>
<td>1.270</td>
<td>2.177</td>
<td>5.858</td>
</tr>
<tr>
<td>(b)</td>
<td>0.0274</td>
<td>0.1164</td>
<td>0.2690</td>
<td>0.817</td>
<td>1.322</td>
<td>2.196</td>
<td>5.871</td>
</tr>
</tbody>
</table>

At least there is a little consistency here.

3.31 \( S' = 0 \) in \( [x_{n-1}, x_n] \) if and only if \( b_n + 2c_n(z - x_n) + 3d_n(z - x_n)^2 = 0 \) for \( x_n \leq z \leq x_{n-1} \). Using the quadratic formula this reduces to the statement in the text. Checking \( b_n b_{n+1} < 0 \) will not detect all zeros of \( S' \), since \( S' \) a piecewise quadratic may have two zeros in a particular \( [x_n, x_{n+1}] \) and consequently \( S'(x_n)S'(x_{n+1}) > 0 \).

3.32 \( S'' = 0 \) in \( [x_n, x_{n+1}] \) if and only if \( 2c_n + 6d_n(z - x_n) = 0 \) for \( x_n \leq z \leq x_{n+1} \). If and only if \( z = x_n - c_n/(3d_n) \) for \( x_n \leq z \leq x_{n+1} \). Since \( S''(x) \) is a straight line in each \( [x_n, x_{n+1}] \), it can have a zero there if and only if \( S''(x_n)S''(x_{n+1}) < 0 \), i.e., \( c_n c_{n+1} < 0 \).

3.33 For the data used in Exercise 3.15 above, the resulting \( S(x) \) had a local maximum at \((4601.3, 0.8360)\), and local minima at \((3514.9, 0.3811)\) and \((4602.5, 0.1353)\).

3.34 Even though the data appears to have only the one local maximum, the \( S(x) \) of Exercise 3.18 has a local maximum at each of \((746.0.669), (897.6, 2.183)\), and \((1044.5, 0.615)\). Only the one at \( x = 897.6 \) is reasonable, but it probably is fairly accurate.

3.35 For the choice of the 12 data points in Exercise 3.20 above, there was one critical point at \( r = 5.5544 \) for which \( S(r) = -12.036 \); there were two inflection points at \((6.199, -8.979)\) and \((9.685, -0.6798)\). The second inflection point is spurious.

3.36 Using the 16 indicated points, the resulting \( S(x) \) had two critical points: a local maximum at \((4.590, 6.578)\) and a local minimum (spurious?) at \((5.574, 4.685)\). There were six inflection points at \( v = 1.340, 2.753, 4.359, 4.811, 5.824, \) and \( 6.186 \).

Exercises for Section 3.6 (Page 127)

3.37 \( a = f_{11}, b = (f_{21} - f_{11})/(x_2 - x_1), c = (f_{12} - f_{11})/(y_2 - y_1), d = (f_{11} + f_{22} - f_{12} - f_{21})/(x_2 - x_1)(y_2 - y_1) \)

Miscellaneous Exercises for Chapter 3 (Page 132)

3.38 \( Q(x, y) = 1 - 3(x + y) + 2(x + y)^2 \)

3.39 \( \{f_i\} = \{0, 1.54, 2.81, 3.65, 4.49, 5.23, 5.78, 6.13, 6.46, 6.76, 7.00\} \). The graph is a spiral in the \( xy \)-plane.