Attractors for Lattice FitzHugh-Nagumo Systems*

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Abstract

The FitzHugh-Nagumo system on infinite lattices is studied. By “tail ends” estimates on solutions, it is proved that the system is asymptotically compact in a weighted $l^2$ space and has a compact global attractor containing traveling wave solutions. The singular limiting behavior of global attractors is also investigated as a singular parameter $\epsilon \to 0$. It is showed that the limiting system for $\epsilon = 0$ has no global attractor, but all the global attractors for perturbed systems are contained in a common compact subset of the phase space when $\epsilon$ is positive but small. Further, a compact local attractor for the limiting system is constructed, and the upper semicontinuity of global attractors is established when $\epsilon \to 0$.

Key words. global attractor, lattice dynamical system, FitzHugh-Nagumo equation.


1 Introduction

In this paper, we study the dynamics of the FitzHugh-Nagumo system on infinite lattices:

$$\frac{dv_i}{dt} = (v_{i-1} - 2v_i + v_{i+1}) + h(v_i) - w_i,$$
$$\frac{dw_i}{dt} = \epsilon (v_i - \gamma w_i),$$

where $i \in \mathbb{Z}$, $v = (v_i)_{i \in \mathbb{Z}}$ and $w = (w_i)_{i \in \mathbb{Z}}$ are two sequences, $\epsilon$ and $\gamma$ are positive constants, $h$ is a smooth nonlinear function. The FitzHugh-Nagumo system arises as a model describing the signal transmission across axons in neurobiology (see, e.g., [8, 25]). In this paper we consider the lattice FitzHugh-Nagumo system which models the propagation of action potentials in myelinated nerve axons (see [21]).

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Lattice differential equations have many applications where the spatial structure has a discrete character. Among such examples are image processing [16, 17, 18], pattern recognition [12, 14], chemical reaction [22, 26], electrical engineering [11], propagation of nerve pulses in myelinated axons [8, 9, 27, 28], etc. Lattice systems also arise from spatial discretizations of partial differential equations. In this respect, the reader may consult Hale [24] for more details.

Recently, the properties of solutions of lattice dynamical systems have been extensively studied. We refer the reader to [1, 4, 5, 13, 19, 20, 22, 30, 31, 38] for traveling wave solutions; [12, 14, 15] for chaotic behavior of solutions; and [6, 7, 10, 24, 37, 39] for long-time behavior of the systems.

In this paper, we investigate global attractors for the infinite-dimensional lattice FitzHugh-Nagumo system. The existence of a compact global attractor for the lattice Reaction-Diffusion equation was proved by the authors [7] by showing that the “tail ends” of solutions on infinite lattices are uniformly small when time is sufficiently large. The idea of “tail ends” estimates was further developed in [37], where the author proved that, if a lattice dynamical system has a bounded absorbing set in a weighted $l^2$ space, then the uniform smallness of “tail ends” of solutions is not only a sufficient condition, but also a necessary condition for existence of a global attractor. In the present paper, we establish the “tail ends” estimates for the lattice FitzHugh-Nagumo system and show that, for every positive $\epsilon$, the lattice FitzHugh-Nagumo system has a global attractor in a weighted $l^2$ space, which is compact as well as contains traveling wave solutions.

Another goal of this paper is to study the singular limiting behavior of global attractors for the lattice FitzHugh-Nagumo system as $\epsilon \rightarrow 0$. We will show that there is a common compact subset that contains all global attractors for all positive but small $\epsilon$. This means that the ultimate dynamics of the system for all small $\epsilon > 0$ is governed by the common compact subset. However, for the limiting system with $\epsilon = 0$, the $w$-component is conserved and therefore the dynamics of the limiting system cannot be confined to a compact subset. This demonstrates that the dynamics of perturbed system is significantly different from the limiting system. Although the limiting system has no global attractors in a phase space, we will construct a compact local attractor for the limiting system and then prove that global attractors of the perturbed systems converge to this local attractor as $\epsilon \rightarrow 0$.

This paper is organized as follows. The next section contains our main results. In Section 3, we define a continuous semigroup in a weighted $l^2$ space for the lattice FitzHugh-Nagumo system.
Section 4 is devoted to the existence of a global attractor for the lattice system. We first show that the system has a bounded absorbing set and then prove that the “tail ends” of solutions are uniformly small when time is large enough. In Section 5, we derive the uniform bounds in $\epsilon$ of global attractors of the perturbed systems, and prove that all global attractors for small $\epsilon > 0$ is contained in a bounded subset of the phase space. In Section 6, we improve the result in the previous section, and show that the union of global attractors for small $\epsilon$ is not only bounded, but also precompact in the norm topology of the phase space. In the last section, we construct a compact local attractor for the limiting system with $\epsilon = 0$, and establish the upper semicontinuity of global attractors.

2 Main Results

Consider the following discrete FitzHugh-Nagumo equations defined on the integer set $\mathbb{Z}$:

$$\frac{dv_i}{dt} = (v_{i-1} - 2v_i + v_{i+1}) + h(v_i) - w_i, \quad (2.1)$$
$$\frac{dw_i}{dt} = \epsilon(v_i - \gamma w_i), \quad (2.2)$$

with the initial data

$$v_i(0) = v_{0,i}, \quad w_i(0) = w_{0,i}, \quad (2.3)$$

where $v = (v_i)_{i \in \mathbb{Z}}$ and $w = (w_i)_{i \in \mathbb{Z}}$ are two sequences, $\epsilon$ and $\gamma$ are positive constants, $h$ is a smooth function that satisfies, for some positive constants $\alpha$, $\beta$ and $\kappa$:

$$h(0) = 0, \quad h(v) \leq -\alpha v^2 + \beta, \quad h'(v) \leq \kappa, \quad \text{for all } v \in \mathbb{R}. \quad (2.4)$$

As a particular example, the cubic function $h = v(1 - v)(v - a)$ for $a > 0$ satisfies (2.4) with

$$\alpha = a, \quad \beta = \frac{27}{256} (a + 1)^4 \quad \text{and} \quad \kappa = \frac{a^2 - a + 1}{3}.$$ 

Another particular example of interest is McKean’s caricature of the cubic (see [32, 33, 21]) for $1 > a > 0$, $h(v) = H(v-a) - v$ where $H$ is the Heaviside step function,

$$H(x) = \begin{cases} 
0, & x < 0, \\
[0,1], & x = 0 \\
1, & x > 0,
\end{cases} \quad (2.5)$$
The McKean $h$ may be smoothly approximated in a small neighborhood of $v = a$ so that the approximation satisfies (2.4).

In this paper, we fix $\sigma > \frac{1}{2}$ and denote by $l^2_{\sigma}$ the weighted space:

$$l^2_{\sigma} = \left\{ u = (u_i)_{i \in \mathbb{Z}} : \sum_{i \in \mathbb{Z}} (1 + |i|^2)^{-\sigma} |u_i|^2 < \infty \right\}$$

with the norm

$$||u||_{\sigma} = \left( \sum_{i \in \mathbb{Z}} (1 + |i|^2)^{-\sigma} |u_i|^2 \right)^{\frac{1}{2}}.$$

Note that $l^2_{\sigma}$ is a Hilbert space with the inner product

$$(u, v)_{\sigma} = \sum_{i \in \mathbb{Z}} (1 + |i|^2)^{-\sigma} u_i v_i, \quad u = (u_i)_{i \in \mathbb{Z}}, \quad v = (v_i)_{i \in \mathbb{Z}} \in l^2_{\sigma}.$$

Due to the weight $(1 + |i|^2)^{-\sigma}$ with $\sigma > \frac{1}{2}$, the space $l^2_{\sigma}$ contains all bounded sequences. Particularly, it contains all traveling wave solutions. Throughout this paper, we use $\phi_\delta$ to denote the function:

$$\phi_\delta(x) = (1 + |\delta x|^2)^{-\sigma}, \quad x \in \mathbb{R}, \quad (2.6)$$

where

$$\delta = \min\{1, \frac{1}{6\sigma 3^\sigma}, \frac{\alpha}{2\sigma 3^\sigma}, \sqrt{\frac{\alpha}{2\sigma 3^\sigma}}\}. \quad (2.7)$$

The condition (2.7) for $\delta$ is a technical condition which is useful for establishing uniform estimates on solutions in time. It is easy to verify that $\phi_\delta$ satisfies the following inequalities:

$$\left| \frac{d}{dx} \phi_\delta(x) \right| \leq \delta \sigma \phi_\delta(x), \quad x \in \mathbb{R}, \quad (2.8)$$

$$||u||_{\sigma}^2 \leq \sum_{i \in \mathbb{Z}} \phi_\delta(i) u_i^2 \leq \delta^{-2\sigma} ||u||_{\sigma}^2, \quad (2.9)$$

$$3^{-\sigma} \phi_\delta(i) \leq \phi_\delta(i \pm 1) \leq 3^{\sigma} \phi_\delta(i), \quad i \in \mathbb{Z}, \quad (2.10)$$

$$|\phi_\delta(i + 1) - \phi_\delta(i)| \leq \delta \sigma 3^{\sigma} \phi_\delta(i), \quad i \in \mathbb{Z}. \quad (2.11)$$

Inequality (2.9) shows that the norm $|| \cdot ||_{\sigma}$ is equivalent to the norm $\left( \sum_{i \in \mathbb{Z}} \phi_\delta(i) u_i^2 \right)^{\frac{1}{2}}$ for $u = (u_i)_{i \in \mathbb{Z}} \in l^2_{\sigma}$. As usual, we denote the norm and the inner product of $l^2$ by $|| \cdot ||$ and $(\cdot, \cdot)$, respectively.

We now describe the main results of this paper. The first result shows that one can define a continuous dynamical system for problem (2.1)-(2.3) in the space $l^2_{\sigma} \times l^2_{\sigma}$. More precisely, we have
Theorem 2.1. Suppose (2.4) holds and \( \sigma > \frac{1}{2} \). Then for every \( \epsilon > 0 \), one can associate problem (2.1)-(2.3) with a continuous semigroup \( S(t)_{t \geq 0} \) in \( l^2_{\sigma} \times l^2_{\sigma} \) such that when the initial datum \( (v_0, w_0) \in l^2 \times l^2 \), \( S(t)(v_0, w_0) \) is the unique solution of problem (2.1)-(2.3).

The next result shows that the dynamical system \( S(t)_{t \geq 0} \) has a global attractor in \( l^2_{\sigma} \times l^2_{\sigma} \).

Theorem 2.2. Suppose (2.4) holds and \( \sigma > \frac{1}{2} \). Then for every \( \epsilon > 0 \), the dynamical system \( S(t)_{t \geq 0} \) has a global attractor \( A_{\epsilon} \) in \( l^2_{\sigma} \times l^2_{\sigma} \), which is a compact invariant set and attracts every bounded set with respect to the norm topology of \( l^2_{\sigma} \times l^2_{\sigma} \).

We now investigate the limiting behavior of the global attractors \( A_{\epsilon} \) when \( \epsilon \to 0 \). Note that when \( \epsilon = 0 \), the limiting system for problem (2.1)-(2.2) is given by

\[
\begin{align*}
\frac{dv_i}{dt} &= (v_{i-1} - 2v_i + v_{i+1}) + h(v_i) - w_i, \\
\frac{dw_i}{dt} &= 0,
\end{align*}
\]

(2.12)

(2.13)

Given initial datum \( (v_0, w_0) \), by (2.13), it follows that \( w(t) = w_0 \) for all \( t \geq 0 \). Therefore, there are no bounded subsets which contain all solutions of problem (2.12)-(2.13). However, for the perturbed system, all the global attractors \( A_{\epsilon} \) for small \( \epsilon > 0 \) are uniformly bounded in \( l^2_{\sigma} \times l^2_{\sigma} \), which is stated as follows.

Theorem 2.3. Suppose (2.4) holds and \( \sigma > \frac{1}{2} \). Then the global attractors \( A_{\epsilon} \) are uniformly bounded in \( \epsilon \) for \( 0 < \epsilon < \min\{1, \frac{\alpha}{2\gamma}\} \) in \( l^2_{\sigma} \times l^2_{\sigma} \). More precisely, for all \( 0 < \epsilon < \min\{1, \frac{\alpha}{2\gamma}\} \) and \( (v, w) \in A_{\epsilon} \),

\[
\|v\|_{l^2_{\sigma}}^2 \leq \frac{2\beta(1 + \alpha\gamma)}{\alpha^2\gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i),
\]

and

\[
\|w\|_{l^2_{\sigma}}^2 \leq \frac{\beta}{\gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i),
\]

where \( \alpha \) and \( \beta \) are the constants in (2.4), \( \delta \) the constant given by (2.7).

Next, we show that the union of all global attractors \( A_{\epsilon} \) is not only bounded, but also precompact in \( l^2_{\sigma} \times l^2_{\sigma} \).

Theorem 2.4. Suppose (2.4) holds, \( \sigma > \frac{1}{2} \) and \( \epsilon_0 = \min\{1, \frac{\alpha}{2\gamma}\} \). Then the union \( \bigcup_{0<\epsilon<\epsilon_0} A_{\epsilon} \) of all global attractors is precompact in the space \( l^2_{\sigma} \times l^2_{\sigma} \).
Based on Theorem 2.4, we will show that the global attractors \( A_\epsilon \) are upper semicontinuous at \( \epsilon = 0 \) in the following sense.

**Theorem 2.5.** Suppose (2.4) holds and \( \sigma > \frac{1}{2} \). Then there exists a local compact attractor \( A_0 \) for the limiting system (2.12)-(2.13) in \( l_2^2 \times l_2^2 \) such that

\[
\lim_{\epsilon \to 0} d_{l_2^2 \times l_2^2}(A_\epsilon, A_0) = 0,
\]

where

\[
d_{l_2^2 \times l_2^2}(A_\epsilon, A_0) = \sup_{a \in A_\epsilon, b \in A_0} \inf_{\epsilon \in A} \|a - b\|_\sigma.
\]

### 3 Dynamical Systems Associated with Problem (2.1)-(2.3)

In this section, we define a dynamical system \( S(t)_{t \geq 0} \) for problem (2.1)-(2.3). To that end, we first show that problem (2.1)-(2.3) is well-posed in the standard \( l^2 \times l^2 \) space, and then extend the solution operator from \( l^2 \times l^2 \) to the weighted space \( l_\sigma^2 \times l_\sigma^2 \).

Next, we reformulate problem (2.1)-(2.3) as an abstract system of ordinary differential equations in \( l^2 \). For each sequence \( v = (v_i)_{i \in \mathbb{Z}} \in l^2 \), define linear operators on \( l^2 \) by

\[
(Bv)_i = v_{i+1} - v_i, \quad (B^*v)_i = v_{i-1} - v_i, \quad i \in \mathbb{Z},
\]

and

\[
(Av)_i = -v_{i-1} + 2v_i - v_{i+1}, \quad i \in \mathbb{Z}.
\]

Then we find that

\[
A = BB^* = B^*B,
\]

\[
(B^*u, v) = (u, Bv), \quad \text{for all } u, v \in l^2,
\]

and

\[
|(B\phi_\delta)(i)| \leq \delta \sigma 3^\sigma \phi_\delta(i), \quad |(B^*\phi_\delta)(i)| \leq \delta \sigma 3^\sigma \phi_\delta(i), \quad i \in \mathbb{Z}, \quad (3.1)
\]

where (3.1) is obtained from (2.11). We now define an operator \( \tilde{h} \) from \( l^2 \) to \( l^2 \) which is associated with \( h \). For each \( v = (v_i)_{i \in \mathbb{Z}} \), let \( \tilde{h}(v) = (h(v_i))_{i \in \mathbb{Z}} \). To simplify notations, we identify \( \tilde{h} \) with \( h \).
and use the same symbol $h$ to denote them. Then problem (2.1)-(2.3) is equivalent to the following system in $l^2 \times l^2$:

\[
\begin{align*}
\frac{dv}{dt} + Av &= h(v) - w, \\
\frac{dw}{dt} &= \epsilon(v - \gamma w),
\end{align*}
\] (3.2) (3.3)

with the initial data

\[v(0) = v_0, \quad w(0) = w_0.\] (3.4)

In what follows, we show that problem (3.2)-(3.4) is well-posed in $l^2 \times l^2$. It is easy to verify that $h$ maps $l^2$ to $l^2$ by simple computations. Further $h$ is locally Lipschitz in the sense that for every bounded set $Y$ in $l^2$, there exists a constant $C$ depending only on $Y$ such that

\[
\|h(u) - h(v)\| \leq C\|u - v\|, \quad \text{for all } u, v \in Y.
\]

Then it follows from the standard theory of ordinary differential equations that there exists a unique local solution $(v, w)$ for problem (3.2)-(3.4) such that $(v, w) \in C([0, T_{\max}), l^2 \times l^2)$, where $[0, T_{\max})$ is the maximal interval of existence of the solution. The next estimates show that the local solution of problem (3.2)-(3.4) is actually defined for all $t \geq 0$.

**Lemma 3.1.** Assume that (2.4) holds. Let $R$ be a positive constant such that $\|(v_0, w_0)\|_{l^2 \times l^2} \leq R$. Then the solution $(v, w)$ of problem (3.2)-(3.4) satisfies, for every $T > 0$,

\[
\|v(t)\| + \|w(t)\| \leq C, \quad \text{for all } 0 \leq t \leq T,
\]

where $C$ is a constant depending on $\epsilon, \kappa, R$ and $T$.

In the sequel, we denote by $C$ any positive constant which may change value from line to line.

**Proof.** Taking the inner product of (3.2) and (3.3) with $v$ and $w$ in $l^2$, respectively, we find that

\[
\frac{1}{2} \frac{d}{dt} \|v\|^2 + \|Bv\|^2 = (h(v), v) - (v, w),
\]

and

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2 = \epsilon(v, w) - \epsilon \gamma \|w\|^2.
\]
Then it follows that

\[ \frac{1}{2} \frac{d}{dt} (\|v\|^2 + \|w\|^2) + \|Bv\|^2 + \epsilon \gamma \|w\|^2 = (h(v), v) + (\epsilon - 1)(v, w). \]  

(3.5)

By (2.4), we find that the nonlinear term on the right-hand side of (3.5) satisfies, for some \( \xi_i \) between 0 and \( v_i \):

\[ (h(v), v) = \sum_{i \in \mathbb{Z}} h(v_i) v_i = \sum_{i \in \mathbb{Z}} (h(v_i) - h(0)) v_i = \sum_{i \in \mathbb{Z}} h'(\xi_i) v_i^2 \leq \kappa \|v\|^2. \]  

(3.6)

By (3.5)-(3.6) there exists a constant \( C_1 \) depending on \( \epsilon \) and \( \kappa \) such that for all \( t \geq 0 \):

\[ \frac{d}{dt} (\|v\|^2 + \|w\|^2) \leq C_1 (\|v\|^2 + \|w\|^2). \]

which along with Gronwall’s inequality concludes the proof.

Note that Lemma 3.1 implies that the solution of problem (3.2)-(3.4) is globally defined in \( l^2 \times l^2 \). Since \( l^2 \times l^2 \) is a subspace of the weighted space \( l^2_\sigma \times l^2_\sigma \) for \( \sigma > \frac{1}{2} \), the solutions in \( l^2 \times l^2 \) are actually functions in \( l^2_\sigma \times l^2_\sigma \). Next, we establish the Lipschitz continuity of the solutions in the weighted space, which is necessary for extension of the solution operator from \( l^2 \times l^2 \) to \( l^2_\sigma \times l^2_\sigma \).

**Lemma 3.2.** Assume that (2.4) holds and \( \sigma > \frac{1}{2} \). If \((v_1, w_1)\) and \((v_2, w_2)\) are two solutions of problem (3.2)-(3.4), then there exists a constant \( C \) depending on \( \epsilon, \kappa, \sigma \) and \( T \) such that for all \( t \in [0, T] \):

\[ \|v_1(t) - v_2(t)\|_\sigma + \|w_1(t) - w_2(t)\|_\sigma \leq C (\|v_1(0) - v_2(0)\|_\sigma + \|w_1(0) - w_2(0)\|_\sigma). \]

**Proof.** Let \( V = v_1 - v_2 \) and \( W = w_1 - w_2 \). Then it follows from (3.2)-(3.3) that

\[ \frac{dV}{dt} + AV = h(v_1) - h(v_2) - W, \]  

(3.7)

and

\[ \frac{dW}{dt} = \epsilon (V - \gamma W). \]  

(3.8)

Taking the inner product of (3.7) with \( V \) in \( l^2_\sigma \), we get

\[ \frac{1}{2} \frac{d}{dt} \|V\|_\sigma^2 + (AV, V)_\sigma = (h(v_1) - h(v_2), V)_\sigma - (W, V)_\sigma. \]  

(3.9)
We next estimate the term \((AV, V)_\sigma\), which can be written as
\[
(AV, V)_\sigma = (BV, B\bar{V}),
\] (3.10)
where \(\bar{V} = (\bar{V}_i)_{i \in \mathbb{Z}} = ((1 + i^2)^{-\sigma} V_i)_{i \in \mathbb{Z}}\), and
\[
(B\bar{V})_i = \bar{V}_{i+1} - \bar{V}_i = (B\phi_1)_i V_{i+1} + \phi_1(i)(BV)_i.
\] (3.11)
where \(\phi_1\) is the function \(\phi_\delta\) with \(\delta = 1\). Substituting (3.11) into (3.10), we get
\[
(AV, V)_\sigma = \sum_{i \in \mathbb{Z}} (BV)_i (B\phi_1)_i V_{i+1} + \sum_{i \in \mathbb{Z}} \phi_1(i)(BV)_i |^2.
\] (3.12)

By (3.1) and (2.10) we have the following bounds for the first term on the right-hand side of (3.12).
\[
| \sum_{i \in \mathbb{Z}} (BV)_i (B\phi_1)_i V_{i+1} | \leq \sum_{i \in \mathbb{Z}} \sigma^3 \phi_1(i)(BV)_i |V_{i+1}|
\]
\[
\leq \frac{1}{2} \sum_{i \in \mathbb{Z}} \phi_1(i)(BV)_i |^2 + \frac{1}{2} \sum_{i \in \mathbb{Z}} (\sigma^3)^2 \phi_1(i) |V_{i+1}|^2
\]
\[
\leq \frac{1}{2} \sum_{i \in \mathbb{Z}} \phi_1(i)(BV)_i |^2 + \frac{1}{2} \sum_{i \in \mathbb{Z}} (\sigma^3)^2 \phi_1(i - 1) |V_i|^2
\]
\[
\leq \frac{1}{2} \sum_{i \in \mathbb{Z}} \phi_1(i)(BV)_i |^2 + \frac{1}{2} (\sigma^3)^2 \sum_{i \in \mathbb{Z}} \phi_1(i) |V_i|^2
\]
\[
= \frac{1}{2} \|BV\|_\sigma^2 + \frac{1}{2} \sigma^2 3^3 \|V\|_\sigma^2.
\]
It follows from the above and (3.12) that
\[
(AV, V)_\sigma \geq \frac{1}{2} \|BV\|_\sigma^2 - \frac{1}{2} \sigma^2 3^3 \|V\|_\sigma^2.
\] (3.13)

By (3.9) and (3.13), we find that
\[
\frac{1}{2} dV^2_\sigma + \frac{1}{2} d\|BV\|_\sigma^2 \leq (h(v_1) - h(v_2), V)_\sigma - (W, V)_\sigma + \frac{1}{2} \sigma^2 3^3 \|V\|_\sigma^2.
\]

Applying (2.4) to the nonlinear term, we obtain that
\[
\frac{1}{2} dV^2_\sigma + \frac{1}{2} d\|BV\|_\sigma^2 \leq \left( \frac{1}{2} + \kappa + \frac{1}{2} \sigma^2 3^3 \right) \|V\|_\sigma^2 + \frac{1}{2} \|W\|_\sigma^2.
\] (3.14)

Taking the inner product of (3.8) with \(W\) in \(l^2_\sigma\), we find
\[
\frac{1}{2} d\|W\|_\sigma^2 = \epsilon (V, W)_\sigma - \epsilon \|W\|_\sigma^2 \leq \frac{1}{2} \epsilon \|V\|_\sigma^2 + \frac{1}{2} \epsilon \|W\|_\sigma^2.
\] (3.15)
By (3.14)-(3.15), we get
\[
\frac{d}{dt} \left( \|V\|_2^2 + \|W\|_2^2 \right) \leq C(\sigma, \kappa, \epsilon) \left( \|V\|_2^2 + \|W\|_2^2 \right),
\]
which together with Gronwall’s inequality implies Lemma 3.2. The proof is complete. \(\square\)

We are now ready to prove Theorem 2.1 to define a dynamical system for problem (2.1)-(2.3).

**Proof of Theorem 2.1.** Given \(T > 0\), By Lemma 3.2, there is a mapping \(G\) from \(l^2 \times l^2\) into \(C([0, T], l^2 \times l^2)\) such that for every \((v_0, w_0) \in l^2 \times l^2\), \(G(v_0, w_0)\) is the unique solution of problem (2.1)-(2.3). Furthermore, the mapping \(G\) is continuous from \(l^2 \times l^2\) into \(C([0, T], l^2 \times l^2)\).

Since \(l^2 \times l^2\) is dense in \(l^2 \times l^2\), \(G\) can be extended uniquely to a mapping \(\hat{G}\) from \(l^2 \times l^2\) into \(C([0, T], l^2 \times l^2)\). We now define a semigroup \(S(t)_{t \geq 0} : l^2 \times l^2 \to l^2 \times l^2\) such that, for every \(t \geq 0\) and \((v_0, w_0) \in l^2 \times l^2\), \(S(t)(v_0, w_0) = \hat{G}(v_0, w_0)(t)\). By Lemma 3.2, \(S(t)_{t \geq 0}\) is a continuous semigroup. The proof is complete.

### 4 Global Attractors

In this section, we prove the existence of a global attractor for problem (2.1)-(2.3) in \(l^2 \times l^2\) for every \(\epsilon > 0\). To that end, we first derive uniform estimates of solutions when time goes to infinity, and then establish the “tail ends” estimates of solutions for the lattice FitzHugh-Nagumo system. The idea of “tail ends” estimates was used in [36] and [7] for continuous and discrete Reaction-Diffusion equation, respectively.

**Lemma 4.1.** Suppose (2.4) holds and \(\sigma > \frac{1}{2}\). Let \(R\) be a positive constant such that \(\|(v_0, w_0)\|_\sigma \leq R\). Then \((v(t), w(t)) = S(t)(v_0, w_0)\) satisfies
\[
\|(v(t), w(t))\|_\sigma \leq M, \quad \text{for all} \quad t \geq T,
\]
where \(M\) is a constant depending on the data \((\epsilon, \alpha, \beta, \gamma, \sigma)\), and \(T\) depending on the data \((\epsilon, \alpha, \beta, \gamma, \sigma)\) and \(R\).

**Proof.** Taking the inner product of (3.2) and (3.3) with \(\epsilon v\) and \(w\) in \(l^2_\sigma\), respectively, we get
\[
\frac{1}{2} \epsilon \frac{d}{dt} \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i|^2 + \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i)(Av)_iv_i = \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i)h(v_i)v_i - \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i)w_iv_i, \quad (4.1)
\]
and
\[ \frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}} \phi_\delta(i) |w_i|^2 + \epsilon \gamma \sum_{i \in \mathbb{Z}} \phi_\delta(i) |w_i|^2 = \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) w_i v_i. \]  
(4.2)

It follows from (4.1)-(4.2) that
\[ \frac{1}{2} \frac{d}{dt} \left( \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |v_i|^2 + \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |w_i|^2 \right) + \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) (Av)_i v_i + \epsilon \gamma \sum_{i \in \mathbb{Z}} \phi_\delta(i) |w_i|^2 = \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) h(v_i) v_i. \]  
(4.3)

By (2.4), the right-hand side of (4.3) is bounded by
\[ \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) h(v_i) v_i \leq -\epsilon \alpha \sum_{i \in \mathbb{Z}} \phi_\delta(i) |v_i|^2 + \epsilon \beta \sum_{i \in \mathbb{Z}} \phi_\delta(i). \]  
(4.4)

Similar to (3.13), the following bounds hold for the second term on the left-hand side of (4.3):
\[ \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) (Av)_i v_i \geq \frac{1}{2} \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |(Bv)_i|^2 - \frac{1}{2} \epsilon \delta^2 \sigma^2 3^\sigma \sum_{i \in \mathbb{Z}} \phi_\delta(i) |v_i|^2. \]  
(4.5)

By (4.3)-(4.5) we obtain that
\[ \frac{1}{2} \frac{d}{dt} \left( \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |v_i|^2 + \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |w_i|^2 \right) + \frac{1}{2} \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |(Bv)_i|^2 + \epsilon \alpha \sum_{i \in \mathbb{Z}} \phi_\delta(i) |v_i|^2 + \epsilon \gamma \sum_{i \in \mathbb{Z}} \phi_\delta(i) |w_i|^2 \leq \frac{1}{2} \epsilon \delta^2 \sigma^2 3^\sigma \sum_{i \in \mathbb{Z}} \phi_\delta(i) |v_i|^2 + \epsilon \beta \sum_{i \in \mathbb{Z}} \phi_\delta(i). \]

Taking (2.7) into account for $\delta$, we have
\[ \frac{d}{dt} \left( \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |v_i|^2 + \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |w_i|^2 \right) + \epsilon \alpha \sum_{i \in \mathbb{Z}} \phi_\delta(i) |v_i|^2 + 2 \epsilon \gamma \sum_{i \in \mathbb{Z}} \phi_\delta(i) |w_i|^2 \leq 2 \epsilon \beta \sum_{i \in \mathbb{Z}} \phi_\delta(i). \]

Let $C_1 = \min\{\alpha, 2 \epsilon \gamma\}$ and $C_2 = 2 \epsilon \beta \sum_{i \in \mathbb{Z}} \phi_\delta(i)$. Then it follows that
\[ \frac{d}{dt} \left( \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |v_i|^2 + \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |w_i|^2 \right) + C_1 \left( \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |v_i|^2 + \epsilon \sum_{i \in \mathbb{Z}} \phi_\delta(i) |w_i|^2 \right) \leq C_2. \]  
(4.6)
By Gronwall’s inequality, we find

\[ \epsilon \sum_{i \in \mathbb{Z}} \phi_{\delta}(i)|v_i(t)|^2 + \sum_{i \in \mathbb{Z}} \phi_{\delta}(i)|w_i(t)|^2 \]
\[ \leq e^{-C_1 t} \left( \epsilon \sum_{i \in \mathbb{Z}} \phi_{\delta}(i)|v_i(0)|^2 + \sum_{i \in \mathbb{Z}} \phi_{\delta}(i)|w_i(0)|^2 \right) + \frac{C_2}{C_1} \]
\[ \leq \delta^{-2\sigma} \max\{\epsilon, 1\} e^{-C_1 t} \|(v_0, w_0)\|_\sigma^2 + \frac{C_2}{C_1} \]
\[ \leq \delta^{-2\sigma} R^2 \max\{\epsilon, 1\} e^{-C_1 t} + \frac{C_2}{C_1} \leq \frac{2C_2}{C_1} \]

for all \( t \geq T = \frac{1}{C_1} \left( \ln(C_1 \delta^{-2\sigma} R^2 \max\{\epsilon, 1\}) - \ln C_2 \right) \), which concludes the proof.

Note that if \( \epsilon \leq \min\{1, \frac{\alpha}{2\gamma}\} \), then the above constant \( C_1 = \min\{\alpha, 2\epsilon \gamma \} = 2\epsilon \gamma \). In this case, for all \( t \geq 0 \), it follows from (4.6) that

\[ \epsilon \sum_{i \in \mathbb{Z}} \phi_{\delta}(i)|v_i(t)|^2 + \sum_{i \in \mathbb{Z}} \phi_{\delta}(i)|w_i(t)|^2 \leq \delta^{-2\sigma} e^{-2\epsilon \gamma t} \|(v_0, w_0)\|_\sigma^2 + \frac{\beta}{\gamma} \sum_{i \in \mathbb{Z}} \phi_{\delta}(i), \tag{4.7} \]

which proves useful to establish the uniform bounds in \( \epsilon \) of global attractors in Section 5. In the sequel, we denote by \( B \) the bounded set in \( l_2^2 \times l_2^2 \):

\[ B = \{(v, w) \in l_2^2 \times l_2^2 : \|(v, w)\|_\sigma \leq M\}, \tag{4.8} \]

where \( M \) is the constant in Lemma 4.1. It follows from Lemma 4.1 that the set \( B \) is an absorbing set for the dynamical system \( S(t), t \geq 0 \) in the phase space \( l_2^2 \times l_2^2 \). Next, we derive the “tail ends” estimates on solutions when \( t \to \infty \), which is crucial for proving the asymptotic compactness of the dynamical system.

**Lemma 4.2.** Suppose (2.4) holds and \( \sigma > \frac{1}{2} \). Let \( R \) be a positive number. Then \((v(t), w(t)) = S(t)(v_0, w_0)\) satisfies

\[ \lim_{k \to \infty} \lim_{t \to \infty} \sup_{\|(v_0, w_0)\|_\sigma \leq R} \sup_{|i| \geq k} \epsilon \phi_{\delta}(i) \left( |v_i(t)|^2 + |w_i(t)|^2 \right) = 0. \]

**Proof.** Let \( \theta \) be a smooth cut-off function satisfying \( 0 \leq \theta(s) \leq 1 \) for \( s \geq 0 \) and

\[ \theta(s) = 0 \text{ for } 0 \leq s \leq 1; \quad \theta(s) = 1 \text{ for } s \geq 2. \]

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Taking the inner product of (3.2) with $\epsilon \left( \theta \left( \frac{|i|}{k} \right) v_i \right)$ in $l_2^2$, we find

$$\frac{1}{2} \frac{d}{dt} \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) |v_i|^2 + \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) (Av)_i v_i$$

$$= \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) h(v_i) v_i - \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) w_i v_i.$$  \hspace{1cm} (4.9)

Note that the second term on the left-hand side of (4.9) can be rewritten as

$$\epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) (Av)_i v_i = \epsilon (Av, v) = \epsilon (Bv, B\bar{v}),$$ \hspace{1cm} (4.10)

where $\bar{v} = (\bar{v}_i)_{i \in Z} = \left( \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) v_i \right)_{i \in Z}$, and

$$(B\bar{v})_i = \bar{v}_{i+1} - \bar{v}_i = \left( \theta \left( \frac{|i+1|}{k} \right) - \theta \left( \frac{|i|}{k} \right) \right) \phi_\delta(i+1) v_{i+1}$$

$$+ \theta \left( \frac{|i|}{k} \right) (B\phi_\delta)_i v_{i+1} + \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) (Bv)_i.$$ \hspace{1cm} (4.11)

By (4.10)-(4.11) we find

$$\epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) (Av)_i v_i = \epsilon \sum_{i \in Z} \left( \theta \left( \frac{|i+1|}{k} \right) - \theta \left( \frac{|i|}{k} \right) \right) \phi_\delta(i+1) v_{i+1} (Bv)_i$$

$$+ \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) (B\phi_\delta)_i v_{i+1} (Bv)_i + \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) |(Bv)_i|^2.$$ \hspace{1cm} (4.12)

By (2.4), the first term on the right-hand side of (4.9) is bounded by

$$\epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) h(v_i) v_i \leq -\epsilon \alpha \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) |v_i|^2 + \epsilon \beta \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i).$$ \hspace{1cm} (4.13)

By (4.9) and (4.12)-(4.13), we find that

$$\frac{1}{2} \frac{d}{dt} \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) |v_i|^2 + \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) |(Bv)_i|^2 + \epsilon \alpha \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) |v_i|^2$$

$$\leq -\epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) w_i v_i - \epsilon \sum_{i \in Z} \left( \theta \left( \frac{|i+1|}{k} \right) - \theta \left( \frac{|i|}{k} \right) \right) \phi_\delta(i+1) v_{i+1} (Bv)_i$$

$$- \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) (B\phi_\delta)_i v_{i+1} (Bv)_i + \epsilon \beta \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i).$$ \hspace{1cm} (4.14)

Taking the inner product of (3.3) with $\left( \theta \left( \frac{|i|}{k} \right) w_i \right)$ in $l_2^2$, we obtain

$$\frac{1}{2} \frac{d}{dt} \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) |w_i|^2 + \epsilon \gamma \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) |w_i|^2 = \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_\delta(i) w_i v_i.$$ \hspace{1cm} (4.15)
Summing up (4.14)-(4.15), we find that

\[
\frac{1}{2} \frac{d}{dt} \left( \epsilon \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i)|v_i|^2 + \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i)|w_i|^2 \right) + \epsilon \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i)|(Bv)_i|^2 + \epsilon \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i)|v_i|^2 + \epsilon \gamma \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i)|w_i|^2 \\
\leq -\epsilon \sum_{i \in \mathbb{Z}} \left( \theta(\frac{|i+1|}{k}) - \theta(\frac{|i|}{k}) \right) \phi_{\delta}(i+1)v_{i+1}(Bv)_i \\
-\epsilon \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) (B\phi_{\delta})_i v_{i+1}(Bv)_i + \epsilon \beta \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i). \tag{4.16}
\]

We now estimate the right-hand side of (4.16). By (2.10), we see that the first term on the right-hand side of (4.16) is bounded by

\[
|\epsilon \sum_{i \in \mathbb{Z}} \left( \theta(\frac{|i+1|}{k}) - \theta(\frac{|i|}{k}) \right) \phi_{\delta}(i+1)v_{i+1}(Bv)_i| \\
\leq \frac{\epsilon}{k} \sum_{i \in \mathbb{Z}} |\theta'(\xi_i)\phi_{\delta}(i+1)v_{i+1}(Bv)_i| \\
\leq \frac{\epsilon}{k} \sum_{i \in \mathbb{Z}} |\theta'(\xi_i)\phi_{\delta}(i+1)|v_{i+1}^2 - v_{i+1}v_i| \\
\leq \frac{\epsilon C_1}{k} \sum_{i \in \mathbb{Z}} \phi_{\delta}(i+1)|v_{i+1}|^2 + \frac{\epsilon C_2}{k} \sum_{i \in \mathbb{Z}} \phi_{\delta}(i+1)|v_i|^2 \\
\leq \left( \frac{\epsilon C_1}{k} + \frac{\epsilon C_2 3^\sigma}{k} \right) \sum_{i \in \mathbb{Z}} \phi_{\delta}(i)|v_i|^2 \leq \frac{\epsilon C_3}{k} \|v(t)\|_2^2 \leq \frac{\epsilon C_4}{k}, \tag{4.17}
\]

for all \( t \geq T \), where the last inequality is obtained by Lemma 4.1, \( C_4 \) is a constant independent of \( \epsilon \) and \( k \). By (3.1) and (2.7), we have the estimates for the second term on the right-hand side of (4.16):

\[
e^{i} \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) (B\phi_{\delta})_i v_{i+1}(Bv)_i \leq e^{i} \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i)|v_{i+1}(Bv)_i| \\
\leq e^{i} \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i) |(Bv)_i + v_i)(Bv)_i| \\
\leq \frac{3}{2} e^{i} \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i)|v_{i+1}(Bv)_i|^2 + \frac{1}{2} e^{i} \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i)|v_i|^2 \\
\leq \frac{1}{4} e^{i} \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i)|v_{i+1}(Bv)_i|^2 + \frac{1}{4} \epsilon \alpha \sum_{i \in \mathbb{Z}} \theta(\frac{|i|}{k}) \phi_{\delta}(i)|v_i|^2. \tag{4.18}
\]
By the property of the cut-off function $\theta$, the last term on the right-hand side of (4.16) is bounded by
\[
\epsilon \beta \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) \leq \epsilon \beta \sum_{|i| \geq k} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) \leq \epsilon \beta \sum_{|i| \geq k} \phi_\delta(i). \tag{4.19}
\]

Then it follows from (4.16)-(4.19) that, for $t \geq T$,
\[
\frac{1}{2} \frac{d}{dt} \left( \epsilon \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |v_i|^2 + \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |w_i|^2 \right)
+ \frac{3}{4} \epsilon \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |(Bv)_i|^2
+ \frac{3}{4} \epsilon \alpha \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |v_i|^2
+ \epsilon \gamma \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |w_i|^2
\leq \frac{\epsilon C_4}{k} + \epsilon \beta \sum_{|i| \geq k} \phi_\delta(i). \tag{4.20}
\]

Let $C_5 = \min\{\alpha, 2\epsilon \gamma\}$. Then from (4.20) we have, for $t \geq T$,
\[
\frac{d}{dt} \left( \epsilon \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |v_i|^2 + \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |w_i|^2 \right)
+ C_5 \left( \epsilon \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |v_i|^2 + \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |w_i|^2 \right)
\leq \frac{2\epsilon C_4}{k} + 2\epsilon \beta \sum_{|i| \geq k} \phi_\delta(i). \tag{4.21}
\]

By Gronwall’s inequality, we obtain from the above, for $t \geq T$,
\[
\epsilon \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |v_i(t)|^2 + \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |w_i(t)|^2
\leq e^{-C_5(t-T)} \left( \epsilon \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |v_i(T)|^2 + \sum_{i \in \mathbb{Z}} \theta\left(\frac{|i|}{k}\right) \phi_\delta(i) |w_i(T)|^2 \right) + \frac{2\epsilon C_4}{C_5 k} + \frac{2\epsilon \beta}{C_5} \sum_{|i| \geq k} \phi_\delta(i)
\leq \delta^{-2\sigma} e^{-C_5(t-T)} \left( \epsilon \|v(T)\|_\delta^2 + \|w(T)\|_\delta^2 \right) + \frac{2\epsilon C_4}{C_5 k} + \frac{2\epsilon \beta}{C_5} \sum_{|i| \geq k} \phi_\delta(i)
\leq C_0 e^{-C_5(t-T)} \max\{1, \epsilon\} + \frac{2\epsilon C_4}{C_5 k} + \frac{2\epsilon \beta}{C_5} \sum_{|i| \geq k} \phi_\delta(i) \tag{4.22}
\]
where the last inequality is obtained by Lemma 4.1. Since $\theta(s) = 1$ for $s \geq 2$, it follows from (4.22) that, for $t \geq T$,
\[
\sum_{|i| \geq 2k} \left( \epsilon \phi_\delta(i) |v_i(t)|^2 + \phi_\delta(i) |w_i(t)|^2 \right) \leq C_0 e^{-C_5(t-T)} \max\{1, \epsilon\} + \frac{2\epsilon C_4}{C_5 k} + \frac{2\epsilon \beta}{C_5} \sum_{|i| \geq k} \phi_\delta(i). \tag{4.23}
\]
Then, Lemma 4.2 follows by first taking the limit of (4.23) as $t \to \infty$, and then taking the limit as $k \to \infty$. The proof is complete.

In what follows, we show that the dynamical system $S(t)_{t \geq 0}$ associated with problem (2.1)-(2.3) has a global attractor in $l^2_\sigma \times l^2_\sigma$. To that end, we recall the following definition in [37]:

**Definition 4.3.** A dynamical system $S(t)_{t \geq 0}$ is said to be asymptotically null in $l^2_\sigma \times l^2_\sigma$ if for any $(v_n, w_n) = ((v_n,i)_{i \in \mathbb{Z}}, (w_n,i)_{i \in \mathbb{Z}})$ bounded in $l^2_\sigma \times l^2_\sigma$ and $t_n \to \infty$, the following holds

$$
\lim_{k \to \infty} \limsup_{n \to \infty} \sum_{|i| \geq k} \phi_1(i)| (S(t_n)(v_n, w_n))_i |^2 = 0,
$$

where $\phi_1$ is the function $\phi_\delta$ given by (2.6) with $\delta = 1$.

It follows from the well-known attractor theory (see, e.g., [2, 3, 23, 29, 34, 35]) that $S(t)_{t \geq 0}$ has a global attractor in $l^2_\sigma \times l^2_\sigma$ if and only if $S(t)_{t \geq 0}$ has a bounded absorbing set and is asymptotically compact. It was proved in [37] that, if $S(t)_{t \geq 0}$ has a bounded absorbing set, then $S(t)_{t \geq 0}$ is asymptotically compact if and only if it is asymptotically null in $l^2_\sigma \times l^2_\sigma$. Therefore, the following conclusion holds.

**Proposition 4.4.** Suppose (2.4) holds and $\sigma > \frac{1}{2}$. Then $S(t)_{t \geq 0}$ has a global attractor in $l^2_\sigma \times l^2_\sigma$ if and only if $S(t)_{t \geq 0}$ has a bounded absorbing set and is asymptotically null in $l^2_\sigma \times l^2_\sigma$.

We are now ready to prove the existence of a global attractor for problem (2.1)-(2.3).

**Proof of Theorem 2.2.** Notice that $S(t)_{t \geq 0}$ has a bounded absorbing set in $l^2_\sigma \times l^2_\sigma$ which is given by (4.8). Further, it follows from Lemma 4.2 that $S(t)_{t \geq 0}$ is asymptotically null in $l^2_\sigma \times l^2_\sigma$. Therefore the existence of a global attractor for $S(t)_{t \geq 0}$ follows from Proposition 4.4. The proof is complete.

## 5 Uniform Bounds of Global Attractors in $\epsilon$

In this section, we show that the global attractors $\mathcal{A}_\epsilon$ are uniformly bounded as $\epsilon \to 0$ which follows from the next estimates.
Lemma 5.1. For any $0 < \epsilon < \min\{1, \frac{a}{2\gamma}\}$, the solution $(v, w)$ of problem (2.1)-(2.3) satisfies, for $t \geq 0$,
\[\|v(t)\|^2_\sigma \leq \frac{2\beta(1 + \alpha \gamma)}{\alpha^2 \gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i) + \left(1 + \frac{2}{\alpha(\alpha - 2\epsilon \gamma)}\right)\delta^{-2\sigma} e^{-2\epsilon \gamma t} \left(\|v_0\|^2_\sigma + \|w_0\|^2_\sigma\right),\]
and
\[\|w(t)\|^2_\sigma \leq \frac{\beta}{\gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i) + \delta^{-2\sigma} e^{-2\epsilon \gamma t} \left(\|v_0\|^2_\sigma + \|w_0\|^2_\sigma\right),\]
where $\alpha$ and $\beta$ are the constants in (2.4), and $\delta$ is given by (2.7).

Proof. Taking the inner product of (3.2) with $v$ in $l^2_\sigma$ and proceeding as the proof of Lemma 4.1, we obtain that
\[\frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i|^2 + \frac{1}{2} \sum_{i \in \mathbb{Z}} \phi_\delta(i)(Bv)_i|^2 + \alpha \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i|^2 \leq \frac{1}{2} \delta^2 \sigma^2 3^{3\sigma} \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i|^2 + \beta \sum_{i \in \mathbb{Z}} \phi_\delta(i) - \sum_{i \in \mathbb{Z}} \phi_\delta(i)w_i v_i. \tag{5.1}\]
By (2.7), the first term on the right-hand side of (5.1) is bounded by
\[\frac{1}{2} \delta^2 \sigma^2 3^{3\sigma} \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i|^2 \leq \frac{1}{4} \alpha \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i|^2, \tag{5.2}\]
and the last term on the right-hand side of (5.1) is less than
\[|\sum_{i \in \mathbb{Z}} \phi_\delta(i)w_i v_i| \leq \frac{1}{4} \alpha \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i|^2 + \frac{1}{\alpha} \sum_{i \in \mathbb{Z}} \phi_\delta(i)|w_i|^2. \tag{5.3}\]
It follows from (5.1)-(5.3) that
\[\frac{d}{dt} \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i|^2 + \sum_{i \in \mathbb{Z}} \phi_\delta(i)(Bv)_i|^2 + \alpha \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i|^2 \leq 2\beta \sum_{i \in \mathbb{Z}} \phi_\delta(i) + \frac{2}{\alpha} \sum_{i \in \mathbb{Z}} \phi_\delta(i)|w_i|^2. \tag{5.4}\]
Since $\epsilon < \min\{1, \frac{a}{2\gamma}\}$, by (4.7) we have
\[\sum_{i \in \mathbb{Z}} \phi_\delta(i)|w_i(t)|^2 \leq \frac{\beta}{\gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i) + \delta^{-2\sigma} e^{-2\epsilon \gamma t} \left(\|v_0\|^2_\sigma + \|w_0\|^2_\sigma\right). \tag{5.5}\]
Substituting (5.5) into (5.4), we find
\[\frac{d}{dt} \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i|^2 + \alpha \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i|^2 \leq \frac{2\beta(1 + \alpha \gamma)}{\alpha \gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i) + \frac{2}{\alpha} \delta^{-2\sigma} e^{-2\epsilon \gamma t} \left(\|v_0\|^2_\sigma + \|w_0\|^2_\sigma\right), \tag{5.6}\]
which implies that
\[
\sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i(t)|^2 \leq e^{-\alpha t} \sum_{i \in \mathbb{Z}} \phi_\delta(i)|v_i(0)|^2 + \frac{2\beta(1 + \alpha \gamma)}{\alpha^2 \gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i) + \frac{2}{\alpha(\alpha - 2 \epsilon \gamma)} \delta^{-2\sigma} e^{-2\epsilon \gamma t} (\|v_0\|_\sigma^2 + \|w_0\|_\sigma^2)
\]
\[
\leq \frac{2\beta(1 + \alpha \gamma)}{\alpha^2 \gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i) + \left(1 + \frac{2}{\alpha(\alpha - 2 \epsilon \gamma)}\right) \delta^{-2\sigma} e^{-2\epsilon \gamma t} (\|v_0\|_\sigma^2 + \|w_0\|_\sigma^2).
\]

(5.6)

Then Lemma 5.1 follows from (5.5) and (5.6). □

Next, we complete the proof of Theorem 2.3 and show that the union of all global attractors is bounded in \(l_\sigma^2 \times l_\sigma^2\).

**Proof of Theorem 2.3.** Let \(\epsilon\) be an arbitrary number with \(0 < \epsilon < \frac{\sigma}{2 \gamma}\), and \((v, w) \in A_\epsilon\). Then we want to show that \(v\) and \(w\) satisfy the bounds given in Theorem 2.3. To that end, we take a sequence \(\{t_n\}_{n=1}^\infty\) with \(t_n \to \infty\). By the invariance of \(A_\epsilon\), there exists a sequence of \((v_n, w_n) \in A_\epsilon\) such that
\[
(v, w) = S(t_n)(v_n, w_n) = (v_n(t_n), w_n(t_n)) \quad \text{for all } n \geq 1.
\]

(5.7)

Since \(A_\epsilon \subseteq B\), where \(B\) is the bounded absorbing set given by (4.8), we have
\[
\|(v_n, w_n)\|_\sigma \leq M, \quad \text{for all } n \geq 1,
\]

(5.8)

where \(M\) is a constant depending on the data \((\epsilon, \alpha, \beta, \gamma, \sigma)\), but independent of \(n\). Applying Lemma 5.1 to \((v_n(t_n), w_n(t_n)) = S(t_n)(v_n, w_n)\), by (5.8) we find that
\[
\|v_n(t_n)\|_\sigma^2 \leq \frac{2\beta(1 + \alpha \gamma)}{\alpha^2 \gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i) + \left(1 + \frac{2}{\alpha(\alpha - 2 \epsilon \gamma)}\right) \delta^{-2\sigma} M^2 e^{-2\epsilon \gamma t_n},
\]

(5.9)

and
\[
\|w_n(t_n)\|_\sigma^2 \leq \frac{\beta}{\gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i) + \delta^{-2\sigma} M^2 e^{-2\epsilon \gamma t_n}.
\]

(5.10)
Taking the limits of (5.9) and (5.10) as $n \to \infty$, it follows from (5.7) that
\[ \|v\|_\sigma^2 = \|v_n(t_n)\|_\sigma^2 \leq \frac{2\beta(1 + \alpha \gamma)}{\alpha^2 \gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i), \]
(5.11)
and
\[ \|w\|_\sigma^2 = \|w_n(t_n)\|_\sigma^2 \leq \frac{\beta}{\gamma} \sum_{i \in \mathbb{Z}} \phi_\delta(i). \]
(5.12)
Note that $(v, w)$ is an arbitrary element in $A_\epsilon$ for any $0 < \epsilon < \frac{\alpha}{2\gamma}$, which, along with (5.11)-(5.12), concludes Theorem 2.3.

6 Compactness of Union of Global Attractors

In this section, we show that all global attractors are contained in a compact subset of $l_\sigma^2 \times l_\sigma^2$, which is based on the fact that the “tail ends” of all sequences in global attractors are uniformly small in $\epsilon$.

**Lemma 6.1.** Suppose (2.4) holds and $\sigma > \frac{1}{2}$. Let $0 < \epsilon < \min\{1, \frac{\alpha}{2\gamma}\}$. Then for every $\eta > 0$, there exist a constant $K(\eta)$ depending only on $\eta$ and the data $(\alpha, \beta, \gamma, \sigma)$, but independent of $\epsilon$, such that for all $(v, w) \in A_\epsilon$, the following inequality holds:
\[ \sum_{|i| \geq K(\eta)} \left( \phi_1(i)|v_i|^2 + \phi_1(i)|w_i|^2 \right) \leq \eta. \]
(6.1)

**Proof.** Let $(v, w) \in A_\epsilon$ and take a sequence $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$. By the invariance of $A_\epsilon$, there exists $(v_n, w_n) \in A_\epsilon$ such that
\[ (v, w) = S(t_n)(v_n, w_n), \quad \text{for } n \geq 1. \]
(6.2)
Set $(v_n(t), w_n(t)) = S(t)(v_n, w_n)$. Then Theorem 2.3 implies that there exists a constant $C_1$ independent of $\epsilon$ such that for all $n \geq 1$ and $t \geq 0$:
\[ \|(v_n(t), w_n(t))\|_{l_\sigma^2 \times l_\sigma^2} \leq C. \]
(6.3)
Using (6.3) and proceeding as the proof of Lemma 4.2, we can obtain that, for all \( t \geq 0 \):

\[
\frac{d}{dt} \left( \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(t)|^2 + \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |w_{n,i}(t)|^2 \right) \\
+ 2\epsilon \gamma \left( \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(t)|^2 + \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |w_{n,i}(t)|^2 \right) \\
\leq \frac{\epsilon C_2}{k} + 2\epsilon \beta \sum_{|i| \geq k} \phi_{\delta}(i), 
\]

(6.4)

where \( C_2 \) is a constant depending on the data \((\alpha, \beta, \gamma, \sigma)\), but independent of \( \epsilon \) and \( k \). Note that (6.4) is an analogue of (4.21). It follows from (6.4) that, given \( \eta > 0 \), there exists \( K_1(\eta) \) depending on \( \eta \) and the data \((\alpha, \beta, \gamma, \sigma)\) such that for all \( k \geq K_1(\eta) \) and \( t \geq 0 \):

\[
\frac{d}{dt} \left( \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(t)|^2 + \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |w_{n,i}(t)|^2 \right) \\
+ 2\epsilon \gamma \left( \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(t)|^2 + \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |w_{n,i}(t)|^2 \right) \\
\leq 2\epsilon \eta.
\]

(6.5)

By Gronwall’s lemma and (6.3), we get, for \( t \geq 0 \) and \( k \geq K_1(\eta) \),

\[
\epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(t)|^2 + \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |w_{n,i}(t)|^2 \\
\leq e^{-2\epsilon \gamma t} \left( \epsilon \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(0)|^2 + \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |w_{n,i}(0)|^2 \right) + \frac{\eta}{\gamma} \\
\leq C_1 e^{-2\epsilon \gamma t} + \frac{\eta}{\gamma}.
\]

(6.6)

Since \( \theta(s) = 1 \) for \( s \geq 2 \), we obtain from (6.6) that for all \( t \geq 0 \) and \( k \geq 2K_1(\eta) \):

\[
\sum_{|i| \geq k} \phi_{\delta}(i) |w_{n,i}(t)|^2 \leq C_1 e^{-2\epsilon \gamma t} + \frac{\eta}{\gamma}.
\]

(6.7)

Next, we derive “tail ends” estimates for \( v_n \). For that purpose, taking the inner product of (3.3) with \((\theta(\frac{|i|}{k})v_{n,i})_{i \in Z} \) in \( l^2_{\sigma} \), we get

\[
\frac{1}{2} \frac{d}{dt} \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(t)|^2 + \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) (Av_n)_i(t) v_{n,i}(t) \\
= \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) h(v_{n,i}(t)) v_{n,i}(t) - \sum_{i \in Z} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) w_{n,i}(t) v_{n,i}(t).
\]

(6.8)
Note that, by (6.7), the last term on the right-hand side of (6.8) is bounded by

\[
\left| \sum_{i \in \mathbb{Z}} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) w_{n,i}(t) v_{n,i}(t) \right| \\
\leq \frac{\alpha}{4} \sum_{i \in \mathbb{Z}} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(t)|^2 + \frac{1}{\alpha} \sum_{i \in \mathbb{Z}} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |w_{n,i}(t)|^2 \\
\leq \frac{\alpha}{4} \sum_{i \in \mathbb{Z}} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(t)|^2 + \frac{C_1}{\alpha} e^{-2\epsilon \gamma t} + \frac{\eta}{\alpha \gamma},
\]

for all \( t \geq 0 \) and \( k \geq 2K_1(\eta) \), where \( C_1 \) is independent of \( \epsilon \). Dealing with other terms on the right-hand side of (6.8) as in the proof of Lemma 4.2, by (6.8)-(6.9) we find that there exists \( K_2(\eta) \) such that for all \( t \geq 0 \) and \( k \geq K_2(\eta) \),

\[
\frac{d}{dt} \sum_{i \in \mathbb{Z}} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(t)|^2 + \alpha \sum_{i \in \mathbb{Z}} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(t)|^2 \leq \frac{(2 + \alpha \gamma) \eta}{\alpha \gamma} + \frac{2C_1}{\alpha} e^{-2\epsilon \gamma t}.
\]

It follows from Gronwall’s lemma and (6.3) that for all \( t \geq 0 \) and \( k \geq K_2(\eta) \),

\[
\sum_{i \in \mathbb{Z}} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(t)|^2 \leq e^{-\alpha t} \sum_{i \in \mathbb{Z}} \theta \left( \frac{|i|}{k} \right) \phi_{\delta}(i) |v_{n,i}(0)|^2 + \frac{(2 + \alpha \gamma) \eta}{\alpha \gamma^2} + \frac{2C_1}{\alpha(\alpha - 2\epsilon \gamma)} e^{-2\epsilon \gamma t},
\]

which implies that for \( t \geq 0 \) and \( k \geq 2K_2(\eta) \):

\[
\sum_{|i| \geq k} \phi_{\delta}(i) |v_{n,i}(t)|^2 \leq C_2 e^{-\alpha t} + \frac{(2 + \alpha \gamma) \eta}{\alpha \gamma^2} + \frac{2C_1}{\alpha(\alpha - 2\epsilon \gamma)} e^{-2\epsilon \gamma t}.
\]

Let \( K_3(\eta) = \max\{2K_1(\eta), 2K_2(\eta)\} \). Then it follows from (6.2), (6.7) and (6.11) that for \( k \geq K_3(\eta) \):

\[
\sum_{|i| \geq k} \phi_{\delta}(i) |w_i|^2 = \sum_{|i| \geq k} \phi_{\delta}(i) |w_{n,i}(t_n)|^2 \leq C_1 e^{-2\epsilon \gamma t_n} + \frac{\eta}{\gamma},
\]

and

\[
\sum_{|i| \geq k} \phi_{\delta}(i) |v_i|^2 = \sum_{|i| \geq k} \phi_{\delta}(i) |v_{n,i}(t_n)|^2 \leq C_2 e^{-\alpha t_n} + \frac{(2 + \alpha \gamma) \eta}{\alpha \gamma^2} + \frac{2C_1}{\alpha(\alpha - 2\epsilon \gamma)} e^{-2\epsilon \gamma t_n}.
\]

Then (6.1) follows by taking the limits of (6.12)-(6.13) as \( n \to \infty \). The proof is complete. \( \square \)

We now establish the precompactness of the union of all global attractors.

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Proof of Theorem 2.4. Given \( \eta > 0 \), we want to show that the set \( \bigcup_{0 < \epsilon < \epsilon_0} A_\epsilon \) has a finite covering of balls of radii less than \( \eta \). It follows from Lemma 6.1 that there exists a constant \( K(\eta) \) depending on \( \eta \) such that for all \((v, w) \in \bigcup_{0 < \epsilon < \epsilon_0} A_\epsilon\):

\[
\sum_{|i| \geq K(\eta)} \phi_1(i) \left(|v_i|^2 + |w_i|^2\right) \leq \frac{\eta}{4}.
\]

(6.14)

On the other hand, by Theorem 2.3, the set \( \bigcup_{0 < \epsilon < \epsilon_0} A_\epsilon \) is bounded in \( l_2^2 \times l_2^2 \), and therefore the set \( \{(\phi_1(i)v_i, \phi_1(i)w_i)_{|i| \leq K(\eta)} : (v, w) \in \bigcup_{0 < \epsilon < \epsilon_0} A_\epsilon\} \) is bounded in the finite-dimensional space \( \mathbb{R}^{2K(\eta)+1} \times \mathbb{R}^{2K(\eta)+1} \) and precompact. In other words, the set \( \{(\phi_1(i)v_i, \phi_1(i)w_i)_{|i| \leq K(\eta)} : (v, w) \in \bigcup_{0 < \epsilon < \epsilon_0} A_\epsilon\} \) has a finite covering of balls of radii less than \( \frac{\eta}{4} \) in \( \mathbb{R}^{2K(\eta)+1} \times \mathbb{R}^{2K(\eta)+1} \), which and (6.14) imply the set \( \bigcup_{0 < \epsilon < \epsilon_0} A_\epsilon \) has a finite covering of balls of radii less than \( \eta \) in \( l_2^2 \times l_2^2 \). The proof is complete.

7 Upper Semicontinuity of Global Attractors

In this section, we establish the upper semicontinuity of global attractors as \( \epsilon \to 0 \). Although the limiting system (2.12)-(2.13) has no global attractors in \( l_2^2 \times l_2^2 \), we will find a local attractor \( A_0 \) for the limiting system such that \( A_\epsilon \) is upper semicontinuous to \( A_0 \) as \( \epsilon \to 0 \).

The local attractor \( A_0 \) can be constructed as follows. Let \( B \) be the limiting set of \( \bigcup_{0 < \epsilon < \epsilon_0} A_\epsilon \) in the following sense:

\[
B = \{(v, w) \in l_2^2 \times l_2^2 : \text{there are } (v_{\epsilon_n}, w_{\epsilon_n}) \in A_{\epsilon_n} \text{ with } \epsilon_n \to 0 \text{ such that } (v_{\epsilon_n}, w_{\epsilon_n}) \to (v, w)\}.
\]

Denote by \( W \) the projection of \( B \) to the \( w \)-space:

\[
W = \{w \in l_2^2 : \text{there is } v \in l_2^2 \text{ such that } (v, w) \in B\}.
\]

By Theorem 2.4, we know that the set \( B \) is compact in \( l_2^2 \times l_2^2 \), and therefore \( W \) is a compact subset of \( l_2^2 \). When the component \( w \) is confined to the compact set \( W \), the limiting system has a local attractor which is stated as follows.

Lemma 7.1. Suppose (2.4) holds and \( \sigma > \frac{1}{2} \). Then the limiting system (2.12)-(2.13) has a local attractor \( A_0 \) which is a compact invariant set and attracts every bounded subset of \( l_2^2 \times W \) with
respect to the norm topology. Further, if we denote by $A_w$ the global attractor of equation (2.12) for a fixed $w \in W$, then $A_0$ can be characterized by

$$A_0 = \{(v, w) \in l^2_w \times l^2_w : w \in W \text{ and } v \in A_w\}.$$ 

**Proof.** The proof is similar to that of Theorem 2.2, and therefore is omitted here. \qed

Next, we establish the relationships between the solutions of problem (2.1)-(2.2) and problem (2.12)-(2.13), which is useful to prove the upper semicontinuity of global attractors.

**Lemma 7.2.** Suppose (2.4) holds, $\sigma > \frac{1}{2}$ and $0 < \epsilon < \min\{1, \frac{\sigma}{2}\}$. If $(v_\epsilon, w_\epsilon)$ is the solution of problem (2.1)-(2.2) with the initial datum $(v_1, w_1) \in l^2 \times l^2$, and $(v, w)$ is the solution of problem (2.12)-(2.13) with the initial datum $(v_2, w_2) \in l^2 \times l^2$, then we have, for $t \geq 0$,

$$\|v_\epsilon(t) - v(t)\|_\sigma^2 + \|w_\epsilon(t) - w(t)\|_\sigma^2 \leq e^{Ct} \left(\|v_1 - v_2\|_\sigma^2 + \|w_1 - w_2\|_\sigma^2\right) + \epsilon \int_0^t e^{C(t-s)} \|v(s) - \gamma w(s)\|_\sigma^2 ds,$$

where $C$ is a constant depending on $\kappa$ and $\sigma$, but independent of $\epsilon$.

**Proof.** Let $V = v_\epsilon - v$ and $W = w_\epsilon - w$. Then it follows from (2.1)-(2.2) and (2.12)-(2.13) that

$$\frac{dV}{dt} + AV = h(v_\epsilon) - h(v) - W, \quad (7.1)$$

$$\frac{dW}{dt} = \epsilon(V - \gamma W) + \epsilon(v - \gamma w). \quad (7.2)$$

Similar to the proof of Lemma 3.2, by (7.1)-(7.2) we can verify that $V$ and $W$ satisfy

$$\frac{d}{dt} \left(\|V\|^2_\sigma + \|W\|^2_\sigma\right) \leq C \left(\|V\|^2_\sigma + \|W\|^2_\sigma\right) + \epsilon\|v - \gamma w\|^2_\sigma. \quad (7.3)$$

Then Lemma 7.2 follows from Gronwall’s lemma and inequality (7.3). \qed

In the sequel, we denote by $S^\epsilon(t)_{t \geq 0}$ the semigroup associated with problem (2.1)-(2.3) for $\epsilon > 0$, and $S(t)_{t \geq 0}$ for $\epsilon = 0$. Note that these semigroups map $l^2_\sigma \times l^2_\sigma$ to $l^2_\sigma \times l^2_\sigma$ which are extensions of solution operators defined in $l^2 \times l^2$. Then it follows from a continuity argument that Lemma 7.2 holds for any initial data $(v_1, w_1), (v_2, w_2) \in l^2_\sigma \times l^2_\sigma$, which implies that if $\epsilon_n \to 0$ and $(v_{\epsilon_n}, w_{\epsilon_n}) \to (v, w)$ in $l^2_\sigma \times l^2_\sigma$, then

$$S^{\epsilon_n}(t)(v_{\epsilon_n}, w_{\epsilon_n}) \to S(t)(v, w) \text{ in } l^2_\sigma \times l^2_\sigma. \quad (7.4)$$

The following lemma is crucial for establishing the upper semicontinuity of attractors.
Lemma 7.3. Suppose (2.4) holds and $\sigma > \frac{1}{4}$. If $(v_{\epsilon_n}, w_{\epsilon_n}) \in \mathcal{A}_{\epsilon_n}$ with $\epsilon_n \to 0$, then there exists $(v_0, w_0) \in \mathcal{A}_0$ such that (up to a subsequence) $(v_{\epsilon_n}, w_{\epsilon_n}) \to (v_0, w_0)$ in $l^2_\sigma \times l^2_\sigma$.

Proof. It follows from Theorem 2.4 that the set $\bigcup_{n=1}^{\infty} \mathcal{A}_{\epsilon_n}$ is precompact. Therefore, there exists $(v_0, w_0) \in l^2_\sigma \times l^2_\sigma$ and a subsequence of $(v_{\epsilon_n}, w_{\epsilon_n})$ (still denoted by $(v_{\epsilon_n}, w_{\epsilon_n})$) such that

$$(v_{\epsilon_n}, w_{\epsilon_n}) \to (v_0, w_0) \text{ in } l^2_\sigma \times l^2_\sigma. \tag{7.5}$$

Take a sequence $\{t_m\}_{m=1}^{\infty}$ with $t_m \to \infty$. By the invariance of $\mathcal{A}_{\epsilon_n}$ we know

$$S^\epsilon_{-t_m}(v_{\epsilon_n}, w_{\epsilon_n}) \in \mathcal{A}_{\epsilon_n} \text{ for all } n, m \geq 1.$$ 

Therefore, for $m = 1$, it follows from the precompactness of $\bigcup_{n=1}^{\infty} \mathcal{A}_{\epsilon_n}$ that there exists $(v_1, w_1)$ and a subsequence $(v_{\epsilon_{n_1}}, w_{\epsilon_{n_1}})$ of $(v_{\epsilon_n}, w_{\epsilon_n})$ such that

$$S^\epsilon_{-t_1}(v_{\epsilon_{n_1}}, w_{\epsilon_{n_1}}) \to (v_1, w_1) \text{ in } l^2_\sigma \times l^2_\sigma \text{ as } n_1 \to \infty.$$ 

For $m = 2$, by the same reason, there exists $(v_2, w_2)$ and a subsequence $(v_{\epsilon_{n_2}}, w_{\epsilon_{n_2}})$ of $(v_{\epsilon_{n_1}}, w_{\epsilon_{n_1}})$ such that

$$S^\epsilon_{-t_2}(v_{\epsilon_{n_2}}, w_{\epsilon_{n_2}}) \to (v_2, w_2) \text{ in } l^2_\sigma \times l^2_\sigma \text{ as } n_2 \to \infty.$$ 

Repeating this procedure and by a standard diagonal argument, it follows that there exists a subsequence $(v_{\epsilon_{n_k}}, w_{\epsilon_{n_k}})$ of $(v_{\epsilon_n}, w_{\epsilon_n})$ such that for all $m \geq 1$:

$$S^\epsilon_{-t_m}(v_{\epsilon_{n_k}}, w_{\epsilon_{n_k}}) \to (v_m, w_m) \text{ in } l^2_\sigma \times l^2_\sigma \text{ as } k \to \infty. \tag{7.6}$$

By (7.4) and (7.6) we find that, as $k \to \infty$:

$$(v_{\epsilon_{n_k}}, w_{\epsilon_{n_k}}) = S^\epsilon_{-t_m}(t_m) \left( S^\epsilon_{-t_{m+k}}(v_{\epsilon_{n_k}}, w_{\epsilon_{n_k}}) \right) \to S(t_m)(v_m, w_m). \tag{7.7}$$

It follows from (7.5) and (7.7) that

$$(v_0, w_0) = S(t_m)(v_m, w_m).$$

Since $t_m \to \infty$, the above implies $(v_0, w_0) \in \mathcal{A}_0$. The proof is complete. \qed

We are now ready to show that the global attractors $\mathcal{A}_{\epsilon}$ are upper semicontinuous to $\mathcal{A}_0$ as $\epsilon \to 0$. 

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Proof of Theorem 2.5. We want to prove

$$\lim_{\epsilon \to 0} d_{l^2_\sigma \times l^2_\sigma} (\mathcal{A}_\epsilon, \mathcal{A}_0) = 0. \quad (7.8)$$

We argue by contradiction. If (7.8) is not true, then there exist $\eta > 0$ and a sequence $(v_{\epsilon_n}, w_{\epsilon_n}) \in \mathcal{A}_{\epsilon_n}$ with $\epsilon_n \to 0$ such that

$$d_{l^2_\sigma \times l^2_\sigma} ((v_{\epsilon_n}, w_{\epsilon_n}), \mathcal{A}_0) \geq \eta. \quad (7.9)$$

But, by Lemma 7.3, we know that there exists a subsequence $(v_{\epsilon_{n_k}}, w_{\epsilon_{n_k}})$ of $(v_{\epsilon_n}, w_{\epsilon_n})$ such that

$$\lim_{k \to \infty} d_{l^2_\sigma \times l^2_\sigma} ((v_{\epsilon_{n_k}}, w_{\epsilon_{n_k}}), \mathcal{A}_0) = 0,$$

which contradicts (7.9). The proof is complete.

Remark. In this paper, we only discuss the lattice FitzHugh-Nagumo system defined on the integer set $\mathbb{Z}$. But with minor changes in the proofs, we see that all results of this paper are also valid for the FitzHugh-Nagumo system defined on the product space $\mathbb{Z}^m$ with $m \geq 1$. In this case, the weighted space $l^2_\sigma$ should be defined by

$$l^2_\sigma = \{ u = (u_i)_{i \in \mathbb{Z}^m} : \sum_{i \in \mathbb{Z}^m} (1 + |i|^2)^{-\sigma} |u_i|^2 < \infty \},$$

with $\sigma > \frac{m}{2}$. Theorems 2.1-2.5 still hold if the condition $\sigma > \frac{1}{2}$ is replaced by $\sigma > \frac{m}{2}$.

References


