For problems involving MATLAB, turn in printouts of your m-files and obtained numerical results. Use format long e of MATLAB to display your results.

**Problem 1.** Show that if a nonsingular \( A \in \mathbb{R}^{m \times m} \) has a LU factorization, that is, \( A = LU \), where \( L \) is unit lower triangular and \( U \) is upper triangular, then such factorization is unique. **Hint:** Consider \( L_1U_1 = L_2U_2 \) which implies \( U_1^{-1}L_1 = U_2^{-1}L_2 \).

**Problem 2.** Let \( A^{(k)} = (a_{ij}^{(k)})_{i,j=1}^m \), \( k = 1, \ldots, m \), be matrices resulting, in the exact arithmetic, from GEPP applied to \( A \in \mathbb{R}^{m \times m} \) (\( A^{(1)} = A \)). Show that if \( A \) is nonsingular, then for each \( k = 1, \ldots, m \), at least one of the coefficients
\[
a_{kk}^{(k)}, \quad i = k, \ldots, m,
\]
is different from zero. **Hint:** Try to prove it by contradiction. (**Remark:** Note that the problem implies that GEPP does not fail on a nonsingular matrix in the exact arithmetic. More precisely, the problem shows that for a nonsingular \( A \) there is always a matrix \( P \), which is a product of permutation matrices, such that \( PA \) has a LU factorization.)

**Problem 3.** A matrix \( B \in \mathbb{R}^{m \times n} \) is said to be strictly column diagonally dominant (scdd) if
\[
|b_{jj}| > \sum_{i=1, i \neq j}^n |b_{ij}|, \quad j = 1, \ldots, n.
\]

(a) Show that, in the exact arithmetic, GE without pivoting preserves scdd property, that is, show that if GE without pivoting is applied to a scdd \( A \in \mathbb{R}^{m \times m} \), then for each \( k = 1, \ldots, m \), the subblock of \( A^{(k)} \) obtained by deleting from \( A^{(k)} \) the first \( k-1 \) rows and the first \( k-1 \) columns is also scdd. **Hint:** Use induction on \( k \) and consider step \( k \) for obtaining \( A^{(k+1)} \) from \( A^{(k)} \). Also you may want to use the inequalities \( |\alpha + \beta| \leq |\alpha| + |\beta| \), \( |\alpha - |\beta| \leq |\alpha - |\beta| \) that hold for any \( \alpha, \beta \in \mathbb{R} \).

(b) Show that, in the exact arithmetic, GE without pivoting will never fail when applied to a scdd \( A \in \mathbb{R}^{m \times m} \) and that \( |l_{ik}| < 1 \) for all \( k = 1, \ldots, m-1 \) and all \( i = k + 1, \ldots, m \).

**Problem 4.** Use Matlab “pascal” to generate \( A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \) compute \( \tilde{b} \) in the system \( A\tilde{x} = \tilde{b} \). Argue that \( A \) and \( \tilde{b} \) are a floating point matrix and a floating point vector, respectively. Solve \( A\tilde{x} = \tilde{b} \) using Matlab “[L,U,P]=lu(A)” and your functions for forward and back substitutions. Compute the relative error \( \|\tilde{x} - \bar{x}\|_2/\|\tilde{x}\|_2 \), where \( \bar{x} \) is the computed solution. Use Matlab “eps” and “cond” compare this error with the theoretical bound \( u\kappa_2(A) \) derived in class.

**Problem 5.** Consider
\[
A\tilde{x} = \tilde{b} \quad \text{with} \quad A = A_1A_2,
\]
where \( A_1, A_2 \in \mathbb{R}^{m \times m} \) are given nonsingular floating point matrices and \( \tilde{b} \in \mathbb{R}^m \) is a given floating point vector. Suppose that solution of (1) is obtained by solving
\[
A_1\tilde{y} = \tilde{b} \quad \text{and} \quad A_2\tilde{x} = \tilde{y}.
\]
Assume that the systems in (2) are solved using a backward stable method, that is, the computed \( \tilde{y} \) and computed \( \tilde{x} \) satisfy
\[
(A_1 + \delta A_1)\tilde{y} = \tilde{b}, \quad (A_2 + \delta A_2)\tilde{x} = \tilde{y},
\]
where \( \|\delta A_i\| \leq c(m)u\|A_i\| \), \( i = 1, 2 \), \( c(m) \) depends mildly on \( m \) (e.g., \( c(m) = m \)), \( u \) is the unit roundoff, and \( \| \cdot \| \) is some induced matrix norm. Determine condition on \( \|A_1\|\|A_2\| \) under which the described method of solving (1) can be viewed as being backward stable.

**Problem 6.** Assume that \( A \in \mathbb{R}^{m \times m} \) is symmetric and positive definite.

(a) Show that if \( X \in \mathbb{R}^{m \times m} \) is nonsingular, then \( X^TAX \) is symmetric and positive definite.

(b) Show that any principal submatrix of \( A \) is symmetric and positive definite. (A principal submatrix of \( A \) is the matrix \( A(j : k, j : k) \), where \( 1 \leq j \leq k \leq m \).)