Math 782, HW 7, Due Wednesday, May 12

For problems involving MATLAB, turn in printouts of your m-files and obtained numerical results. Use format long e of MATLAB to display your results.

**Problem 1.** Let $T$ be the $n \times n$ tridiagonal matrix with 2 on the main diagonal and $-1$ on the first superdiagonal and the first sub-diagonal. It is known that the eigenvalues and the corresponding orthonormal eigenvectors of $T$ are given, respectively, by

$$
\lambda_j = 4 \sin^2 \frac{j\pi}{2(n+1)}, \quad j = 1, \ldots, n,
$$

$$
\tilde{q}_j = \sqrt{\frac{2}{n+1}} \left[ \sin \frac{j\pi}{n+1}, \sin \frac{2j\pi}{n+1}, \sin \frac{3j\pi}{n+1}, \ldots, \sin \frac{n j\pi}{n+1} \right]^T, \quad j = 1, \ldots, n.
$$

For $6 \times 6$ matrix $T$ and $j = 1, \ldots, 6$, use Matlab and one iteration of inverse iteration with the shift $\mu = \lambda_j$ to compute the corresponding orthonormal eigenvector of $T$. Use “rand” to select $\tilde{v}^{(0)}$ (remember to normalize it) and “\backslash” for the solution of the required linear system. For $j = 1, \ldots, 6$, compare each computed eigenvector $\tilde{q}_j$ with the corresponding exact eigenvector $\tilde{q}_j$ by printing horizontally $\tilde{q}_j$, $\tilde{q}_j$, and $\tilde{q}_j - (\pm \tilde{q}_j) \approx 0$. Comment on what you observe.

**Problem 2.** Let $A \in \mathbb{R}^{n \times m}$ be upper Hessenberg, that is, $a_{ij} = 0$ for $i > j + 1$. Assume that a $QR$ step without shift is applied to $A$, that is,

$$
A = QR, \quad B = RQ,
$$

where $QR$ in the first equation is a $QR$ factorization of $A$. Describe the computation of $R$ and $B$ using Givens rotations. Show that $B$ is upper Hessenberg and that cost of computing $B$ is proportional to $m^2$.

**Problem 3.** (We showed in class that if Givens rotations are used at each step of the QR algorithm without shifts, then the symmetric tridiagonal form of a matrix is preserved. The purpose of this problem is to show that the symmetric tridiagonal form is preserved for an unreduced symmetric tridiagonal matrix regardless of how $QR$ factorization is computed at each step of the QR algorithm.)

**Definition.** A symmetric tridiagonal $T \in \mathbb{R}^{n \times n}$ is unreduced if all its elements on the first subdiagonal (equivalently all elements on the first superdiagonal) are nonzero.

Assume that $T \in \mathbb{R}^{m \times m}$ is symmetric tridiagonal and unreduced and let $T = QR$ be a $QR$ factorization of $T$, that is, $Q \in \mathbb{R}^{m \times m}$ is orthogonal and $R \in \mathbb{R}^{m \times m}$ is real upper triangular.

(a) Show that the first $m-1$ columns of $R$ are linearly independent. **Hint:** First show that the first $m-1$ columns of $T$ are linearly independent.

(b) Let $\hat{T}, \hat{Q} \in \mathbb{R}^{m \times (m-1)}$ consist of the first $m-1$ columns of $T$ and $Q$, respectively, and let $\hat{R} \in \mathbb{R}^{(m-1) \times (m-1)}$ be obtained from $R$ by deleting its last column and last row. Show that $\hat{R}$ is nonsingular and that $\hat{T} = \hat{Q}\hat{R}$. Using these facts prove that $Q$ is upper Hessenberg.

(c) Show that $RQ$ is tridiagonal. **Hint:** First show that $RQ$ is upper Hessenberg and then use symmetry of $RQ$.

**Problem 4.** Use $6 \times 6$ matrix $T$ of Problem 1 and Matlab with “[Q,R]=qr(X)” to:

(a) perform 50 iterations of the QR algorithm without shifts. Print (using, for example, “rem(k,10)”) $T^{(k)}$ for $k = 10, 20, 30, 40, 50$. Comment on the convergence of the diagonal elements of $T^{(k)}$ to the eigenvalues of $T$.

(b) perform 3 iterations of the QR algorithm with Wilkinson shifts. Print $T^{(k)}$ for $k = 1, 2, 3$ and comment on the convergence of $T^{(k)}_{6,6}$ to one of the eigenvalues of $T$.

**Problem 5.** Use matlab to perform 50 iterations of the QR algorithm without shifts on the Hessenberg matrices

$$
H = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 0 & 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix}.
$$

For each $H$, print $H^{(50)}$ and use it to determine the eigenvalues of $H$. Compare them with the eigenvalues of $H$ obtained using matlab “eig”.