GENERIC REGULARITY AND LIPSCHITZ METRIC FOR THE HUNTER–SAXTON TYPE EQUATIONS

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Abstract. The Hunter–Saxton equation determines a flow of conservative solutions taking values in the space $H^1(\mathbb{R}^+)$. However, the solution typically includes finite time gradient blowups, which make the solution flow not continuous w.r.t. the natural H^1 distance. The aim of this paper is to first study the generic properties of conservative solutions of some initial boundary value problems to the Hunter–Saxton type equations. Then using these properties, we give a new way to construct a Finsler type metric which renders the flow uniformly Lipschitz continuous on bounded subsets of $H^1(\mathbb{R}^+)$.

Keywords. Hunter–Saxton equations; generic regularity; Lipschitz metric; conservative solution.

1. Introduction

In this paper, we study the one and two–component Hunter–Saxton equations, in the region $(t,x)\in\mathbb{R}^+\times\mathbb{R}^+$. More precisely, the two–component Hunter–Saxton equations which can be used to model the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal, are given as follows

$$\begin{cases} u_t+uu_x=\dfrac{1}{2}\displaystyle\int_0^x (u_z^2+\rho^2)(z)\,dz,\\ \rho_t+(u\rho)_x=0, \end{cases} \tag{1.1}$$

for $(t,x)\in\mathbb{R}^+\times\mathbb{R}^+$. Here the variable $u=u(t,x)$ describes the horizontal velocity of the fluid, $\rho=\rho(t,x)$ describes the horizontal deviation of the surface from equilibrium.

When $\rho\equiv 0$, the above system becomes the one–component Hunter–Saxton equation

$$u_t+uu_x=\frac{1}{2}\int_0^x u_z^2(z)\,dz, \tag{1.2}$$

which is an asymptotic equation of the variational wave equation used to model nematic liquid crystal [1, 11].

The Hunter–Saxton equation (1.2) was first derived in [11] as an asymptotic equation of the variational wave equation, which was considered in [1, 2, 3, 4, 8, 21, 22], for the nematic liquid crystals. The global existences of weak conservative and dissipative solutions of (1.2) were first proved by Hunter and Zheng in [12, 13] on the initial value problem, by studying the self-similar solutions, then were treated by several other methods including the Young measure method by Zhang and Zheng in [20], and the characteristic method by Bressan and Constantin [5] and Bressan, Zhang and Zheng [7] on the initial value or initial boundary value problem. The uniqueness of conservative solution can be found in [7]. Especially, in [5, 7], by introducing some “energy related” variables, the equation (1.2) can be written into a new semi-linear system under some characteristic coordinates. Furthermore, by studying this semi-linear system, one can prove the global existence of H^1 solution for the original system.

However, due to the energy concentration when the finite time gradient blowup happens, the solution flow of (1.2) is in general not Lipschitz continuous with respect to the natural H^1 distance. To obtain the Lipschitz property, one needs to introduce some new metric. One natural choice is to use an optimal transport metric measuring the cost in transporting from one

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solution to another one. Before this paper, there were two major ways available in constructing such a metric for the initial value problem of the one-component Hunter-Saxton equation. First, in [5], the Lipschitz metric was constructed by optimizing the direct transportation from one dissipative solution to another one. Secondly in [6], Bressan, Holden and Raynaud established a geodesic distance for energy conservative solutions,, where their construction relies on the analysis of the semi-linear system, used also in the existence proof, on energy related variables under the characteristic coordinates.

The two–component Hunter–Saxton system is a generalization of the Hunter–Saxton equation. It is also a special case of the Gurevich-Zybin system modelling the dynamics of non-dissipative dark matter [16]. Its local well–posedness, global existence and blow–up phenomena were discussed recently in [18]. Moreover, Munsch [19] proved that there exist global dissipative solutions to the two–component Hunter–Saxton system on $\mathbb{R}$. In [15], Nordli established the existence of conservative solutions and the Lipschitz continuous dependence of the solutions with respect to the initial data using similar method as in [6].

In this paper, we provide another way to establish a Finsler type distance which renders the conservative solution flows of (1.1) and (1.2) uniformly Lipschitz continuous on bounded subsets of $H^1(\mathbb{R}^+)$. We first define the metric for smooth solutions, purely using functions on the original (t, x)-coordinates, then prove the uniform Lipschitz continuity of the flow until the blowup time. On the other hand, we establish a generic regularity result, roughly speaking, which can be understood as that piecewise smooth solutions with only generic singularities are dense in the whole solution set. Using this result, we could extend the metric for smooth solutions to a metric for general weak solutions in $H^1(\mathbb{R}^+)$, where some extra efforts need to be done in order to extending the Lipschitz properties from smooth solutions to piecewise smooth solutions with only generic singularities. This framework was first established by Bressan and Chen in [2, 3] for the variational wave equation.

Actually, the generic regularity results in this paper are new. These results give us a thorough understand on the properties of generic singularities, which are among the most physically relevant singularities, for systems (1.1) and (1.2). As stated before, some geodesic Lipschitz metrics for conservative solutions, using variables under new coordinates instead of (t, x) coordinate, were established in [5, 15]. One reason why we still wish to construct the Lipschitz metric through our method, under the help of the generic regularity results, is because our method which works mainly on the original (t, x)-coordinates can provide readers a new intuitive way to understand the construction of metrics through other ways. Especially, we expect that the construction of the metrics in this paper, especially the metrics for the smooth cases (in Subsections 2.3.1 and 3.3.1), can be easily understood even by readers in a broader field. In this paper, we study the initial boundary value problems, instead of the initial value problems considered by [5, 15].

In this paper, we deal with the Hunter–Saxton equation (1.2) and the two–component Hunter–Saxton equations (1.1) in Sections 2 and 3, respectively. First we review the existence and uniqueness of conservative solutions to these two type equations in Subsections 2.1 and 3.1, respectively.

Next, in Subsections 2.2 and 3.2, we consider the generic regularity of conservative solutions of these two types of equations, respectively. As mentioned before, the Hunter–Saxton equation is a special case of the two–component equations. From this point of view, the two–component Hunter–Saxton equations should inherit all the singular behavior from the Hunter–Saxton equation. However, we reveal that these two type equations have quite different generic singularity behaviors. We will see from Theorem 2.2 and Corollary 3.1 that the solutions of the Hunter–Saxton equation may form singularity in finitely many piecewise $\mathcal{C}^2$ curves in the domain $[0, T] \times \mathbb{R}$, for any $T > 0$, while the singularity of the two–component Hunter–Saxton equations generically only occurs at finitely many isolated points within the same domain. Moreover, when the density ρ is positive, there is an upper bound of $|u_x|$, which means that there is no blow–up phenomenon in the domain when $\rho > 0$ for (1.1). To prove this generic regularity result, we use the method first provided in a recent paper [2] for the variational wave equation and then used for one and two-component Camassa-Holm equations in [14], where one of the main ideas is to use the Thom's Transversality Theorem.

At last, we will study Lipschitz continuous dependence of solutions of the initial boundary value problem by constructing a Lipschitz metric on two types of equations in Subsections 2.3 and 3.3, respectively. More specifically, In 2.3.1 and 3.3.1, we construct a Finsler norm on tangent vector and show how the norm evolves in time for smooth solutions to two systems. Then we extend the metric to piecewise smooth solutions with only generic singularities in 2.3.2 and 3.3.2. Finally, in 2.3.3 and 3.3.3, we extend the metric to general weak solutions to two systems by the "dense" results obtained in Subsections 2.2 and 3.2, respectively.

2. The Hunter–Saxton equation

In this section, we first study the generic regularity of conservative solutions to the Hunter–Saxton equation (1.1). Based on this result, we will construct a Finsler distance which renders Lipschitz continuous the unique flow generated by (1.2), c.f. [7]. To this end, we start from the existence and uniqueness result established in [7].

2.1. **Preliminaries.**

We consider the Hunter–Saxton equation

$$u_t + uu_x = \frac{1}{2}\int_0^x u_z^2(z)\,dz, \tag{2.1}$$

with initial–boundary conditions

$$u(0,x) = u_0(x), \quad u(t,0) = 0, \tag{2.2}$$

and a compatibility conditions

$$u_0(0) = 0, \quad u_0'(0) = 0. \tag{2.3}$$

For smooth solution, formally differentiating equation (2.1) with respect to the spatial variable x, we obtain

$$u_{xt} + (uu_x)_x = \frac{1}{2}u_x^2. \tag{2.4}$$

One can easily check that every smooth solution satisfies a conservation law, namely,

$$(u_x^2)_t + (uu_x^2)_x = 0. \tag{2.5}$$

Integrating (2.5) with respect to x, we see that

$$E(t) := \int_0^\infty u_x^2(t,x)\,dx = E_0 \tag{2.6}$$

is constant in time.

Definition 2.1. *A function $u = u(t,x)$ defined on $[0,T] \times \mathbb{R}^+$ is a solution of the initial–boundary value problem* (2.1)–(2.3) *if the following holds.*

(i) *The function u is locally Hölder continuous with respect to both variables t, x. The initial and boundary conditions* (2.2) *and* (2.3) *hold pointwise. For each time $t \in [0,T]$, the map $x \mapsto u(t,x)$ is absolutely continuous with $u_x(t,\cdot) \in L^2(\mathbb{R}^+)$.*

(ii) *For any $M > 0$, consider the restriction of u to the interval $x \in [0,M]$. Then the map $t \mapsto u(t,\cdot) \in L^2([0,M])$ is absolutely continuous and satisfies the equation $\frac{d}{dt}u(t,\cdot) = -uu_x + \frac{1}{2}\int_0^x u_z^2(z)\,dz$ for a.e. $t \in [0,T]$. Here equality is understood in the sense of functions in $L^2([0,M])$.*

Now, we review the existence and uniqueness result to the Hunter–Saxton equation, c.f.[7, 10].

Theorem 2.1. ([7]) *For any initial data $u_0 \in H^1(\mathbb{R}^+)$, the initial–boundary value problem* (2.1)–(2.3) *admits a global unique conservative solution $u = u(t,x)$. More precisely, there exists a family of Radon measures $\{\mu_{(t)}; t \in \mathbb{R}^+\}$, depending continuously on time with respect to the topology of weak convergence of measures, such that the following properties hold.*

(i) *The functions u provides a solution of* (2.1)–(2.3) *in the sense of Definition 2.1.*

(ii) *There exists a null set $A \subset \mathbb{R}$ with* meas*(A)=0 such that for every $t \notin A$ the measure $\mu_{(t)}$ is absolutely continuous and has density $u_x^2(t,\cdot)$ with respect to Lebesgue measure.*

(iii) *The family* $\{\mu_{(t)}; t \in \mathbb{R}^+\}$ *provides a measure–valued solution* $\varpi = u_x^2$ *to the linear transport equation with source* $\varpi_t + (u\varpi)_x = 0$.

Remark 2.1. (1) *For smooth solution, along the characteristic* $\frac{dx(t)}{dt} = u(t, x(t))$, *we know from* (2.1) *that the value of the solution* u *is*

$$u(t, x(t)) = u_0(x) + \frac{1}{2}\int_0^t \int_0^x u_z^2(z)\,dz\,dt.$$

More precisely, for later references, one has the bound

$$\|u(x)\|_{L^\infty} \le \|u_0(x)\|_{L^\infty} + \frac{t}{2}E_0.$$

(2) *One can find a global existence result to a more general scalar equation in* [10].

2.2. **Generic regularity of solutions to the Hunter–Saxton equation.** Now, we review some basic setup used in [10]. We first introduce an energy variable $\xi \in \mathbb{R}^+$ by setting

$$\xi := \int_0^{\bar{y}(\xi)} (1 + u_x^2(0, x'))\,dx',$$

where $t \mapsto y(t, \xi)$ is the characteristic starting at $\bar{y}(\xi)$, so that

$$\frac{dy(t,\xi)}{dt} = u(t, y(t, \xi)), \qquad y(0, \xi) = \bar{y}(\xi),$$

where we write $u(t, \xi) := u(t, y(t, \xi))$. Then we define two dependent variables $r := r(t, \xi)$ and $q := q(t, \xi)$ as

$$r = 2\arctan u_x \quad \text{and} \quad q = (1 + u_x^2)\cdot\frac{\partial y}{\partial \xi},$$

then one obtains a semi–linear system

$$\begin{cases} u_\xi(t,\xi) = \frac{1}{2}q\sin r, \\ r_t(t,\xi) = -\sin^2\frac{r}{2}, \\ q_t(t,\xi) = \frac{1}{2}q\sin r. \end{cases} \tag{2.7}$$

Furthermore, if we set

$$S := u_t + uu_x,$$

we have another semi–linear system

$$\begin{cases} u_t(t,\xi) = S, \\ S_\xi(t,\xi) = \frac{1}{2}q\sin^2\frac{r}{2}, \\ r_t(t,\xi) = -\sin^2\frac{r}{2}, \\ q_t(t,\xi) = \frac{1}{2}q\sin r. \end{cases} \tag{2.8}$$

Here the initial data on $(0, \xi)$ with $\xi \ge 0$ are

$$\begin{cases} u(0,\xi) = u_0(x(0,\xi)), \\ r(0,\xi) = 2\arctan u_{0,x}(x(0,\xi)), \\ q(0,\xi) = 1. \end{cases} \tag{2.9}$$

The boundary conditions on $(t, 0)$ with $t \ge 0$ are

$$\begin{cases} u(t,0) = 0, \\ r(t,0) = 0, \\ q(t,0) = 1. \end{cases} \tag{2.10}$$

It suffices to express the solution $u(t, \xi)$ in terms of the original variables (t, x), then we have, according to the results in [10],

Lemma 2.1. *Let $(x,u,r,q)(t,\xi)$ be the solution to the system* (2.8)–(2.10) *with $q>0$. Then the set of points*

$$\{(t,x(t,\xi),u(t,\xi)); \quad (t,\xi)\in\mathbb{R}^+\times\mathbb{R}^+\} \tag{2.11}$$

is the graph of a conservative solution to the Hunter–Saxton equation (2.1).

Now, we begin with the construction of perturbed solutions.

Lemma 2.2. *Let (u,r,q) be a smooth solution of the semilinear system* (2.8) *and given a point $(t_0,\xi_0)\in\mathbb{R}^+\times\mathbb{R}^+$. If $(r,r_\xi,r_{\xi\xi})(t_0,\xi_0)=(\pi,0,0)$, then there exists a 3–parameter family of smooth solutions $(u^\vartheta,r^\vartheta,q^\vartheta)$ of* (2.8)*, depending smoothy on ϑ with ϑ in a small enough open ball centered at the origin in $\mathbb{R}^3$, such that the following holds.*

(i) *when $\vartheta=0\in\mathbb{R}^3$, one recovers the original solution, namely $(u^0,r^0,q^0)=(u,r,q)$.*

(ii) *At the point (t_0,ξ_0), when $\vartheta=0$, one has*

$$\textbf{rank}\ D_\vartheta(r^\vartheta,r^\vartheta_\xi,r^\vartheta_{\xi\xi})=3. \tag{2.12}$$

Proof. Let (u,r,q) be a smooth solution of the semilinear system (2.8). Now, we construct families of solutions $(u^\vartheta,r^\vartheta,q^\vartheta)$ to system (2.8) with perturbations on the initial data as

$$\begin{cases} u^\vartheta(0,\xi)=u(0,\xi)+\sum\limits_{i=1,2,3}\vartheta_iU_i(0,\xi),\\ r^\vartheta(0,\xi)=r(0,\xi)+\sum\limits_{i=1,2,3}\vartheta_iR_i(0,\xi),\\ q^\vartheta(0,\xi)=q(0,\xi)+\sum\limits_{i=1,2,3}\vartheta_iQ_i(0,\xi),\end{cases} \tag{2.13}$$

respectively, for some suitable functions $U_i(0,\xi),R_i(0,\xi),Q_i(0,\xi)\in\mathcal{C}_c^\infty(\mathbb{R}^+)$. The boundary conditions are always

$$\begin{cases} u^\vartheta(t,0)=0,\\ r^\vartheta(t,0)=0,\\ q^\vartheta(t,0)=1.\end{cases} \tag{2.14}$$

We note that in any bounded time interval, the singularity, that is $r=\pi$, can only happen in a region uniformly away from the t-axis.

One has

$$\begin{cases} \frac{\partial}{\partial\xi}u^\vartheta=f_1^\vartheta=\frac12 q^\vartheta\sin r^\vartheta,\\ \frac{\partial}{\partial t}r^\vartheta=f_2^\vartheta=-\sin^2\frac{r^\vartheta}{2},\\ \frac{\partial}{\partial t}q^\vartheta=f_3^\vartheta=\frac12 q^\vartheta\sin r^\vartheta,\end{cases} \tag{2.15}$$

where $f_1^\vartheta,f_1^\vartheta,f_3^\vartheta$ are the perturbations of the right hand side of $(2.7)_1,(2.7)_2$ and $(2.7)_3$. In light of [10], for each $\vartheta\in\mathbb{R}^3$, $U_i(0,\xi),R_i(0,\xi),Q_i(0,\xi)\in\mathcal{C}_c^\infty(\mathbb{R}^+)$, we obtain a unique solution

$$\begin{cases} u^\vartheta(t,\xi)=u(t,\xi)+\sum\limits_{i=1,2,3}\vartheta_iU_i(t,\xi),\\ r^\vartheta(t,\xi)=r(t,\xi)+\sum\limits_{i=1,2,3}\vartheta_iR_i(t,\xi),\\ q^\vartheta(t,\xi)=q(t,\xi)+\sum\limits_{i=1,2,3}\vartheta_iQ_i(t,\xi),\end{cases}$$

of the semilinear system (2.15).

On the other hand, taking derivatives to the equation of r in (2.8), we have

$$\frac{\partial}{\partial t}r^\vartheta_\xi=-\frac12 r^\vartheta_\xi\sin r=:f_4^\vartheta, \tag{2.16}$$

and

$$\frac{\partial}{\partial t}r^\vartheta_{\xi\xi}=-\frac12(r^\vartheta_{\xi\xi}\sin r^\vartheta+(r^\vartheta_\xi)^2\cos r^\vartheta)=:f_5^\vartheta. \tag{2.17}$$

Thus, the equations $(2.8)_3$, (2.16) and (2.17) form a complete system. Then consider the ODE system

$$\frac{\partial}{\partial t}\begin{pmatrix} r^\vartheta \\ r^\vartheta_\xi \\ r^\vartheta_{\xi\xi} \end{pmatrix} = \begin{pmatrix} f^\vartheta_2 \\ f^\vartheta_4 \\ f^\vartheta_5 \end{pmatrix}.$$

Then it is easy to check that the terms on right hand side of $(2.15)_2$, (2.16) and (2.17) are Lipschitz continuous, so we can choose suitable perturbation R_i, $i = 1, 2, 3$, such that at the point (t_0, ξ_0) and $\vartheta = 0$, the Jacobian matrix has full rank, that is,

$$\textbf{rank}\ D_\vartheta \begin{pmatrix} r^\vartheta \\ r^\vartheta_\xi \\ r^\vartheta_{\xi\xi} \end{pmatrix} = 3. \tag{2.18}$$

Let's give a little more details. In fact, it is easy to get that

$$\frac{\partial}{\partial t}\begin{pmatrix} R_i \\ (R_i)_\xi \\ (R_i)_{\xi\xi} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sin r & 0 & 0 \\ * & -\frac{1}{2}\sin r & 0 \\ * & * & -\frac{1}{2}\sin r \end{pmatrix}\begin{pmatrix} R_i \\ (R_i)_\xi \\ (R_i)_{\xi\xi} \end{pmatrix}$$

where the matrix in the right hand side is a lower triangular matrix and $*$ denotes any number. Then we only have to choose R_i such that at (t_0, ξ_0), $R_1 = O(1)$; $R_2 = 0$ and $(R_2)_\xi = O(1)$; $R_3 = 0$, $(R_3)_\xi = 0$ and $(R_3)_{\xi\xi} = O(1)$, then we can prove (2.18). This completes the proof of Lemma 2.2. □

To prove the main theorem in this subsection, we first have the following lemma following from Lemma 2.2, which shows for almost all of the solutions the level sets $\{(t, \xi);\ r(t, \xi) = \pi\}$ satisfies a generic property. The proof is based on Lemma 2.2 and the transversality argument, which is similar to those given in [2, 14]. We omit it here for brevity.

Lemma 2.3. *Let a compact domain*

$$\Omega := \{(t, \xi); 0 \le t \le T, 0 \le \xi \le M\},$$

and define $\mathcal{S}$ be the family of all $\mathcal{C}^2$ solutions (u, r, q) to the semilinear system (2.8)*, with $q > 0$ for all $(t, \xi) \in [0, T] \times \mathbb{R}^+$. Moreover, define $\mathcal{S}' \subset \mathcal{S}$ be the subfamily of all solutions (u, r, q), such that for $(t, \xi) \in \Omega$, the value*

$$(r, r_\xi, r_{\xi\xi}) = (\pi, 0, 0) \tag{2.19}$$

cannot be attained. Then $\mathcal{S}'$ is a relatively open and dense subset of $\mathcal{S}$, in the topology induced by $\mathcal{C}^2(\Omega)$.

Now, we study the structure of solutions. Roughly speaking, we prove that, for generic smooth initial data, the solution is piecewise smooth. Its gradient u_x blows up along finitely many smooth curves in the t–x plane. The main result reads as

Theorem 2.2 (Generic regularity)**.** *Let $T > 0$ be given, then there exists an open dense set of initial data*

$$\mathcal{D} \subset \Big(\mathcal{C}^3(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)\Big),$$

such that, for $u_0 \in \mathcal{D}$, the solution $u = u(t, x)$ of (2.1) *is twice continuously differentiable in the complement of finitely many characteristic curves, within the domain $[0, T] \times \mathbb{R}^+$.*

Proof. For the future use, we define the space

$$\mathcal{M} := \mathcal{C}^3(\mathbb{R}^+) \cap H^1(\mathbb{R}^+),$$

with norm

$$\|u_0\|_{\mathcal{M}} := \|u_0\|_{\mathcal{C}^3} + \|u_0\|_{H^1}.$$

Let the initial data $\hat{u}_0 \in \mathcal{M}$ be given and set the open ball

$$B_\delta := \{u_0 \in \mathcal{M};\ \|u_0 - \hat{u}_0\|_{\mathcal{M}} < \delta\}.$$

Now, we prove our result by six steps.

1. Since $u_0 \in \mathcal{M}$, by the definition of the space $\mathcal{M}$, we have

$$u_0(x) \to 0 \quad \text{and} \quad u_{0,x}(x) \to 0, \quad \text{as } x \to \infty.$$

Hence, there exists $h > 0$ sufficiently large, such that $u_0(x), u_{0,x}(x)$ being uniformly bounded for all $x \geq h$. By a standard comparison argument, we deduce $u_x(t,x)$ remains uniformly bounded on a domain of the form $\{(t,x);\ t \in [0,T], x \geq h + T\|u\|_{L^\infty}\}$. This means that the singularity of $u(t,x)$ in the set $[0,T] \times \mathbb{R}^+$ only occur on the compact set

$$\mathcal{N} := [0,T] \times [0, h + T\|u\|_{L^\infty}].$$

Next, for any $u_0 \in B_\delta$, denote Λ be the map of $(t,\xi) \mapsto \Lambda(t,\xi) := (t, x(t,\xi))$, and let Ω be a domain as in Lemma 2.3. Then we can obtain the inclusion $\mathcal{N} \subset \Lambda(\Omega)$ by choosing M large enough and by possibly shrinking the radius δ.

More specifically, we define the subset $\tilde{\mathcal{D}} \subset B_\delta$ as: $u_0 \in \tilde{\mathcal{D}}$ if $u_0 \in B_\delta$ and for the corresponding solution (u,r,q) of (2.8), the value (2.19) is never attained for any (t,ξ) such that $(t, x(t,\xi)) \in \mathcal{N}$. Later in this proof, we will validate $\tilde{\mathcal{D}}$ is an open dense set.

2. To begin with, we claim the set $\tilde{\mathcal{D}}$ is open, in the topology of $\mathcal{C}^3$. Indeed, consider a sequence of initial data $(u_0^\nu)_{\nu \geq 1}$ such that the sequence converges to u_0, with $u_0^\nu \notin \tilde{\mathcal{D}}$. By the definition of $\tilde{\mathcal{D}}$, there exist points (t^ν, ξ^ν) such that the corresponding solutions (u^ν, r^ν, q^ν) satisfy

$$(r^\nu, r^\nu_\xi, r^\nu_{\xi\xi})(t^\nu, \xi^\nu) = (\pi, 0, 0), \quad (t^\nu, x^\nu(t^\nu, \xi^\nu)) \in \mathcal{N},$$

for all $\nu \geq 1$. Recall that the domain $\mathcal{N}$ is compact, we can choose a subsequence, denote still by (t^ν, ξ^ν) at which $(t^\nu, \xi^\nu) \to (\bar{t}, \bar{\xi})$ for some point $(\bar{t}, \bar{\xi})$. By continuity,

$$(r, r_\xi, r_{\xi\xi})(\bar{t}, \bar{\xi}) = (\pi, 0, 0), \quad (t, x(\bar{t}, \bar{\xi})) \in \mathcal{N},$$

which implies $u_0 \notin \tilde{\mathcal{D}}$. This means $\tilde{\mathcal{D}}$ is an open set.

3. Now, we prove the set $\tilde{\mathcal{D}}$ is dense in B_δ. Let $u_0 \in B_\delta$ be given, by a small perturbation, we can assume that $u_0 \in \mathcal{C}^\infty$. By virtue of Lemma 2.3, we can construct a sequence of solutions (u^ν, r^ν, q^ν) of (2.8), such that,

(i) for every $\nu \geq 1, (t,\xi) \in \Omega$, the value in (2.19) is never attained.

(ii) The $\mathcal{C}^k, k \geq 1$ norm of the difference satisfies

$$\lim_{\nu \to \infty} \|(u^\nu - u, r^\nu - r, q^\nu - q, x^\nu - x)\|_{\mathcal{C}^k(I)} = 0,$$

for every bounded set $I \subset [0,T] \times \mathbb{R}^+$. Thus, for $t = 0$, the corresponding sequence of initial value satisfies

$$\lim_{\nu \to \infty} \|u_0^\nu - u_0\|_{\mathcal{C}^k([a,b])} = 0, \tag{2.20}$$

for every bounded set $[a,b] \subset \mathbb{R}^+$.

Introduce a cutoff function $\psi(x) \in \mathcal{C}_c^\infty$, such that

$$\begin{cases} \psi(x) = 1, & \text{if } 0 \leq x \leq l, \\ \psi(x) = 0, & \text{if } x \geq l + 1, \end{cases}$$

where $l \gg h + T\|u\|_{L^\infty}$ is large enough. Then for every $\nu \geq 1$, consider the following initial data

$$\tilde{u}_0^\nu := \psi u_0^\nu + (1 - \psi) u_0.$$

With the help of (2.20), we can easily get

$$\lim_{\nu \to \infty} \|\tilde{u}_0^\nu - u_0\|_{\mathcal{M}} = 0.$$

Now, we choose $l > 0$ sufficiently large, such that for any $(t,x) \in \mathcal{N}$, we have

$$\tilde{u}^\nu(t,x) = u^\nu(t,x).$$

Notice that $\tilde{u}^\nu(t,x)$ is $\mathcal{C}^2$ on the outer domain $\{(t,x);\ t \in [0,T], x \geq h + T\|u\|_{L^\infty}\}$. Thus, for every $\nu \geq 1$ sufficiently large, $\tilde{u}_0^\nu \in \tilde{D}$. This concludes that $\tilde{D}$ is dense in B_δ.

4. At last, we need to show that for every initial data $u_0 \in \tilde{D}$, the corresponding solution $u(t,x)$ of (2.1) is piecewise $\mathcal{C}^2$ on the domain $[0,T] \times \mathbb{R}^+$. Toward this goal, we recall that u

is $\mathcal{C}^2$ on the outer domain $\{(t,x);\ t\in[0,T], x\geq h+T\|u\|_{L^\infty}\}$, so it remains to consider the singularity of u on the inner domain $\mathcal{N}$.

According to step 1, every point in $\mathcal{N}$ is contained in the image of the domain Ω. Hence, for every point $(t_0,\xi_0)\in\Omega$, we have two cases.

case I. $r(t_0,\xi_0)\neq\pi$. From the coordinate change $x_\xi=q\cos^2\frac{r}{2}$, we know the map $(t,\xi)\mapsto(t,x)$ is locally invertible in a neighborhood of (t_0,ξ_0). Therefore, the function u is $\mathcal{C}^2$ in a neighborhood of $(t_0,x(t_0,\xi_0))$.

case II. $r(t_0,\xi_0)=\pi$. By the equation of r in (2.7), we have $r_t(t_0,\xi_0)\neq 0$.

5. By continuity, there exists $\eta>0$, such that the value in (2.19) is never attained in the open neighborhood

$$\Omega':=\{(t,\xi);\ 0\leq t\leq T, 0\leq\xi\leq M+\eta\}.$$

Thanks to the implicit function theorem, we derive that the set

$$S^r:=\{(t,\xi)\in\Omega';\ r(t,\xi)=\pi\}$$

is 1–dimensional embedded manifold of class $\mathcal{C}^2$.

Now, we claim that the number of connected components of S^r that intersect the compact set Ω is finite. Assume, by contradiction, that $P_1,P_2,\cdots$ is a sequence of points in $S^r\cap\Omega$ belonging to distinct components. Thus, we can choose a subsequence P_i, such that $P_i\to\bar{P}$ for some $\bar{P}\in S^r\cap\Omega$. By assumption, $(r_t,r_\xi)(\bar{P})\neq(0,0)$.

Hence, by the implicit function theorem, there is a neighborhood $\mathcal{U}$ of $\bar{P}$ such that $\gamma:=S^r\cap\mathcal{U}$ is a connected $\mathcal{C}^2$ curve. Thus, P_i on all i large enough, providing a contradiction.

6. To complete the proof, we need to study in more detail the image of the singular set S^r, since the set of points (t,x) where u is singular coincides with the image of the set S^r under the C^2 map $(t,\xi)\mapsto\Lambda(t,\xi)=(t,x(t,\xi))$.

By the argument in step 5, inside the compact set Ω, there are only finite many points where $r=\pi, r_\xi=0, r_t\neq 0$, say $P_i=(t_i,\xi_i), i=1,\cdots,m$.

From the analysis in step 5, the set $S^r\backslash\{P_1,\cdots,P_m\}$ has finitely many connected components which intersect Ω. Consider any one of these components. This is a connected curve, say γ_j, such that $r=\pi, r_\xi\neq 0$ for any $(t,\xi)\in\gamma_j$. Thus, this curve can be expressed in the form

$$\gamma_j=\{(t,\xi);\ \xi=\phi_j(t), a_j<t<b_j\},$$

for a suitable function ϕ_j.

At this stage, we claim that the image $\Lambda(\gamma_j)$ is a $\mathcal{C}^2$ curve in the t–x plane. Indeed, it suffices to show that, on the open interval (a_j,b_j), the differential of the map $t\mapsto(t,x(t,\phi_j(t)))$ does not vanish. This is true, because

$$\frac{d}{dt}x(t,\phi_j(t))=1+x_\xi\phi_j'=1>0,$$

since $x_\xi=q\cos^2\frac{r}{2}=0$ when $r=\pi$. Hence, the singular set $\Lambda(S^r)$ is thus the union of the finitely points $p_i=\Lambda(P_i), i=1,\cdots,m$, together with finitely many $\mathcal{C}^2$–curve $\Lambda(\gamma_j)$. This completes the proof of Theorem 2.2. □

In the next subsection, we will construct a distance which renders Lipschitz continuous the semigroup of conservative solutions of (2.1). Toward this goal, one needs a dense set of piecewise smooth paths of solutions, whose weighted length can be controlled in time. Hence, we now study families of conservative solutions $u^\theta=u(t,x,\theta)$ of (2.1) with initial data $u(0,x,\theta)=u_0(x,\theta)=:u_0^\theta(x)$, depending smoothly on an additional parameter $\theta\in[0,1]$. More precisely, these paths of initial data will lie in the space

$$\mathcal{X}_1:=\mathcal{C}^3\big(\mathbb{R}^+\times[0,1]\big)\cap L^\infty\big([0,1];H^1(\mathbb{R}^+)\big).$$

Now, we have the following generic regularity for the 1–parameter family of solution. The proof is similar to [2], we omit it here for brevity.

Theorem 2.3. *Let $T > 0$ be given, then for any 1–parameter family of initial data $\tilde{u}_0^\theta \in \mathcal{X}_1$ and any $\varepsilon > 0$, there exists a perturbed family $(x, \theta) \mapsto u_0^\theta(x)$ such that*

$$\|u_0^\theta - \tilde{u}_0^\theta\|_{\mathcal{X}_1} < \varepsilon,$$

and moreover, for all except at most finitely many $\theta \in [0, 1]$, the conservative solution $u^\theta = u(t, x, \theta)$ of (2.1) is smooth in the complement of finitely many points and finitely many $\mathcal{C}^2$ curves in the domain $[0, T] \times \mathbb{R}^+$.

2.3. **Lipschitz metric for the Hunter–Saxton equation.** In this subsection, we establish a new metric which renders Lipschitz continuous on bounded subsets of $H^1(\mathbb{R}^+)$. Our distance will be determined by the minimum cost to transport an energy measure from one solution to the other. To define a suitable transportation distance between two solutions, we start from the case of smooth solutions of (2.1).

2.3.1. *The norm of tangent vector for smooth solutions.* Now, we introduce a Finsler norm on the solution flow and its tangent vector. Then by some elaborate estimates, we obtain the key estimate describing how the norm grows in time. To this end, let $u(x)$ be a smooth solution to (2.1) and consider a family of perturbed solutions of the form

$$u^\epsilon(x) = u(x) + \epsilon v(x) + o(\epsilon). \tag{2.21}$$

In terms of (2.1), the first order perturbation v must satisfy the equations

$$v_t + uv_x + vu_x = \int_0^x (u_z v_z)(z)\, dz, \tag{2.22}$$

and

$$v_{xt} + uv_{xx} + u_x v_x + vu_{xx} = 0. \tag{2.23}$$

To measure the cost of transporting u to u^ϵ on the x-u plane, we notice that the tangent flow v only measures the vertical displacement between two solutions. In order to giving enough freedom of this planar transport, we also need to add a quantity, named as w, to measure the (horizontal) shift on x as

$$x^\epsilon := x + \epsilon w(x) + o(\epsilon). \tag{2.24}$$

Here $w(t, x)$ can be obtained by propagating along characteristics the shifts $w_0(x)$ as the initial data. That is, we require that when $x(t)$ is a characteristic starting from x_0 then $x^\epsilon(t)$ is also a characteristic starting from x_0^ϵ, so

$$\frac{d}{dt}x^\epsilon(t) = u^\epsilon(x^\epsilon) \qquad \text{when} \qquad \frac{d}{dt}x(t) = u(x).$$

Then, using (2.21), (2.24) and taking limit $\epsilon \to 0$, we have

$$w_t + uw_x = v + u_x w. \tag{2.25}$$

Thus, we can introduce a Finsler norm for the tangent vector as

$$\|v\|_u := \inf_{w \in \mathcal{A}} \|(w, v)\|_u, \tag{2.26}$$

where

$$\mathcal{A} = \{\text{solutions } w(t, x) \text{ of (2.25) with smooth initial data } w_0(x)\}.$$

Note u is always smooth in this section, hence w solved in (2.25). The norm is defined as

$$\begin{aligned} &\|(w, v)\|_u \\ &= \int_0^\infty \{|w|(e^{-x} + u_x^2) + |v + u_x w|(e^{-x} + u_x^2) + |2u_x(v_x + u_{xx}w) + u_x^2 w_x|\}\, dx \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{2.27}$$

Remark 2.2. *We insert e^{-x}, instead of a constant, in the integrands of I_1 and I_2, because the integrals are on an infinite interval. If u is in L^1, then one could use a constant instead, with w_0 also assumed to be in L^1.*

A norm very similar to (2.27) *for smooth solutions of Hunter-Saxton equation (2.1) was proposed in A. Bressan's unpublished research note. We collect this norm here for the completeness of this paper.*

The explanation of each integrand in (2.27) is given in the following.

(a) For I_1, it can be interpreted as the cost for transporting the base measure with density $e^{-x}+u_x^2$ from the point x to the point $x+\epsilon w(x)$.

(b) I_2 accounts for the change in u times the density $e^{-x}+u_x^2$ of the base measure. Indeed, the change in u can be estimated as

$$\frac{u^\epsilon(x+\epsilon w(x))-u(x)}{\epsilon}\approx v(x)+u_x(x)w(x).$$

(c) I_3 accounts for the change in the base measure with density u_x^2. More precisely,

$$\begin{aligned}&\lim_{\epsilon\to0}\frac{(u_x^\epsilon)^2(x+\epsilon w(x))\,dx^\epsilon-u_x^2(x)\,dx}{\epsilon}\\ =&\lim_{\epsilon\to0}\frac{\Big((u_x^\epsilon)^2(x+\epsilon w(x))-u_x^2(x)\Big)\,dx^\epsilon-u_x^2(x)\big(\,dx^\epsilon-\,dx\big)}{\epsilon}\\ =&\Big(2u_x(v_x+u_{xx}w)(x)+u_x^2(x)w_x(x)\Big)\,dx.\end{aligned}$$

Our main goal of this subsection is to estimate how the norm defined in (2.26) changes in time.

Theorem 2.4. *Let $u=u(t,x)$ be a smooth solution to* (2.1)–(2.3)*, and assume that the first order perturbation v satisfies the corresponding linear equation* (2.22)*. Then for any $\tau\in[0,T]$, we have*

$$\|v(\tau)\|_{u(\tau)}\le e^{C\tau}\|v(0)\|_{u_0},\tag{2.28}$$

for some constant $C>0$ depending only on T and $\|u_0(x)\|_{H^1}$.

Proof. It suffices to show that

$$\frac{d}{dt}\|(w,v)(t)\|_{u(t)}\le C\|(w,v)(t)\|_{u(t)},\tag{2.29}$$

for any v and w satisfying (2.22) and (2.25). Here and in this subsection, $C>0$ is a generic constant only depending on T and $\|u_0(x)\|_{H^1}$ which may vary in different estimates.

To prove (2.29), first, notice that for any L^1 smooth function f, we have

$$\frac{d}{dt}\int_{\mathbb{R}}|f|\,dx=\int_{\mathbb{R}}(|f|)_t+(u|f|)_x\,dx=\int_{\mathbb{R}}\ \mathrm{sign}(f)[f_t+(uf)_x]\,dx\le\int_{\mathbb{R}}|f_t+(uf)_x|\,dx.$$

Now, we devote to the estimate of $f_t+(uf)_x$, with f being $w(e^{-x}+u_x^2)$, $(v+u_xw)(e^{-x}+u_x^2)$ and $2u_x(v_x+u_{xx}w)+u_x^2w_x$, respectively.

1. We first treat the time derivative of I_1, It follows from (2.5) and (2.25) that

$$\begin{aligned}&\Big(w(e^{-x}+u_x^2)\Big)_t+\Big(uw(e^{-x}+u_x^2)\Big)_x\\ =&(w_t+uw_x)(e^{-x}+u_x^2)+w\big[(e^{-x}+u_x^2)_t+(u(e^{-x}+u_x^2))_x\big]\\ =&(v+u_xw)(e^{-x}+u_x^2)+w(u_xe^{-x}-ue^{-x}).\end{aligned}$$

This together with Remark 2.1 yields

$$\frac{d}{dt}\int_0^\infty|w|(e^{-x}+u_x^2)\,dx\le\int_0^\infty|v+u_xw|(e^{-x}+u_x^2)\,dx+C\int_0^\infty|w|(e^{-x}+u_x^2)\,dx.\tag{2.30}$$

2. To estimate the time derivative of I_2, with the help (2.1), (2.5), (2.22) and (2.25), we obtain

$$
\begin{aligned}
&\Big((v+u_xw)(e^{-x}+u_x^2)\Big)_t+\Big(u(v+u_xw)(e^{-x}+u_x^2)\Big)_x\\
=&\big[v_t+uv_x+u_x(w_t+uw_x)+w(u_{xt}+uu_{xx})\big](e^{-x}+u_x^2)\\
&+(v+u_xw)[(e^{-x}+u_x^2)_t+(u(e^{-x}+u_x^2))_x]\\
=&\big[-vu_x+\int_0^x(u_zv_z)(z)\,dz+u_x(v+u_xw)-\frac{1}{2}u_x^2w\big](e^{-x}+u_x^2)\\
&+(v+u_xw)(u_xe^{-x}-ue^{-x})\\
=&\big[\int_0^x(u_zv_z)(z)\,dz+\frac{1}{2}u_x^2w\big](e^{-x}+u_x^2)+(v+u_xw)(u_xe^{-x}-ue^{-x}).
\end{aligned}
\tag{2.31}
$$

Notice that the estimate of the term $\frac{1}{2}u_x^2w(e^{-x}+u_x^2)$ fails to close directly, we rewrite

$$
\begin{aligned}
\frac{1}{2}u_x^2w=&\frac{1}{2}\int_0^x\big(u_z^2w\big)_z(z)\,dz\\
=&\frac{1}{2}\int_0^x(2u_zu_{zz}w+u_z^2w_z)(z)\,dz.
\end{aligned}
\tag{2.32}
$$

Thanks to (2.32), the first two terms on the right hand side of (2.31) can be treated as

$$
\begin{aligned}
&\big[\int_0^x(u_zv_z)(z)\,dz+\frac{1}{2}u_x^2w\big](e^{-x}+u_x^2)\\
=&\frac{1}{2}(e^{-x}+u_x^2)\int_0^x\big(2u_z(v_z+u_{zz}w)+u_z^2w_z\big)(z)\,dz.
\end{aligned}
\tag{2.33}
$$

In view of (2.31) and (2.33), we obtain the estimate

$$
\begin{aligned}
\frac{d}{dt}\int_0^\infty|v+u_xw|(e^{-x}+u_x^2)\,dx\le&C\int_0^\infty|2u_x(v_x+u_{xx}w)+u_x^2w_x|\,dx\\
&+C\int_0^\infty|v+u_xw|(e^{-x}+u_x^2)\,dx.
\end{aligned}
\tag{2.34}
$$

3. We now turn to the time derivative of I_3, using (2.1), (2.23) and (2.25) to get

$$
\begin{aligned}
&\big[2u_x(v_x+u_{xx}w)+u_x^2w_x\big]_t+\big[u(2u_x(v_x+u_{xx}w)+u_x^2w_x)\big]_x\\
=&2(u_{xt}+uu_{xx})(v_x+u_{xx}w)+2u_x\big[v_{xt}+(uv_x)_x+u_{xx}(w_t+uw_x)+w(u_{xxt}+(uu_{xx})_x\big]\\
&+2u_xw_x(u_{xt}+uu_{xx})+u_x^2(w_{xt}+(uw_x)_x)\\
=&-u_x^2(v_x+u_{xx}w)+2u_x\big[-vu_{xx}+u_{xx}(v+u_xw)-wu_xu_{xx}\big]\\
&-u_x^3w_x+u_x^2(v_x+u_{xx}w+u_xw_x)\\
=&0.
\end{aligned}
$$

This yields the estimate

$$
\frac{d}{dt}\int_0^\infty|2u_x(v_x+u_{xx}w)+u_x^2w_x|\,dx\le 0
\tag{2.35}
$$

Combining the estimates (2.30), (2.34) and (2.35), we deduce the desired estimate (2.29). This completes the proof of Theorem 2.4. □

2.3.2. *Length of path of solutions in transformed coordinates.* The analysis in the previous sub–subsection has provided an estimate on how the norm increases in time for smooth solution to (2.1). However, even for smooth initial data, the quantity u_x may blow up in finite time. When this happens, a tangent vector may no longer exist, even if it does exist, it is not obvious that the estimate (2.28) holds. Therefore, we should examine these issues. Indeed, in subsection 2.2 we have proved the following.

• Every path of solutions $\theta \mapsto u^\theta$ can be uniformly approximated by a second path $\theta \mapsto \tilde{u}^\theta$, such that, for all but finitely many values of $\theta \in [0,1]$, the corresponding solution $\tilde{u}^\theta$ remains piecewise smooth on the domain $[0,T] \times \mathbb{R}^+$.

Thus, here we need to show that

• If all solutions u^θ are piecewise smooth, with generic singularities along finitely many points in the t–x plane, then the tangent vectors are still well defined and their norms can be estimated as before.

In what follows, we are interested not in a single solution, but a path of solutions $\theta \mapsto u^\theta$, $\theta \in [0,1]$. To this end, we introduce suitable regularity condition that allows us to define the tangent vector and hence compute the length of the path of solutions.

Definition 2.2. *We say that a solution $u = u(t,x)$ of (2.1) has generic singularities for $t \in [0,T]$ if it admits a representation of the form (2.11), where*

(i) *the functions $(x,u,r,q)(t,\xi)$ are $\mathcal{C}^\infty$,*

(ii) *for $t \in [0,T]$, the following generic condition holds*

$$r = \pi, r_\xi = 0 \Longrightarrow r_t \neq 0, r_{\xi\xi} \neq 0. \tag{2.36}$$

Definition 2.3. *We say that a path of initial data $\gamma_0^1 : \theta \mapsto u_0^\theta$, $\theta \in [0,1]$ is a piecewise regular path if the following conditions hold*

(i) *There exists a continuous map $(\xi,\theta) \mapsto (x,u,r,q)$ such that the semilinear system (2.8)–(2.10) holds for $\theta \in [0,1]$, and the function $u^\theta(x,t)$ whose graph is*

$$Graph\ (u^\theta) = \{(t, x(t,\xi,\theta), u(t,\xi,\theta)); \quad (t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^+\}$$

provides a conservative solution of (2.1) with initial data $u^\theta(0,x) = u_0^\theta(x)$.

(ii) *There exist finitely many values $0 = \theta_0 < \theta_1 < \cdots < \theta_N = 1$ such that the map $(\xi,\theta) \mapsto (x,u,r,q)$ is $\mathcal{C}^\infty$ for $\theta \in (\theta_{i-1},\theta_i), i = 1,\cdots,N$, and the solution $u^\theta = u^\theta(t,x)$ has only generic singularities at time $t = 0$.*

In addition, if for all $\theta \in [0,1]\backslash\{\theta_1,\cdots,\theta_N\}$, the solution u^θ has generic singularities for $t \in [0,T]$, then we say the path of solution $\gamma_t^1 : \theta \mapsto u^\theta$ is **piecewise regular** *for $t \in [0,T]$.*

As a consequence of Theorem 2.3 and Lemma 2.3, we obtain the following Corollary. We refer the readers to [2] for more details on how to get this corollary.

Corollary 2.1. *Given $T > 0$, let $\theta \mapsto (x^\theta, u^\theta, r^\theta, q^\theta), \theta \in [0,1]$, be a smooth path of solutions to the system (2.8)–(2.10). Then there exists a sequence of paths of solution $\theta \mapsto (x_n^\theta, u_n^\theta, r_n^\theta, q_n^\theta)$, such that*

(i) *For each $n \geq 1$, the path of corresponding solution of (2.1) $\theta \mapsto u_n^\theta$ is regular for $t \in [0,T]$, according to Definition 2.3.*

(ii) *For any bounded domain Ω in the t–ξ space, the functions $(x_n^\theta, u_n^\theta, r_n^\theta, q_n^\theta)$ converge to $(x^\theta, u^\theta, r^\theta, q^\theta)$ uniformly in $\mathcal{C}^k([0,1] \times \Omega)$, for every $k \geq 1$, as $n \to \infty$.*

Thanks to Corollary 2.1, now we devote to proving that the weighted length of a regular path satisfies the same estimates as the smooth paths considered in subsection 2.3.1. To begin with, we derive an expression for the norm of a tangent vector in t–ξ coordinates. To continue, giving a reference solution $u(t,x)$ of (2.1) and a family of perturbed solutions $u^\varepsilon(t,x)$, we consider the corresponding smooth solutions of (2.8)–(2.10), say $(x^\varepsilon, u^\varepsilon, r^\varepsilon, q^\varepsilon)$. Assume the perturbed solutions take the form

$$(x^\varepsilon, u^\varepsilon, r^\varepsilon, q^\varepsilon)(t,\xi) = (x,u,r,q)(t,\xi) + \varepsilon(X,U,R,Q)(t,\xi) + o(\varepsilon).$$

By the smooth coefficients of (2.8), we have that the first order perturbations satisfy a linearized system which are well defined for $(t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^+$. Now we express the terms I_1–I_3 of (2.27) in terms of (X,U,R,Q).

(1) We begin with the shift in x as

$$w = \lim_{\varepsilon \to 0} \frac{x^\varepsilon(t,\xi^\varepsilon) - x(t,\xi)}{\varepsilon} = X + x_\xi \cdot \frac{\partial \xi^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}. \tag{2.37}$$

(2) The change in u is

$$v+u_x w=\lim_{\varepsilon\to 0}\frac{u^\varepsilon(t,\xi^\varepsilon)-u(t,\xi)}{\varepsilon}=U+u_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}. \tag{2.38}$$

(3) To achieve the change in the base measure with density u_x^2, first, we have

$$\frac{d}{d\varepsilon}q^\varepsilon|_{\varepsilon=0}=\lim_{\varepsilon\to 0}\frac{q^\varepsilon(t,\xi^\varepsilon)-q(t,\xi)}{\varepsilon}=Q+q_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}.$$

Then the integrand in I_3 is calculated as

$$\begin{aligned}&\frac{d}{d\varepsilon}(q^\varepsilon\sin^2\frac{r^\varepsilon}{2}+q\sin^2\frac{r}{2}x_\xi\xi_x^\varepsilon)|_{\varepsilon=0}\\=&\Big(Q+q_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}+qx_\xi\cdot\frac{\partial\xi_x^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}\Big)\sin^2\frac{r}{2}+\frac{q}{2}\sin r\big[R+r_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}\big].\end{aligned} \tag{2.39}$$

Notice that

$$(1+u_x^2)\,dx=q\,d\xi.$$

Hence, the relations (2.37)–(2.39) imply the weighted norm of a tangent vector (2.27) can be written as

$$\|(w,v)\|_u=\sum_{\ell=1}^3\int_0^\infty|J_\ell(t,\xi)|\,d\xi, \tag{2.40}$$

where

$$\begin{aligned}J_1&=\big[X+x_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}\big]\big(e^{-y(t,\xi)}\cos^2\frac{r}{2}+\sin^2\frac{r}{2}\big)q,\\J_2&=\big[U+u_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}\big]\big(e^{-y(t,\xi)}\cos^2\frac{r}{2}+\sin^2\frac{r}{2}\big)q,\\J_3&=\Big(Q+q_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}+q\,x_\xi\cdot\frac{\partial\xi_x^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}\Big)\sin^2\frac{r}{2}+\frac{q}{2}\sin r\big[R+r_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}\big].\end{aligned}$$

Note, the horizontal shift variable w in (2.24) can be obtained by propagating along characteristics the shift w_0. And on the other hand, ξ is a constant on any given characteristic. So $\frac{\partial\xi^\varepsilon}{\partial\varepsilon}(t,\xi)=\frac{\partial\xi^\varepsilon}{\partial\varepsilon}(0,\xi)$, hence is a continuous function. Then, it is easy to verify that each integrand J_ℓ is continuous, for $\ell=1,2,3$.

Now, we introduce the definition of the length of piecewise regular path $\gamma_t^1:\theta\mapsto u^\theta(t)$ and examine the appearance of the generic singularity will not impact the Lipschitz property of the metric.

Definition 2.4. *The length $\|\gamma_t^1\|$ of the piecewise regular path $\gamma_t^1:\theta\mapsto u^\theta$ is defined as*

$$\|\gamma_t^1\|=\inf_{\gamma_t^1}\int_0^1\sum_{\ell=1}^3\int_0^\infty|J_\ell^\theta(t,\xi)|\,d\xi\,d\theta,$$

where the infimum is taken over all piecewise regular path.

The next theorem proves the main goal of this subsection.

Theorem 2.5. *Given any $T>0$, consider a path of solutions $\theta\mapsto u^\theta$ of (2.1), which is piecewise regular for $t\in[0,T]$. Moreover, the estimate $\int_{\mathbb{R}^+}(u_x^\theta)^2(x)\,dx$ is less than some constant $C_E>0$. Then its length satisfies*

$$\|\gamma_t^1\|\le C\|\gamma_0^1\|, \tag{2.41}$$

for some constant $C>0$ depends only on T and $H^1(\mathbb{R}^+)$–norm of initial data.

Proof. By the definition of piecewise regular path, we know u^θ has generic regularity for any $\theta\in[0,1]\backslash\{\theta_1,\cdots,\theta_N\}$. Then the solution u^θ is smooth in the t–ξ variables and piecewise smooth in the t–x variables, thus the existence of the tangent vector is obvious. Now, we claim that, for $\theta\in[0,1]\backslash\{\theta_1,\cdots,\theta_N\}$, we have

$$\|v^\theta(t)\|_{u^\theta(t)}\le e^{C_1t}\|v^\theta(0)\|_{u^\theta(0)}, \tag{2.42}$$

where the constant $C_1 > 0$ depends only on T and $H^1(\mathbb{R}^+)$–norm of initial data.

In accordance with the definition 2.4, fix $\eta > 0$, there exists some ξ, such that, at time $t = 0$, we have

$$\int_0^1 \sum_{\ell=1}^3 \int_0^\infty |J_\ell^\theta(0,\xi)|\, d\xi\, d\theta \le \|\gamma_0^1\| + \eta.$$

Integrating (2.42) over $\theta \in [0,1]$, one deduces that

$$\|\gamma_t^1\| \le C_2(\|\gamma_0^1\| + \eta),$$

which yields (2.41), since $\eta > 0$ is arbitrary.

The main issue here is to prove the estimate (2.42) satisfies. According to (2.28), (2.42) holds, if u^θ is smooth in the t–x variables. Thus, it remains to show the same result can be applied if u^θ is piecewise smooth with generic singularities. Towards this goal, we first observe that there exist at most finitely many points $W_j = (t_j, \xi_j), j = 1, \cdots, N$, such that $r^\theta = \pi$, $r^\theta_\xi = 0$, because of the the generic condition (2.36) and the fact that for any bounded time interval singularity only happens in a bounded set of the (t,x)-plane as proved in the part 1 of Theorem 2.2. Furthermore, at each point W_j, the map

$$t \mapsto \int_0^1 \sum_{\ell=1}^3 \int_0^\infty |J_\ell^\theta(t,\xi)|\, d\xi\, d\theta$$

is continuous at each time $t = t_j$. Hence, the metric will not be impacted at (at most finite) time $t = t_j$ when there exist singularities such that $r^\theta = \pi$, $r^\theta_\xi = 0$. In fact, these singularities are corresponding to the starting and ending points of the singularity curve: $r^\theta = \pi$.

Now, it remains to show that, at $t \ne t_j$, the generic singularity does not affect the estimate (2.28), that is, we will show the time derivative

$$\frac{d}{dt} \sum_{\ell=1}^3 \int_0^\infty |J_\ell^\theta(t,\xi)|\, d\xi$$

will not be affected by the presence of singularity. Indeed, for a fixed time $\tau \in [0,T]$, let the point $(t_\varepsilon, \xi_\varepsilon)$ be the intersection of the $\Gamma_{\tau-\varepsilon} = \{(t,\xi);\ t = \tau - \varepsilon\}$ and $\{(t,\xi);\ r^\theta(t,\xi) = \pi\}$, and the point $(t'_\varepsilon, \xi'_\varepsilon)$ be the intersection of the $\Gamma_{\tau+\varepsilon} = \{(t,\xi);\ t = \tau + \varepsilon\}$ and $\{(t,\xi);\ r^\theta(t,\xi) = \pi\}$. We denote

$$\begin{cases} \Lambda_\varepsilon^+ := \Gamma_{\tau+\varepsilon} \cap \{(t,\xi); \xi \in [\xi'_\varepsilon, \xi_\varepsilon]\}, \\ \Lambda_\varepsilon^- := \Gamma_{\tau-\varepsilon} \cap \{(t,\xi); \xi \in [\xi'_\varepsilon, \xi_\varepsilon]\}. \end{cases}$$

Then, we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(\int_{\Lambda_\varepsilon^+} - \int_{\Lambda_\varepsilon^-} \Big) \sum_{\ell=1}^3 |J_\ell^\theta(t,\xi)|\, d\xi = 0,$$

since each integrand is continuous and $|\xi_\varepsilon - \xi'_\varepsilon| = O(\varepsilon)$, because at time $t \ne t_j$, we know that $r^\theta = \pi$, $r^\theta_\xi \ne 0$ at singularity. Thus, (2.28) follows even in the presence of singular curve where $r^\theta = \pi$. This completes the proof of Theorem 2.5. □

2.3.3. *Construction of the geodesic distance.* Now, we are ready to generalize the metric to the space $H^1(\mathbb{R}^+)$ and prove the Lipschitz property. Then we will compare our distance with some familiar distance determined by various norm.

Of course, by a small perturbation on the initial data u_0^θ with $\theta \in [0,1]$, we can establish a path of conservative solutions $\theta \mapsto u^\theta(t,x)$, which remain piecewise smooth, for all except finitely many values of $\theta \in [0,1]$. Namely, for all $t \in [0,T]$, the length of the path $\theta \mapsto u^\theta$ is well defined by the formula

$$\|\gamma_t^1\| := \int_0^1 \|\frac{d}{d\theta} u^\theta(t)\|_{u^\theta(t)}\, d\theta,$$

where $\|\cdot\|_u$ is defined as in (2.27) or equivalently as in (2.40).

Now, to continue, we construct a geodesic distance $d_1(\cdot,\cdot)$ on the space $H^1(\mathbb{R}^+)$. In light of Theorem 2.2, there exists an open dense set $\mathcal{D}\subset\mathcal{C}^3(\mathbb{R}^+)\cap H^1(\mathbb{R}^+)$, such that, for $u_0\in\mathcal{D}$, the solution of (2.1) has only generic singularities. Now, on $\mathcal{D}^\infty:=\mathcal{C}_0^\infty\cap\mathcal{D}$, we construct a geodesic distance, defined as the infimum among the weighted lengths of all piecewise regular paths connecting two given points. Thus, by continuity, this distance can be extended from $\mathcal{D}^\infty$ to a larger space, defined as the completion of $\mathcal{D}^\infty$ with respect to the distance $d_1(\cdot,\cdot)$. In particular, this completion will contain the space H^1.

Assume two data $u,\tilde u\in\mathcal{D}^\infty$, denote

$$\mathcal{E}(u):=\int_0^\infty u_x^2(t,x)\,dx,\quad \mathcal{E}(\tilde u):=\int_0^\infty \tilde u_x^2(t,x)\,dx.$$

Then fix any constant $E_1>0$, denote the set

$$\Sigma_{E_1}:=\{u\in H^1(\mathbb{R}^+);\ \mathcal{E}(u)\le E_1\}.$$

Our distance functional $d_1(\cdot,\cdot)$ is now defined by optimizing over all piecewise regular paths connecting two solutions of (2.1).

Definition 2.5. *For solutions with initial data in $\mathcal{D}^\infty\cap\Sigma_{E_1}$, we define the geodesic distance $d_1(u,\tilde u)$ as the infimum among the weighted lengths of all piecewise regular paths $\theta\mapsto u^\theta$, which connect u with $\tilde u$, that is, for any time t,*

$$d_1(u,\tilde u):=\inf\{\|\gamma_t^1\|;\ \gamma_t^1 \text{ is a piecewise regular path}, \gamma_t^1(0)=u,\gamma_t^1(1)=\tilde u,\\ \mathcal{E}(u^\theta)\le E_1,\ \text{for all } \theta\in[0,1]\}.$$

Now, we can define the metric for the general weak solutions.

Definition 2.6. *Let u_0 and $\tilde u_0$ in $H^1(\mathbb{R}^+)$ be two absolute continuous initial data as required in the existence Theorem 2.1. Denote u and $\tilde u$ to be the corresponding global weak solutions, then we define, for any time t,*

$$d_1(u,\tilde u):=\lim_{n\to\infty} d_1(u^n,\tilde u^n),$$

for any two sequences of solutions u^n and $\tilde u^n$ in $\mathcal{D}^\infty\cap\Sigma_{E_1}$ with

$$\|u^n-u\|_{H^1}\to 0,\quad \text{and}\quad \|\tilde u^n-\tilde u\|_{H^1}\to 0.$$

The limit in the definition is independent on the selection of sequences, because the solution flows are Lipschitz in $\mathcal{D}^\infty\cap\Sigma_{E_1}$, so the definition is well-defined. Note when

$$\|u_0^n-u_0\|_{H^1}\to 0,$$

it is easy to show that the corresponding solutions satisfy, for any $t>0$,

$$\|u^n-u\|_{H^1}\to 0$$

by the semi-linear equations (2.8). Thus the Lipschitz property in Theorem 2.5 can be extended to the general solutions. Notice that the concatenation of two piecewise regular paths is still a piecewise regular path (after a suitable re–parameterization), so $d_1(\cdot,\cdot)$ is a distance. With the help of Theorem 2.5, we have

Theorem 2.6. *The geodesic distance $d_1(\cdot,\cdot)$ renders Lipschitz continuous the flow generated by initial boundary value problem (2.1)–(2.3). In particular, let u_0 and $\tilde u_0$ be two $H^1(\mathbb{R}^+)$ initial data, then for arbitrarily given $T>0$, when $t\in[0,T]$, the corresponding solutions $u(t,x)$ and $\tilde u(t,x)$ satisfy*

$$d_1\big(u(t),\tilde u(t)\big)\le Cd_1(u_0,\tilde u_0),$$

where the constant C depends only on T and $H^1(\mathbb{R}^+)$–norm of initial data.

Finally, we study the relations among our distance $d_1(\cdot,\cdot)$ and other distances such as Sobolev distance and Kantorovich-Rubinstein or Wasserstein distance in the next two Propositions.

Proposition 2.1. *For any $u, \tilde{u} \in H^1$, there exists some constant C depends only on E_1, such that,*

$$d_1(u,\tilde{u}) \le C\Big(\|u-\tilde{u}\|_{L^\infty} + \|u_x - \tilde{u}_x\|_{L^2}\Big). \tag{2.43}$$

Proof. For $\theta \in [0,1]$, consider the path $\gamma_t^1 : \theta \mapsto u^\theta$ as

$$u^\theta = \theta\tilde{u} + (1-\theta)u. \tag{2.44}$$

Obviously, when $\theta = 0, 1$, u^θ coincides with u and $\tilde{u}$, respectively. Moreover, the estimate u_x^θ satisfies

$$\begin{aligned}\int_0^\infty (u_x^\theta)^2(t,x)\,dx &= \int_0^\infty [\theta\tilde{u}_x + (1-\theta)u_x]^2\,dx\\ &\le \int_0^\infty \Big([\theta^2+\theta(1-\theta)]\tilde{u}_x^2 + [(1-\theta)^2+\theta(1-\theta)]u_x^2\Big)\,dx\\ &\le \max\{\mathcal{E}(u),\mathcal{E}(\tilde{u})\} \le E_1.\end{aligned} \tag{2.45}$$

On the other hand, by virtue of (2.44), we obtain

$$v^\theta = \frac{du^\theta}{d\theta} = \tilde{u} - u. \tag{2.46}$$

To derive an upper bound for the weighted length $\|\gamma_t^1\|$, we can choose the shift $w = 0$ in (2.27). Indeed, by (2.44)–(2.46) and the definition of the weighted length of the path γ_t^1, we have

$$\begin{aligned}\|\gamma_t^1\| &= \int_0^1 \|v^\theta\|_{u^\theta}\,d\theta\\ &= \int_0^1\int_0^\infty \Big(|v^\theta|(e^{-x} + (u_x^\theta)^2) + 2|u_x^\theta v_x^\theta|\Big)\,dx\,d\theta\\ &\le \|\tilde{u}-u\|_{L^\infty}\int_0^1\int_0^\infty (e^{-x}+(u_x^\theta)^2)\,dx\,d\theta + 2\|\tilde{u}_x - u_x\|_{L^2}\int_0^1 \|u_x^\theta\|_{L^2}^2\,d\theta\\ &\le C\Big(\|u-\tilde{u}\|_{L^\infty} + \|u_x-\tilde{u}_x\|_{L^2}\Big),\end{aligned}$$

which implies that (2.43) holds. This completes the proof of Proposition 2.1. □

Proposition 2.2. *For any $u, \tilde{u} \in H^1(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$, there exists some constant C depends only on E_1, such that,*

$$\|u_0 - \tilde{u}_0\|_{L^1_{loc}} \le C \cdot d_1(u,\tilde{u}), \tag{2.47}$$

$$\sup_{\|f\|_{\mathcal{C}^1}\le 1} \Big|\int f\,d\mu - \int f\,d\tilde{\mu}\Big| \le d_1(u,\tilde{u}), \tag{2.48}$$

where $\mu, \tilde{\mu}$ are the measures with densities u_x^2 and $\tilde{u}_x^2$ with respect to Lebesgue measure.

Proof. Let $\gamma_t^1 : \theta \mapsto u^\theta$ be a regular path connecting u with $\tilde{u}$.

1. Thanks to the inequality

$$|v| = |v + u_x w - u_x w| \le |v + u_x w| + |u_x w|,$$

for any bounded domain A, it follows from the definition 2.6, (2.26) and (2.46) that

$$d_1(u,\tilde{u}) \ge C_3 \inf_{\gamma_t^1}\int_0^1\int_A |v^\theta|\,dx\,d\theta = C_3\inf_{\gamma_t^1}\int_0^1\int_A \Big|\frac{du^\theta}{d\theta}\Big|\,dx\,d\theta \ge C_4\|u-\tilde{u}\|_{L^1(A)},$$

for some constants $C_3, C_4 > 0$. This yields (2.47).

2. For any function f with $\|f\|_{\mathcal{C}^1} \le 1$, denote μ^θ be the measures with density $(u_x^\theta)^2$ with respect to Lebesgue measure, then the following holds

$$\begin{aligned}\Big|\int_0^1 \frac{d}{d\theta}\int f\,d\mu^\theta\,d\theta\Big| \le& \int_0^1\int_0^\infty |f'|\cdot|w^\theta|(u_x^\theta)^2\,dx\,d\theta\\ &+ \int_0^1\int_0^\infty |f|\cdot|2u_x^\theta(v_x^\theta + u_{xx}^\theta w^\theta) + (u_x^\theta)^2 w_x^\theta|\,dx\,d\theta,\end{aligned} \tag{2.49}$$

where the two integrands on the right hand side of (2.49) are less than I_1 and I_3 of (2.27). Hence, we get the estimate (2.48). This completes the proof of Proposition 2.2. □

The metric (2.48) is a Kantorovich-Rubinstein distance, which is equivalent to a Wasserstein distance by a duality theorem [17].

3. The two–component Hunter–Saxton equations

Aim of this section is to study the generic regularity and stability of solutions to the two–component Hunter–Saxton equations. The procedure is actually as same as in Section 2, while the details are different due to the presence of ρ.

Note that the symbols in this section have no relation to the previous section.

3.1. Preliminaries.

Recall the two–component Hunter–Saxton equations

$$\begin{cases} u_t + uu_x = \frac{1}{2}\int_0^x (u_z^2+\rho^2)(z)\,dz, \\ \rho_t + (u\rho)_x = 0. \end{cases} \tag{3.1}$$

with initial–boundary conditions

$$u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x), \quad u(t,0) = 0, \tag{3.2}$$

and a compatibility conditions

$$u_0(0) = 0, \quad u_0'(0) = 0. \tag{3.3}$$

For smooth solutions, differentiating equation $(3.1)_1$ with respect to the spatial variable x, we obtain

$$u_{xt} + (uu_x)_x = \frac{1}{2}(u_x^2+\rho^2). \tag{3.4}$$

Multiplying (3.4) by u_x and $(3.1)_2$ by ρ, then summing up the resultant equations, we have

$$(u_x^2+\rho^2)_t + \big(u(u_x^2+\rho^2)\big)_x = 0. \tag{3.5}$$

Thus, integrating (3.5) with respect to the x–variable, we see that for smooth solutions the total energy

$$\mathcal{E}(t) := \int_0^\infty (u_x^2+\rho^2)(t,x)\,dx = \mathcal{E}(0) \tag{3.6}$$

is constant in time.

Now, we state the existence results of conservative solutions to the two–component Hunter-Saxton system (3.1)–(3.3), c.f. [15].

Theorem 3.1. *For any initial data $u_0 \in H^1(\mathbb{R}^+), \rho_0 \in L^2(\mathbb{R}^+)$, the two–component Hunter–Saxton equations* (3.1)–(3.3) *admits a global conservative solution $u = u(t,x), \rho = \rho(t,x)$. More precisely, there exists a family of Radon measures $\{\mu_{(t)}; t \in \mathbb{R}^+\}$, depending continuously on time with respect to the topology of weak convergence of measures, such that the following properties hold.*

(i) *The equations* (3.1) *holds in the sense of distribution.*

(ii) *There exists a null set $A \subset \mathbb{R}$ with* meas*(A)=0 such that for every $t \notin A$ the measure $\mu_{(t)}$ is absolutely continuous and has density $u_x^2(t,\cdot) + \rho^2(t,\cdot)$ with respect to Lebesgue measure.*

(iii) *The function $\varpi = u_x^2 + \rho^2$ provides a distributional solution to the balance law $\varpi_t + (u\varpi)_x = 0$.*

(iii) *The family $\{\mu_{(t)}; t \in \mathbb{R}^+\}$ provides a measure–valued solution ϖ to the linear transport equation with source $\varpi_t + (u\varpi)_x = 0$.*

Remark 3.1. *For smooth solution, along the characteristic $\frac{dx(t)}{dt} = u(t,x(t))$, we know from $(3.1)_1$ that the value of the solution u is*

$$u(t,x(t)) = u_0(x) + \frac{1}{2}\int_0^t \int_0^x \big(u_z^2+\rho^2\big)(z)\,dz\,dt.$$

More precisely, one has the bound

$$\|u(t,x)\|_{L^\infty} \le \|u_0(x)\|_{L^\infty} + \frac{t}{2}\mathcal{E}(0).$$

3.2. **Generic regularity of solutions to the two–component Hunter–Saxton equation.** In this subsection, we prove the generic regularity for solutions of (3.1)–(3.3), using some virtues of proof in [2, 14].

Similar to Subsection 2.2, we begin our subsection by introducing new coordinates (t,ξ), where ξ is implicitly defined as

$$\xi := \int_0^{\bar{y}(\xi)} (1 + u_{0,x}^2 + \rho_0^2)\, dx.$$

The characteristic is corresponding to the curve on which ξ equals to a constant,

$$\partial_t y(t,\xi) = u(t, y(t,\xi)), \quad y(0,\xi) = \bar{y}(\xi).$$

Some suitable transformation of variables are

$$p = (1 + u_x^2 + \rho^2)\cdot \frac{\partial y}{\partial \xi}, \quad L = \frac{p}{1+u_x^2+\rho^2}, \quad \alpha = \frac{u_x p}{1+u_x^2+\rho^2}, \quad \beta = \frac{\rho p}{1+u_x^2+\rho^2}. \tag{3.7}$$

then one obtains a semi–linear system

$$\begin{cases} u_t = S, \\ S_\xi = \frac{1}{2}(p-L), \\ p_t = \alpha, \\ L_t = \alpha, \\ \alpha_t = \frac{1}{2}(p-L), \\ \beta_t = 0. \end{cases} \tag{3.8}$$

We derive the semilinear system as follows. For any $t > \tau > 0$ and ξ_1, ξ_2, by (3.5) and (3.7), the equation $(3.8)_3$ can be established as

$$\begin{aligned} &\int_{\xi_1}^{\xi_2}\int_\tau^t \frac{\partial}{\partial s} p(s,\eta)\, ds\, d\eta = \int_{\xi_1}^{\xi_2} \big(p(t,\eta) - p(\tau,\eta)\big)\, d\eta \\ =& \int_{x(t,\xi_1)}^{x(t,\xi_2)} (1+u_x^2+\rho^2)(t,y)\, dy - \int_{x(\tau,\xi_1)}^{x(\tau,\xi_2)} (1+u_x^2+\rho^2)(\tau,y)\, dy \\ =& \int_\tau^t \frac{\partial}{\partial s}\int_{x(s,\xi_1)}^{x(s,\xi_2)} (1+u_x^2+\rho^2)(s,y)\, dy\, ds = \int_\tau^t \int_{x(s,\xi_1)}^{x(s,\xi_2)} u_x(s,y)\, dy\, ds \\ =& \int_\tau^t \int_{\xi_1}^{\xi_2} \frac{u_x p}{1+u_x^2+\rho^2}(s,\eta)\, d\eta\, ds = \int_\tau^t\int_{\xi_1}^{\xi_2} \alpha(s,\eta)\, d\eta\, ds. \end{aligned}$$

The other terms can be estimated similarly, we omit it here for brevity.

By expressing the solution $(u,\rho)(t,\xi)$ in terms of the original variables (t,x), one obtains a solution of the initial–boundary problem (3.1)–(3.3). Indeed, we have the following results, c.f. [15].

Lemma 3.1. *Let $(x,u,L,\alpha,\beta,p)(t,\xi)$ be a smooth solution to the system* (3.8) *with $p > 0$. Then the set of points*

$$\text{Graph } (u,\rho) := \{(t, x(t,\xi), u(t,\xi), \rho(t,\xi)); \quad (t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^+\} \tag{3.9}$$

is the graph of a conservative solution to the two–component Hunter–Saxton equations (3.1)–(3.3)*.*

As a consequence of the semilinear system (3.8), we obtain the following estimates.

Remark 3.2. *From the above semilinear system, we have the following exponential estimates*

$$e^{-C_1 t} \le p, L \le e^{C_1 t}, \quad \text{and} \quad C_2 e^{-C_1 t} \le \alpha \le C_2 e^{C_1 t},$$

where the constant C_1, C_2 are related with initial data.

We provide the consistency condition validated on system (3.8).

Lemma 3.2. *For any fixed ξ, if L, α, β, p satisfy*

$$L^2 + \alpha^2 + \beta^2 = Lp \tag{3.10}$$

at initial time, then it also holds for any time t.

Proof. By virtue of system (3.8), we can get

$$\begin{aligned}\frac{d}{dt}(L^2 + \alpha^2 + \beta^2 - Lp) &= 2LL_t + 2\alpha\alpha_t + 2\beta\beta_t - Lp_t - L_t p \\ &= 2L\alpha + \alpha(p - L) - L\alpha - \alpha p \\ &= 0.\end{aligned}$$

This completes the proof of Lemma 3.2. □

Before proving the generic property of the two–component Hunter–Saxton equations, we indicate the key point when the singularity happens.

Lemma 3.3. *The singularity only appears on the characteristics where $\beta(0) = 0$.*

Proof. From $(3.8)_6$, on the characteristics where $\beta(0, \xi) \neq 0$, we know $\beta(t) = \beta(0) \neq 0$. Solving L from (3.10), we have

$$L = \frac{p \pm \sqrt{p^2 - 4(\alpha^2 + \beta^2)}}{2}.$$

Then the smaller root has the following property

$$L = \frac{p - \sqrt{p^2 - 4(\alpha^2 + \beta^2)}}{2} = \frac{2(\alpha^2 + \beta^2)}{p + \sqrt{p^2 - 4(\alpha^2 + \beta^2)}} \geq \frac{\alpha^2 + \beta^2}{p} \geq \frac{\beta^2}{p} \geq \beta^2(0)e^{-C_1 t}.$$

Thus, by the definition of L and Remark 3.2, it holds that

$$|u_x(t, \xi)| \leq \frac{1}{\beta(0)} e^{C_1 t},$$

which implies that the singularity only appears on the characteristics where $\beta(0) = 0$. This completes the proof of Lemma 3.3. □

Then we introduce the following technical lemma for later references. Please find the proof of this lemma in [14].

Lemma 3.4. *Consider an ODE system*

$$\frac{d}{dt}\mathbf{u}^\varepsilon = f(\mathbf{u}^\varepsilon), \quad \mathbf{u}^\varepsilon(0) = \mathbf{u}_0 + \varepsilon_1 \mathbf{v}_1 + \cdots + \varepsilon_m \mathbf{v}_m,$$

where $\mathbf{u}^\varepsilon(t) : \mathbb{R} \to \mathbb{R}^n$, f is a Lipschitz function. The system is well–posed in $[0, T^*)$. Assume the matrix

$$D_\varepsilon \mathbf{u}^\varepsilon(0) = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m) \in \mathbb{R}^{n \times m},$$

and the rank of this matrix is **rank** *$(D_\varepsilon \mathbf{u}^\varepsilon(0) = k$. Then for any $t \in [0, T^*)$,*

$$\mathbf{rank}\ (D_\varepsilon \mathbf{u}^\varepsilon(t)) = k.$$

Similar to Lemma 2.2, we can construct several families of perturbations of a given solutions to (3.8).

Lemma 3.5. *Let (u, L, α, β, p) be a smooth solution of the semilinear system (3.8) and given a point $(t_0, \xi_0) \in \mathbb{R}^+ \times \mathbb{R}^+$.*

(1) If $(L, L_\xi, L_{\xi\xi})(t_0, \xi_0) = (0, 0, 0)$, then there exists a 3–parameter family of smooth solutions $(u^\vartheta, L^\vartheta, \alpha^\vartheta, \beta^\vartheta, p^\vartheta)$ of (3.8), depending smoothy on ϑ with ϑ in a small enough open ball centered at the origin in $\mathbb{R}^3$, such that the following holds.

(i) when $\vartheta = 0 \in \mathbb{R}^3$, one recovers the original solution, namely $(u^0, L^0, \alpha^0, \beta^0, p^\theta) = (u, L, \alpha, \beta, p)$.

(ii) *At the point* (t_0, ξ_0), *when* $\vartheta = 0$, *one has*

$$\textbf{rank}\ D_\vartheta(L^\vartheta, L^\vartheta_\xi, L^\vartheta_{\xi\xi}) = 3. \tag{3.11}$$

(2) *If* $(L, \alpha_\xi, \alpha_{\xi\xi})(t_0, \xi_0) = (0, 0, 0)$, *then there exists a 3–parameter family of smooth solutions* $(u^\vartheta, L^\vartheta, \alpha^\vartheta, \beta^\vartheta, p^\vartheta)$ *of* (3.8), *depending smoothy on* ϑ *with* ϑ *in a small enough open ball centered at the origin in* $\mathbb{R}^3$, *satisfying* (i) (ii) *above with* (3.11) *replaced by*

$$\textbf{rank}\ D_\vartheta(L^\vartheta, \alpha^\vartheta_\xi, \alpha^\vartheta_{\xi\xi}) = 3.$$

(3) *If* $(L, \beta_\xi, \beta_{\xi\xi})(t_0, \xi_0) = (0, 0, 0)$, *then there exists a 3–parameter family of smooth solutions* $(u^\vartheta, L^\vartheta, \alpha^\vartheta, \beta^\vartheta, p^\vartheta)$ *of* (3.8), *depending smoothy on* ϑ *with* ϑ *in a small enough open ball centered at the origin in* $\mathbb{R}^3$, *satisfying* (i) (ii) *above with* (3.11) *replaced by*

$$\textbf{rank}\ D_\vartheta(L^\vartheta, \beta^\vartheta_\xi, \beta^\vartheta_{\xi\xi}) = 3.$$

In light of Lemma 3.4, this lemma can be proved by showing boundedness of coefficient matrix of (3.8) and the following two equations, that is, equations of first order derivatives

$$\begin{cases} \dfrac{\partial}{\partial t} L_\xi = \alpha_\xi, \\ \dfrac{\partial}{\partial t} \alpha_\xi = \dfrac{1}{2}(p_\xi - L_\xi), \\ \dfrac{\partial}{\partial t} p_\xi = \alpha_\xi, \end{cases}$$

and equations of second order derivatives

$$\begin{cases} \dfrac{\partial}{\partial t} L_{\xi\xi} = \alpha_{\xi\xi}, \\ \dfrac{\partial}{\partial t} \alpha_{\xi\xi} = \dfrac{1}{2}(p_{\xi\xi} - L_{\xi\xi}), \\ \dfrac{\partial}{\partial t} p_{\xi\xi} = \alpha_{\xi\xi}. \end{cases}$$

The next lemma can be proved in a similar way with Lemma 2.3 of the one–component Hunter–Saxton equation or [14].

Lemma 3.6. *Let a compact domain of the form*

$$\Omega := \{(t, \xi); 0 \le t \le T, 0 \le \xi \le M\},$$

and define $\mathcal{S}$ *be the family of all* $\mathcal{C}^2$ *solutions* (u, L, α, β, p) *to the semilinear system* (3.8), *with* $p > 0$ *for all* $(t, \xi) \in [0, T] \times \mathbb{R}^+$. *Moreover, define* $\mathcal{S}' \subset \mathcal{S}$ *be the subfamily of all solutions* (u, L, α, β, p), *such that for* $(t, \xi) \in \Omega$, *none of the the following values is attained*

$$(L, L_\xi, L_{\xi\xi}) = (0, 0, 0), \quad (L, \alpha_\xi, \alpha_{\xi\xi}) = (0, 0, 0), \quad (L, \beta_\xi, \beta_{\xi\xi}) = (0, 0, 0).$$

Then $\mathcal{S}'$ *is a relatively open and dense subset of* $\mathcal{S}$, *in the topology induced by* $\mathcal{C}^2(\Omega)$.

Now, we are ready to prove the main result of this subsection.

Theorem 3.2 (Generic regularity). *For any initial data* $u_0(x) \in H^1(\mathbb{R}^+), \rho_0(x) \in \mathcal{C}^2(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$. *Assume* $\rho_0(x)$ *has finitely number of zero points* $\{x_1, \cdots, x_n\}$. $u_0(x)$ *is* $\mathcal{C}^3$ *in each interval* $I_i := (x_i, x_{i+1}), (i = 0, 1, \cdots, n)(x_0 = -\infty, x_{n+1} = +\infty)$, *and* $\rho_0(x)$ *is* $\mathcal{C}^2$ *in each interval* I_i. *Then the solution* $u(t, x)$ *is three times continuously differentiable and* $\rho(t, x)$ *is twice continuously differentiable in the complement of finitely many isolated points within the domain* $[0, T] \times \mathbb{R}^+$.

Proof. This theorem can be proved in an entirely similar way as in Theorem 2.2. To make our paper brief, we pick up a proof similar to the one in [14], using which we can also find more detailed structures of singularities.

First, formally differentiating $(3.1)_2$ and (3.4) with respect to the spatial variable x, we obtain

$$\begin{cases} u_{xxt} + uu_{xxx} + 2u_x u_{xx} - \rho\rho_x = 0, \\ \rho_{xt} + u\rho_{xx} + u_{xx}\rho + 2u_x\rho_x = 0. \end{cases}$$

or we can express it as

$$\frac{d}{dt}\begin{pmatrix} u_{xx} \\ \rho_x \end{pmatrix} = \begin{pmatrix} -2u_x & \rho \\ -\rho & -2u_x \end{pmatrix}\begin{pmatrix} u_{xx} \\ \rho_x \end{pmatrix}.$$

Set the Lyapunov function $W(t) := u_{xx}^2(t) + \rho_x^2(t)$, a direct computation shows that

$$\begin{aligned} \frac{d}{dt}W(t) =& 2u_{xx}\frac{du_{xx}}{dt} + 2\rho_x\frac{d\rho_x}{dt} \\ =& 2u_{xx}(\rho\rho_x - 2u_x u_{xx}) - 2\rho_x(u_{xx}\rho + 2u_x\rho_x) \\ =& -4u_x(u_{xx}^2 + \rho_x^2) \\ \le& \frac{4}{\beta(0)}e^{C_1 t}W(t), \end{aligned}$$

On the characteristic where $\beta(0) \neq 0$, the above estimate together with the Gronwall inequality yields

$$W(t) \le W(0)e^{Ce^{C_1 t}}, \tag{3.12}$$

where the constant C is a constant related with initial data. On the other hand, u_{xxx} and ρ_{xx} satisfy

$$\begin{cases} u_{xxxt} + uu_{xxxx} + 3u_x u_{xxx} + 2u_{xx}^2 - \rho_x^2 - \rho\rho_{xx} = 0, \\ \rho_{xxt} + u\rho_{xxx} + 3u_{xx}\rho_x + 3u_x\rho_{xx} + u_{xxx}\rho = 0, \end{cases}$$

or we can express it as

$$\frac{d}{dt}\begin{pmatrix} u_{xxx} \\ \rho_{xx} \end{pmatrix} = \begin{pmatrix} -3u_x & \rho \\ -\rho & -3u_x \end{pmatrix}\begin{pmatrix} u_{xxx} \\ \rho_{xx} \end{pmatrix} + \begin{pmatrix} \rho_x^2 - 2u_{xx}^2 \\ -3u_{xx}\rho_x \end{pmatrix}.$$

Similarly, consider the Lyapunov function $V(t) := u_{xxx}^2 + \rho_{xx}^2$, it holds that

$$\begin{aligned} \frac{d}{dt}V(t) =& 2u_{xxx}\frac{du_{xxx}}{dt} + 2\rho_{xx}\frac{d\rho_{xx}}{dt} \\ =& 2u_{xxx}(\rho_x^2 + \rho\rho_{xx} - 3u_x u_{xxx} - 2u_{xx}^2) - 2\rho_{xx}(3u_{xx}\rho_x + 3u_x\rho_{xx} + \rho u_{xxx}) \\ =& -6u_x(u_{xxx}^2 + \rho_{xx}^2) - 6u_{xx}\rho_x\rho_{xx} + 2u_{xxx}(\rho_x^2 - 2u_{xx}^2) \\ \le& \Big(\frac{6}{\beta(0)}e^{C_1 t} + C\Big)V(t) + CW^2(t). \end{aligned}$$

Thus, by using (3.12) and the Gronwall inequality on the above estimate, we obtain

$$V(t) \le \Big(V(0) + C\int_0^t W^2(s)e^{Ce^{C_1 t}}\,dt\Big)e^{Ce^{C_1 t}},$$

where the constant C is related with initial data. So we have proved the point–wise estimate of higher order derivatives of u and ρ when $x \in I_i$, for every $i = 0, 1, \cdots, n$. This also shows that whenever u_x is bounded, $W(t)$ and $V(t)$ are bounded.

Assume (t_0, ξ_0) is an singular point, then by the definition of L, α, β, we have

$$L(t_0, \xi_0) = \alpha(t_0, \xi_0) = \beta(t_0, \xi_0) = 0.$$

By the equation of α, there holds $\alpha_t \neq 0$ at point (t_0, ξ_0). So we can take a small neighborhood of t_0 such that at time $t = t_0 + \varepsilon$, $\alpha(t_0 + \varepsilon, \xi_0) \neq 0$. This together with (3.10) implies that $L(t_0 + \varepsilon, \xi_0) \neq 0$. Thus, it follows from the definition of L that there is a time T_0, such that u_x is bounded in the interval $[t_0 + \varepsilon, T_0 - \varepsilon]$. Hence u_x is bounded in any proper sub–interval of (t_0, T_0) along the characteristic $\xi = \xi_0$, where we have used the fact that ε is arbitrarily small.

This concludes that

- Singular points are isolated on the characteristics where $\rho = 0$.

Now, notice by (3.8) that

$$(p - L)_t = 0,$$

while $p(0, \xi) \equiv 1$ and $L(0, \xi) < 1$ for any $\xi \geq 0$. So $(p - L)(t, \xi)$ is always a positive constant. Hence, on any given characteristic, there exists a positive constant $\delta > 0$, such that, at any singularity point on the characteristic, $\alpha = 0$, $L = 0$ while $p > \delta > 0$, hence $\alpha_t = \frac{1}{2}(p - L) > \frac{1}{2}\delta$. Using this transversality property, one can derive that

- There are only finitely many singular points on each characteristic in the interval $[0, T]$.

This completes the proof of Theorem 3.2. □

Comparing to the generic regularity for the one–component Hunter–Saxton in Theorem 2.2, we have the following corollary, the proof is similar to [14], we omit it here for brevity.

Corollary 3.1 (Generic regularity)**.** *Let $T > 0$ be given, then there exists an open dense set*

$$\mathcal{D} \subset \{(u_0, \rho_0);\ u_0 \in \mathcal{C}^3(\mathbb{R}^+) \cap H^1(\mathbb{R}^+), \rho_0 \in C^2(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)\},$$

such that, for $(u_0, \rho_0) \in \mathcal{D}$, the solution $(u(t, x), \rho(t, x))$ of (3.1) satisfies that $u(t, x)$ is three times continuously differentiable and $\rho(t, x)$ is twice continuously differentiable in the complement of finitely many isolated points within the domain $[0, T] \times \mathbb{R}^+$.

At this stage, we study families of conservative solutions $(u^\theta, \rho^\theta) = (u, \rho)(t, x, \theta)$ of (3.1) with initial data $(u, \rho)(0, x, \theta) = (u_0, \rho_0)(x, \theta) =: (u_0^\theta, \rho_0^\theta)(x)$, depending smoothly on an additional parameter $\theta \in [0, 1]$, More precisely, these paths of initial data will lie in the space

$$\mathcal{X}_2 := \Big(\mathcal{C}^3\big(\mathbb{R}^+ \times [0, 1]\big) \cap L^\infty\big([0, 1];\ H^1(\mathbb{R}^+)\big)\Big) \times \Big(\mathcal{C}^2\big(\mathbb{R}^+ \times [0, 1]\big) \cap L^\infty\big([0, 1];\ L^2(\mathbb{R}^+)\big)\Big).$$

Now, we have the following generic regularity for 1–parameter family of solutions. Roughly speaking, for a 1–parameter family of initial $\theta \mapsto (\hat{u}_0^\theta, \hat{\rho}_0^\theta)$, with $\theta \in [0, 1]$, it can be uniformly approximated by a second path of initial data $\theta \mapsto (u_0^\theta, \rho_0^\theta)$, such that the corresponding solutions $(u^\theta, \rho^\theta)(t, x)$ of (3.1) are piecewise smooth in the domain $[0, T] \times \mathbb{R}^+$. The argument is similar to Theorem 3.2 and Corollary 3.1, we omit it here for brevity.

Theorem 3.3. *Let $T > 0$ be given, then for any 1–parameter family of initial data $(\hat{u}_0^\theta, \hat{\rho}_0^\theta) \in \mathcal{X}_2$ and any $\varepsilon > 0$, there exists a perturbed family $(x, \theta) \mapsto (u_0^\theta, \rho_0^\theta)(x)$ such that*

$$\|(u_0^\theta - \hat{u}_0^\theta, \rho_0^\theta - \hat{\rho}_0^\theta)\|_{\mathcal{X}_2} < \varepsilon,$$

and moreover, for all except at most finitely many $\theta \in [0, 1]$, the conservative solution $(u^\theta, \rho^\theta)(t, x)$ of (3.1) is smooth in the complement of finitely many points in the domain $[0, T] \times \mathbb{R}^+$.

3.3. **Lipschitz metric for the two–component Hunter–Saxton equations.** Now, we study stability of solutions to (3.1)–(3.3) by constructing a Lipschitz metric.

3.3.1. *The norm of tangent vectors for smooth solutions.* Similar to sub–subsection 2.3.1, we establish a Finsler norm on tangent vector for smooth solutions first. Let $(u, \rho)(t, x)$ be a smooth solution to (3.1) and consider a family of perturbed solutions of the form

$$u^\epsilon(x) = u(x) + \epsilon v(x) + o(\epsilon), \quad \rho^\epsilon(x) = \rho(x) + \epsilon\varrho(x) + o(\epsilon). \tag{3.13}$$

A straightforward calculation yields that the first order perturbations v and ϱ satisfy

$$v_t + uv_x + vu_x = \int_0^x (u_z v_z + \rho\varrho)(z)\, dz, \tag{3.14}$$

and

$$\varrho_t + u\varrho_x + \varrho u_x + v\rho_x + \rho v_x = 0. \tag{3.15}$$

Differentiating (3.14) with respect to x, one obtain

$$v_{xt} + uv_{xx} + u_x v_x + vu_{xx} - \rho\varrho = 0. \tag{3.16}$$

In a similar fashion as in (2.25), we need to add a quantity $w(t, x)$ measuring the horizontal shift which is satisfying

$$w_t + uw_x = v + u_x w, \quad w(0, x) = w_0(x). \tag{3.17}$$

Introducing a Finsler norm

$$\|(v,\varrho)\|_{(u,\rho)} := \inf_{w\in\mathcal{A}} \|(w,v,\varrho)\|_{(u,\rho)},$$

where

$$\mathcal{A} = \{\text{solutions } w(t,x) \text{ of (3.17) with smooth initial data } w_0(x)\}.$$

Here the norm is defined as

$$\begin{aligned}
&\|(w,v,\varrho)\|_{(u,\rho)}\\
=&\int_0^\infty \big\{|w|(e^{-x}+u_x^2+\rho^2)+|v+u_xw|(e^{-x}+u_x^2+\rho^2)+|\varrho+\rho_xw+\rho w_x|e^{-x}\big\}\,dx\\
&+\int_0^\infty |2u_x(v_x+u_{xx}w)+2\rho(\varrho+\rho_xw)+(u_x^2+\rho^2)w_x|\,dx\\
=:&\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.
\end{aligned}\tag{3.18}$$

The meaning for $\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_4$ in (3.18) are similar as I_1,I_2,I_3 in (2.27). While $\mathcal{I}_3$ accounts for the change in base measure with density ρ. In fact, we are looking at

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{\rho^\epsilon(x+\epsilon w(x))\,dx^\epsilon-\rho(x)\,dx}{\epsilon}\\
=&\lim_{\epsilon\to 0}\frac{\Big(\rho^\epsilon(x+\epsilon w(x))-\rho(x)\Big)\,dx^\epsilon-\rho(x)\big(\,dx^\epsilon-\,dx\big)}{\epsilon}\\
=&\Big(\varrho+\rho_xw+\rho w_x\Big)\,dx.
\end{aligned}$$

Now, we will make good use of (3.1), (3.5), (3.14)–(3.16) and (3.17) to prove the following theorem.

Theorem 3.4. *Let* (u,ρ) *be a smooth solution to* (3.1)–(3.3), *and assume that the first order perturbations* (v,ϱ) *satisfy the equations* (3.14)–(3.16). *Then for any* $\tau>0$, *we have*

$$\|(v,\sigma)(\tau)\|_{(u,\rho)(\tau)} \le e^{C\tau}\|(v,\sigma)(0)\|_{(u_0,\rho_0)},\tag{3.19}$$

for some constant $C>0$ *depending only on* T *and* $H^1(\mathbb{R}^+)\times L^2(\mathbb{R}^+)$*–norm of initial data..*

Proof . We prove Theorem 3.4 by four steps.

1. Exactly as in (2.30), we begin with the time derivative of $\mathcal{I}_1$, due to (3.5) and (3.17), it is easy to get

$$\begin{aligned}
&\big[w(e^{-x}+u_x^2+\rho^2)\big]_t+\big[uw(e^{-x}+u_x^2+\rho^2)\big]_x\\
=&(w_t+uw_x)(e^{-x}+u_x^2+\rho^2)+w[(e^{-x}+u_x^2+\rho^2)_t+(u(e^{-x}+u_x^2+\rho^2))_x]\\
=&(v+u_xw)(e^{-x}+u_x^2+\rho^2)+w(u_xe^{-x}-ue^{-x}).
\end{aligned}$$

This yields the estimate

$$\frac{d}{dt}\int_0^\infty |w|(e^{-x}+u_x^2+\rho^2)\,dx \le \int_0^\infty |v+u_xw|(e^{-x}+u_x^2+\rho^2)\,dx + C\int_0^\infty |w|(e^{-x}+u_x^2+\rho^2)\,dx.\tag{3.20}$$

2. Next, we turn to estimate the time derivative of $\mathcal{I}_2$, using $(3.1)_1$, (3.5), (3.14) and (3.17), we obtain

$$\begin{aligned}
&\big[(v+u_xw)(e^{-x}+u_x^2+\rho^2)\big]_t+\big[u(v+u_xw)(e^{-x}+u_x^2+\rho^2)\big]_x\\
=&[v_t+uv_x+u_x(w_t+uw_x)+w(u_{xt}+uu_{xx})](e^{-x}+u_x^2+\rho^2)\\
&+(v+u_xw)[(e^{-x}+u_x^2+\rho^2)_t+(u(e^{-x}+u_x^2+\rho^2))_x]\\
=&\big[-vu_x+\int_0^x(u_zv_z+\rho\varrho)(z)\,dz+u_x(v+u_xw)+\frac12w(\rho^2-u_x^2)\big](e^{-x}+u_x^2+\rho^2)\\
&+(v+u_xw)(u_xe^{-x}-ue^{-x})\\
=&\big[\frac12(u_x^2w+\rho^2w)+\int_0^x(u_zv_z+\rho\varrho)(z)\,dz\big](e^{-x}+u_x^2+\rho^2)+(v+u_xw)(u_xe^{-x}-ue^{-x}).
\end{aligned}\tag{3.21}$$

As in (2.32), we can easily express the first term on the right hand side of (3.21) as

$$\begin{aligned}\frac{1}{2}(u_x^2 w+\rho^2 w)(x) =&\frac{1}{2}\int_0^x \big(u_z^2 w+\rho^2 w\big)_z(z)\,dz\\ =&\frac{1}{2}\int_0^x \big(2u_z u_{zz} w+2\rho\rho_z w+(u_z^2+\rho^2)w_z\big)(z)\,dz.\end{aligned}$$

which, in combination with the second term on the right hand side of (3.21) that

$$\begin{aligned}&\big[\frac{1}{2}(u_x^2 w+\rho^2 w)+\int_0^x (u_z v_z+\rho\varrho)(z)\,dz\big](e^{-x}+u_x^2+\rho^2)\\ =&\frac{1}{2}(e^{-x}+u_x^2+\rho^2)\int_0^x \big[2u_z(v_z+u_{zz}w)+2\rho(\varrho+\rho_z w)+(u_z^2+\rho^2)w_z\big](z)\,dz.\end{aligned} \tag{3.22}$$

Plugging (3.22) into (3.21), it holds that

$$\begin{aligned}\frac{d}{dt}\int_0^\infty |v+u_x w|(e^{-x}+u_x^2+\rho^2)\,dx \le& C\int_0^\infty |v+u_x w|(e^{-x}+u_x^2+\rho^2)\,dx\\ &+C\int_0^\infty |2u_x(v_x+u_{xx}w)+2\rho(\varrho+\rho_x w)+(u_x^2+\rho^2)w_x|\,dx.\end{aligned} \tag{3.23}$$

3. Now, we deal with the time derivative of $\mathcal{I}_3$, with the help of $(3.1)_2$, (3.15) and (3.17), we can get

$$\begin{aligned}&[\varrho+\rho_x w+\rho w_x]_t+\big[u(\varrho+\rho_x w+\rho w_x)\big]_x\\ =&\varrho_t+(u\varrho)_x+\rho_x(w_t+uw_x)+w(\rho_{xt}+(u\rho_x)_x)+\rho(w_{xt}+(uw_x)_x)+w_x(\rho_t+u\rho_x)\\ =&-v\rho_x-v_x\rho+\rho_x(v+u_x w)-w(u_x\rho_x+u_{xx}\rho)+\rho(v_x+u_{xx}w+u_x w_x)-w_x u_x\rho\\ =&0.\end{aligned}$$

Thus, we have

$$\big[(\varrho+\rho_x w+\rho w_x)e^{-x}\big]_t+\big[ue^{-x}(\varrho+\rho_x w+\rho w_x)\big]_x=-e^{-x}u(\varrho+\rho_x w+\rho w_x),$$

which implies

$$\frac{d}{dt}\int_0^\infty |\varrho+\rho_x w+\rho w_x|e^{-x}\,dx \le C\int_0^\infty |\varrho+\rho_x w+\rho w_x|e^{-x}\,dx. \tag{3.24}$$

4. Finally, we repeat the same argument on $\mathcal{I}_4$, recall (3.1), (3.15), (3.16) and (3.17), we compute

$$\begin{aligned}&\big[2u_x(v_x+u_{xx}w)+2\rho(\varrho+\rho_x w)+(u_x^2+\rho^2)w_x\big]_t\\ &+\big[u\big(2u_x(v_x+u_{xx}w)+2\rho(\varrho+\rho_x w)+(u_x^2+\rho^2)w_x\big)\big]_x\\ =&2(u_{xt}+uu_{xx})(v_x+u_{xx}w)+2u_x\big[v_{xt}+(uv_x)_x+u_{xx}(w_t+uw_x)+w(u_{xxt}+(uu_{xx})_x\big]\\ &+2(\rho_t+u\rho_x)(\varrho+\rho_x w)+2\rho\big[\varrho_t+(u\varrho)_x+\rho_x(w_t+uw_x)+w(\rho_{xt}+(u\rho_x)_x\big]\\ &+2u_x w_x(u_{xt}+uu_{xx})+2\rho w_x(\rho_t+u\rho_x)+(u_x^2+\rho^2)(w_{xt}+(uw_x)_x)\\ =&(\rho^2-u_x^2)(v_x+u_{xx}w)+2u_x[\rho\varrho-vu_{xx}+u_{xx}(v+u_x w)+w(\rho\rho_x-u_x u_{xx})]\\ &-2u_x\rho(\varrho+\rho_x w)+2\rho[-v\rho_x-\rho v_x+\rho_x(v+u_x w)-w(u_{xx}\rho+u_x\rho_x)]\\ &+u_x w_x(\rho^2-u_x^2)-2u_x\rho^2 w_x+(u_x^2+\rho^2)(v_x+u_x w_x+u_{xx}w)=0.\end{aligned}$$

This yields

$$\frac{d}{dt}\int_0^\infty |2u_x(v_x+u_{xx}w)+2\rho(\varrho+\rho_x w)+(u_x^2+\rho^2)w_x|\,dx\le 0. \tag{3.25}$$

Thus, putting together the estimates (3.20), (3.23)–(3.25), we conclude the desired inequality (3.19). This completes the proof of Theorem 3.4. □

3.3.2. *Length of a path of solutions in transformed coordinates.* Assume (t_0, ξ_0) is the point of singularity, then by definition (3.7), we have $L(t_0, \xi_0) = \alpha(t_0, \xi_0) = \beta(t_0, \xi_0) = 0$. Considering the consistency condition (3.10), differentiating it with respect to ξ, one obtains

$$2LL_\xi + 2\alpha\alpha_\xi + 2\beta\beta_\xi = L_\xi p + Lp_\xi. \tag{3.26}$$

At the singular point, we have $L_\xi = 0$, since $p > 0$. Taking derivative ∂_ξ to (3.26), then at the singular point, we finally derive $2\alpha_\xi^2 + 2\beta_\xi^2 = L_{\xi\xi}q$. Since the generic singularity is $L_\xi = 0, L_{\xi\xi} \neq 0$, we can conclude generically α_ξ and β_ξ cannot be zero at the same time. Thus, we give the following definitions.

Definition 3.1. *We say that a solution* $(u, \rho)(t, x)$ *of* (3.1) *has generic singularities for* $t \in [0, T]$ *if it admits a representation of the form* (3.9)*, where*

(i) *the functions* $(x, u, L, \alpha, \beta, q)(t, \xi)$ *are* $\mathcal{C}^\infty$*,*

(ii) *for* $t \in [0, T]$*, the following generic conditions hold*

$$(\mathrm{G}_1).\quad L = 0, L_\xi = 0, \alpha_\xi = 0 \Longrightarrow \beta_\xi \neq 0, L_{\xi\xi} \neq 0, \alpha_{\xi\xi} \neq 0,$$
$$(\mathrm{G}_2).\quad L = 0, L_\xi = 0, \beta_\xi = 0 \Longrightarrow \alpha_\xi \neq 0, L_{\xi\xi} \neq 0, \beta_{\xi\xi} \neq 0.$$

Definition 3.2. *We say that a path of initial data* $\gamma_0^2 : \theta \mapsto (u_0^\theta, \rho_0^\theta)$*,* $\theta \in [0, 1]$ *is a piecewise regular path if the following conditions hold*

(i) *There exists a continuous map* $(\xi, \theta) \mapsto (x, u, L, \alpha, \beta, p)$ *such that the semilinear system* (3.8) *holds for* $\theta \in [0, 1]$*, and the function* $(u^\theta, \rho^\theta)(x, t)$ *whose graph is*

$$\textit{Graph } (u^\theta, \rho^\theta) = \{(t, x(t, \xi, \theta), u(t, \xi, \theta), \rho(t, \xi, \theta)); \quad (t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^+\}$$

provides the conservation solution of (3.1) *with initial data* $u^\theta(0, x) = u_0^\theta(x), \rho^\theta(0, x) = \rho_0^\theta(x)$*.*

(ii) *There exist finitely many values* $0 = \theta_0 < \theta_1 < \cdots < \theta_N = 1$ *such that the map* $(\xi, \theta) \mapsto (x, u, L, \alpha, \beta, p)$ *is* $\mathcal{C}^\infty$ *for* $\theta \in (\theta_{i-1}, \theta_i), i = 1, \cdots, N$*, and the solution* $(u^\theta, \rho^\theta)(t, x)$ *has only generic singularities at time* $t = 0$*.*

In addition, if for all $\theta \in [0, 1] \backslash \{\theta_1, \cdots, \theta_N\}$*, the solution* $(u^\theta, \rho^\theta)(t)$ *has only generic singularities for* $t \in [0, T]$*, then we say the path of solution* $\gamma_t^2 : \theta \mapsto (u^\theta, \rho^\theta)$ *is* **piecewise regular** *for* $t \in [0, T]$*.*

Thanks to the argument in Subsection 3.2, we now have the following corollary, which shows that the set of piecewise regular paths is dense.

Corollary 3.2. *Given* $T > 0$*, let* $\theta \mapsto (x^\theta, u^\theta, L^\theta, \alpha^\theta, \beta^\theta, p^\theta), \theta \in [0, 1]$*, be a smooth path of solutions to the system* (3.8)*. Then there exists a sequence of paths of solutions* $\theta \mapsto (x_n^\theta, u_n^\theta, L_n^\theta, \alpha_n^\theta, \beta_n^\theta, p_n^\theta)$*, such that*

(i) *For each* $n \geq 1$*, the path of corresponding solutions of* (3.1) $\theta \mapsto (u_n^\theta, \rho_n^\theta)$ *is regular for* $t \in [0, T]$*, according to Definition 3.2.*

(ii) *For any bounded domain* Ω *in the* t*–*ξ *space, the functions* $(x_n^\theta, u_n^\theta, L_n^\theta, \alpha_n^\theta, \beta_n^\theta, p_n^\theta)$ *converge to* $(x^\theta, u^\theta, L^\theta, \alpha^\theta, \beta^\theta, p^\theta)$ *uniformly in* $\mathcal{C}^k([0, 1] \times \Omega)$*, for every* $k \geq 1$*, as* $n \to \infty$*.*

Similar to sub–subsection 2.3.2, we first derive an expression for the norm of a tangent vector in the t–ξ coordinates. To this end, for a reference solution (u, ρ) of (3.1) and a family of perturbed solutions $(u^\varepsilon, \rho^\varepsilon)$, we assume that, in the t–ξ coordinates, these define a smooth family of solutions of (3.8), say $(x^\varepsilon, u^\varepsilon, L^\varepsilon, \alpha^\varepsilon, \beta^\varepsilon, p^\varepsilon)$. Consider the perturbed solutions of the form

$$(x^\varepsilon, u^\varepsilon, L^\varepsilon, \alpha^\varepsilon, \beta^\varepsilon, p^\varepsilon) = (x, u, L, \alpha, \beta, p) + \varepsilon(X, U, \mathcal{L}, \mathcal{A}, \mathcal{B}, P) + o(\varepsilon).$$

By the smooth coefficients of (3.8), we have that the first order perturbations satisfy a linearized system and are well defined for $(t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^+$. In the following, our main goal is to express the quantities w, v, ϱ appearing in (3.18) in terms of $(X, U, \mathcal{L}, \mathcal{A}, \mathcal{B}, P)$. Indeed,

(1) The shift in x is computed by

$$w = \lim_{\varepsilon \to 0} \frac{x^\varepsilon(t, \xi^\varepsilon) - x(t, \xi)}{\varepsilon} = X + x_\xi \cdot \frac{\partial \xi^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}. \tag{3.27}$$

(2) We will calculate the change in u as

$$v+u_x w=\lim_{\varepsilon\to 0}\frac{u^\varepsilon(t,\xi^\varepsilon)-u(t,\xi)}{\varepsilon}=U+u_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}. \tag{3.28}$$

(3) To achieve the change in base measure with density ρ, first, we have

$$\frac{d}{d\varepsilon}\beta^\varepsilon|_{\varepsilon=0}=\lim_{\varepsilon\to 0}\frac{\beta^\varepsilon(t,\xi^\varepsilon)-\beta(t,\xi)}{\varepsilon}=\mathcal{B}+\beta_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}.$$

Then the integrand in $\mathcal{I}_3$ is calculated as

$$\frac{d}{d\varepsilon}(\beta^\varepsilon+\beta L\cdot\xi^\varepsilon_x)|_{\varepsilon=0}=\mathcal{B}+\beta_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}+\beta L\cdot\frac{\partial\xi^\varepsilon_x}{\partial\varepsilon}|_{\varepsilon=0}. \tag{3.29}$$

(4) To complete the analysis, we have to concern the term due to the change in base measure with density $u_x^2+\rho^2$. Indeed, it follows

$$\frac{d}{d\varepsilon}(p^\varepsilon-L^\varepsilon+pL\cdot\xi^\varepsilon_x-L^2\cdot\xi^\varepsilon_x)|_{\varepsilon=0}=P-\mathcal{L}+(p_\xi-L_\xi)\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}+(p-L)L\cdot\frac{\partial\xi^\varepsilon_x}{\partial\varepsilon}|_{\varepsilon=0}, \tag{3.30}$$

where

$$\frac{d}{d\varepsilon}p^\varepsilon|_{\varepsilon=0}=\lim_{\varepsilon\to 0}\frac{p^\varepsilon(t,\xi^\varepsilon)-p(t,\xi)}{\varepsilon}=P+p_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0},$$

and

$$\frac{d}{d\varepsilon}L^\varepsilon|_{\varepsilon=0}=\lim_{\varepsilon\to 0}\frac{L^\varepsilon(t,\xi^\varepsilon)-L(t,\xi)}{\varepsilon}=\mathcal{L}+L_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}.$$

Notice that

$$(1+u_x^2+\rho^2)\,dx=p\,d\xi.$$

Consequently, relations (3.27)–(3.30) imply the weighted norm of a tangent vector (3.18) can be written as

$$\|(w,v,\varrho)\|_{(u,\rho)}=\sum_{\ell=1}^{4}\int_0^\infty|\mathcal{J}_\ell(t,\xi)|\,d\xi, \tag{3.31}$$

where

$$\begin{aligned}
\mathcal{J}_1&=(X+x_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0})(e^{-y(t,\xi)}L+p-L),\\
\mathcal{J}_2&=(U+u_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0})(e^{-y(t,\xi)}L+p-L),\\
\mathcal{J}_3&=\big(\mathcal{B}+\beta_\xi\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}+\beta L\cdot\frac{\partial\xi^\varepsilon_x}{\partial\varepsilon}|_{\varepsilon=0}\big)e^{-y(t,\xi)},\\
\mathcal{J}_4&=P-\mathcal{L}+(p_\xi-L_\xi)\cdot\frac{\partial\xi^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}+(p-L)L\cdot\frac{\partial\xi^\varepsilon_x}{\partial\varepsilon}|_{\varepsilon=0}.
\end{aligned}$$

Since ξ^ε equals to a constant in time along the characteristic, it is clear that the integrand $\mathcal{J}_\ell$ is continuous, for $\ell=1,2,3,4$.

Now, we are ready to define the length of the piecewise regular path.

Definition 3.3. *The length $\|\gamma_t^2\|$ of the piecewise regular path $\gamma_t^2:\theta\mapsto(u^\theta,\rho^\theta)$ is defined as*

$$\|\gamma_t^2\|=\inf_{\gamma_t^2}\int_0^1\sum_{\ell=1}^{4}\int_0^\infty|\mathcal{J}_\ell^\theta(t,\xi)|\,d\xi\,d\theta,$$

where the infimum is taken over all piecewise regular path.

At this stage, we have the following theorem, the proof is similar to Theorem 2.5 in Subsection 2.3.2, we omit it here for brevity.

Theorem 3.5. *Given any $T > 0$, consider a path of solutions $\theta \mapsto (u^\theta, \rho^\theta)$ of (3.1), which is piecewise regular for $t \in [0, T]$. Moreover, the total energy is less than a constant $E_2 > 0$. Then there exists some constant $C > 0$, such that*

$$\|\gamma_t^2\| \le C\|\gamma_0^2\|,$$

where C depends only on T and $H^1(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$–norm of initial data.

3.3.3. *Construction of the geodesic distance.* In this sub–subsection, we are in a position to show that the flow generated by the two–component Hunter–Saxton (3.1)–(3.3) is Lipschitz continuous with respect to the geodesic distance defined in Definition 3.5. In light of Corollary 3.1, for an open dense set of initial data $\mathcal{D} \subset \{(u_0, \rho_0); u_0 \in \mathcal{C}^3(\mathbb{R}^+) \cap H^1(\mathbb{R}^+), \rho_0 \in \mathcal{C}^2(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)\}$, the corresponding solution $(u, \rho)(t, x)$ of (3.1) is piecewise smooth, with singularities occurring in finitely many isolated points. Now, on $\mathcal{D}^\infty := \big(\mathcal{C}_0^\infty \times \mathcal{C}_0^\infty\big) \cap \mathcal{D}$, we construct a geodesic distance, defined as the infimum among the weighted lengths of all piecewise regular paths connecting two given points. Thus, by continuity, this distance can be extended from $\mathcal{D}^\infty$ to a larger space, defined as the completion of $\mathcal{D}^\infty$ with respect to the distance $d_2(\cdot, \cdot)$.

Let two data $(u, \rho), (\hat{u}, \hat{\rho}) \in \mathcal{D}^\infty$ be given, we introduce the quantities

$$\mathcal{E}(u, \rho) := \int_0^\infty (u_x^2 + \rho^2)(t, x)\, dx, \quad \mathcal{E}(\hat{u}, \hat{\rho}) := \int_0^\infty (\hat{u}_x^2 + \hat{\rho}^2)(t, x)\, dx.$$

Then fix any constant $E_2 > 0$, denote the set

$$\Sigma_{E_2} := \{u \in H^1(\mathbb{R}^+), \rho \in L^2(\mathbb{R}^+);\ \mathcal{E}(u, \rho) \le E_2\}.$$

Definition 3.4. *For solutions with initial data in $\mathcal{D}^\infty \cap \Sigma_{E_2}$, we define the geodesic distance $d_2\Big((u, \rho), (\hat{u}, \hat{\rho})\Big)$ as the infimum among the weighted lengths of all piecewise regular paths $\theta \mapsto (u^\theta, \rho^\theta)$, which connect (u, ρ) with $(\hat{u}, \hat{\rho})$, that is, for any time t,*

$$d_2\Big((u, \rho), (\hat{u}, \hat{\rho})\Big) := \inf\{\|\gamma_t^2\|;\ \gamma_t^2 \text{ is a piecewise regular path}, \gamma_t^2(0) = (u, \rho), \\ \gamma_t^2(1) = (\hat{u}, \hat{\rho}), \mathcal{E}(u^\theta, \rho^\theta) \le E_2 \text{ for all } \theta \in [0, 1]\}.$$

Now, we can define the metric for the general weak solutions.

Definition 3.5. *Let (u_0, ρ_0) and $(\hat{u}_0, \hat{\rho}_0)$ in $H^1(\mathbb{R}^+) \times L^1(\mathbb{R}^+)$ be two absolute continuous initial data as required in the existence Theorem 3.1. Denote (u, ρ) and $(\hat{u}, \hat{\rho})$ to be the corresponding global weak solutions, then we define, for any time t,*

$$d_2\Big((u, \rho), (\hat{u}, \hat{\rho})\Big) := \lim_{n \to \infty} d_2\Big((u^n, \rho^n), (\hat{u}^n, \hat{\rho}^n)\Big),$$

for any two sequences of solutions (u^n, ρ^n) and $(\hat{u}^n, \hat{\rho}^n)$ in $\mathcal{D}^\infty \cap \Sigma_{E_2}$ with

$$\|u^n - u\|_{H^1} \to 0, \|\rho^n - \rho\|_{L^2} \to 0 \quad \text{and} \quad \|\hat{u}^n - \hat{u}\|_{H^1} \to 0, |\hat{\rho}^n - \hat{\rho}\|_{L^2} \to 0.$$

The limit in the definition is independent on the selection of sequences, because the solution flows are Lipschitz in $\mathcal{D}^\infty \cap \Sigma_{E_2}$, so the definition is well-defined. Since the concatenation of two piecewise regular paths is still a piecewise regular path (after a suitable re–parameterization), it is clear that $d_2(\cdot, \cdot)$ is a distance. By the fact that $\mathcal{D}^\infty \cap \Sigma$ is a dense set in the solution space, one could easily extend the Lipschitz metric to the general initial data. As a consequence of Theorem 3.5, we report directly the following result.

Theorem 3.6. *The geodesic distance $d_2(\cdot, \cdot)$ renders Lipschitz continuous the flow generated by the equation (3.1). In particular, let (u_0, ρ) and $(\hat{u}_0, \hat{\rho}_0)$ be two $H^1(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$ initial data, then for every $t \in [0, T]$, the corresponding solutions $(u, \rho)(t, x)$ and $(\hat{u}, \hat{\rho})(t, x)$ satisfy*

$$d_2\Big((u(t), \rho(t)), (\hat{u}(t), \hat{\rho}(t))\Big) \le C d_2\Big((u_0, \rho_0), (\hat{u}_0, \hat{\rho}_0)\Big),$$

where the constant $C > 0$ depends only on T and $H^1(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$–norm of initial data.

Now, we compare the distance $d_2(\cdot,\cdot)$ with the familiar distance in Sobolev space and the Wasserstein distance between energy measure.

Proposition 3.1. *(1) For any $(u,\rho),(\hat u,\hat\rho)\in\mathcal{D}^\infty\cap\Sigma_{E_2}$, there exists some positive constant C depends only on E_2, such that,*

$$d_2\Big((u,\rho),(\hat u,\hat\rho)\Big)\le C\Big(\|u-\hat u\|_{L^\infty}+\|u_x-\hat u_x\|_{L^2}+\|\rho-\hat\rho\|_{L^2}+\|\rho-\hat\rho\|_{L^1}\Big). \tag{3.32}$$

(2) For any $u,\hat u\in L^1_{loc}(\mathbb{R}^+)\cap H^1(\mathbb{R}^+)$ and $\rho,\hat\rho\in L^2(\mathbb{R}^+)$, there exists some constant $C>0$ depends only on E_2, such that,

$$\|u-\hat u\|_{L^1_{loc}}\le C\cdot d_2\Big((u,\rho),(\hat u,\hat\rho)\Big), \tag{3.33}$$

$$|meas\ \lambda - meas\ \hat\lambda|\le d_2\Big((u,\rho),(\hat u,\hat\rho)\Big), \tag{3.34}$$

$$\sup_{\|f\|_{C^1}\le 1}|\int f\,d\mu-\int f\,d\hat\mu|\le d_2\Big((u,\rho),(\hat u,\hat\rho)\Big), \tag{3.35}$$

where $\lambda,\hat\lambda$ are the measures with densities ρe^{-x} and $\hat\rho e^{-x}$ with respect to Lebesgue measure, and $\mu,\hat\mu$ are the measures with densities $(u_x)^2+\rho^2$ and $(\hat u_x)^2+\hat\rho^2$ with respect to Lebesgue measure.

Proof. The estimates (3.32), (3.33) and (3.35) can be bounded similarly as in Proposition 2.1 and 2.2, respectively. It remains to show (3.34). Let $\gamma_t^2:\theta\mapsto(u^\theta,\rho^\theta)$ be a regular path connecting (u,ρ) with $(\hat u,\hat\rho)$. Call λ^θ be the measure with density $\rho^\theta e^{-x^\theta}$ with respect to Lebesgue measure, it is clear to see that

$$|\int_0^1\frac{d}{d\theta}\int d\lambda^\theta\,d\theta|\le\int_0^1\int_0^\infty|\varrho^\theta+\rho^\theta_x w^\theta+\rho^\theta w^\theta_x|e^{-x}\,dx\,d\theta+\int_0^1\int_0^\infty|w^\theta|(e^{-x^\theta}+(\rho^\theta)^2)\,dx\,d\theta, \tag{3.36}$$

where the integrands on the right hand side of (3.36) are less than the integrands of $\mathcal{I}_1$ and $\mathcal{I}_3$ in (3.18). (3.34) holds immediately. This completes the proof of Proposition 3.1. □

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