**Problem 1.** The characteristic polynomial of $A \in \mathbb{R}^{m \times m}$ is defined by $p_A(z) = \det(zI - A)$, $z \in \mathbb{C}$. Show that

$$p_A(z) = z^m + a_{m-1}z^{m-1} + a_{m-2}z^{m-2} + \ldots + a_1z + a_0, \quad z \in \mathbb{C},$$

where $a_{m-1} = -\text{tr}A$ and $a_j \in \mathbb{R}$, $j = 0, \ldots, m - 2$. *(Remark: Note that this problem shows that $p_A(z)$ is a polynomial of degree $m$ with the coefficient of $z^m$ equal to 1 and the coefficient of $z^{m-1}$ equal to $-\text{tr}A$.)

**Problem 2.** Assume that $A \in \mathbb{R}^{m \times m}$. Show the following:

(a) If $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ and $\mu \in \mathbb{R}$, then $\lambda - \mu$ is an eigenvalue of $A - \mu I$.

(b) If $A$ is nonsingular and $\lambda \in \mathbb{C}$ is an eigenvalue of $A$, then $\lambda \neq 0$ and $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

(c) If $A$ is diagonalizable (similar to a diagonal matrix) and all eigenvalues of $A$ are equal, then $A$ is diagonal.

(d) $A$ and $A^T$ have the same eigenvalues.

(e) If $\lambda \in \mathbb{C}$ and $\vec{v} = [v_1, \ldots, v_m]^T \in \mathbb{C}^m$ is an eigenvector of $A$, then so is $\overline{\lambda}$ and $\overline{\vec{z}} = [\overline{v_1}, \ldots, \overline{v_m}]^T$.

**Problem 3.** Use the real Schur decomposition theorem presented in class to show that if $A \in \mathbb{R}^{m \times m}$ and $A = A^T$, then there is an orthogonal $Q \in \mathbb{R}^{m \times m}$ and a diagonal $D \in \mathbb{R}^{m \times m}$ such that $A = QDQ^T$. **Hint:**  First show that all eigenvalues of $A$ are real.

**Problem 4.** Assume $A \in \mathbb{R}^{m \times m}$, $A = A^T$, and that $\lambda_1, \ldots, \lambda_m$ are eigenvalues of $A$. Let $E \in \mathbb{R}^{m \times m}$ be a perturbation of $A$, and let $\lambda \in \mathbb{R}$ be an eigenvalue of $A + E$. Show that for at least one $\lambda_j$, $|\lambda - \lambda_j| \leq \|E\|_2$. **Hint:** Use $\|E\|_2 = \max_{\vec{v} \neq \vec{0} \in \mathbb{R}^m} \|E\vec{v}\|_2/\|\vec{v}\|_2$ and $(A + E)\vec{x} = \lambda\vec{x}$, where $\vec{x} \neq \vec{0} \in \mathbb{R}^m$. *(Remark: Note that this result shows that small perturbations of a real symmetric matrix lead to equally small perturbations in the eigenvalues, that is, the problem of computing eigenvalues of a real symmetric matrix is well-conditioned in absolute sense. The relative error in some, or all, of the eigenvalues may still be large.)*

**Problem 5.** Use Matlab “hess” to reduce $A = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 4 & 3 \end{bmatrix}$ to the orthogonally similar upper Hessenberg matrix $H$. Compare the computed $\hat{H}$ with the exact $H = \begin{bmatrix} 5 & -7/\sqrt{5} & 1/\sqrt{5} \\ -\sqrt{5} & 24/5 & 13/5 \\ 0 & -2/5 & 1/5 \end{bmatrix}$ by computing $H - \hat{H}$.

**Problem 6.** Assume $A \in \mathbb{R}^{m \times m}$ and $A = A^T$. Consider step 1 in reducing $A$ to an orthogonally similar tridiagonal matrix $T$,

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \vec{x}^T \\ \vec{x} & \hat{A} \\ \vec{e}_1 & 0 \\ 0 & \vec{e}\end{bmatrix} = \begin{bmatrix} a_{11} + \|\vec{x}\|_2 & 0 & \cdots & 0 \\ 0 & \|\vec{x}\|_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B \end{bmatrix},$$

where $F_1 = I - \gamma \vec{v}_1\vec{v}_1^T$, $\gamma = 2/\vec{v}_1^T\vec{v}_1$, and $B = F_1\hat{A}F_1$.

(a) Show that $B = A + \vec{w}\vec{v}_1^T + \vec{v}_1\vec{w}^T$, where $\vec{w} = \vec{u} + \delta\vec{v}_1$, $\vec{u} = -\gamma\hat{A}\vec{v}_1$, and $\delta = -(1/2)\gamma\vec{v}_1^T\vec{u}$.

(b) Using (a) explain how to perform step 1 so that the number of multiplications in this step $\sim 2(m - 1)^2$.

**Hint:** Note that $(\vec{v}_1\vec{v}_1^T)^T = \vec{w}\vec{v}_1^T$.

(c) Use (b) to show that the number of multiplications in reducing $A$ to $T \sim (2/3)m^3$.

*(Remark: Since the number of additions $\sim (2/3)m^3$, this shows that the total number of flops in reducing $A$ to $T \sim (4/3)m^3$.)