3D COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH TEMPERATURE DEPENDENT HEAT CONDUCTIVITY

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Abstract. We prove the existence of unique local-in-time strong solution with vacuum and arbitrarily large data to the three dimensional, compressible Navier-Stokes equations for the heat conducting polytropic flow, when heat conductivity depends on temperature in a power law of Chapman-Enskog. Moreover, a minimum principle of the temperature for our system is shown in bounded smooth domain.

1. Introduction

Let $\mathbb{V} \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. We consider strong solutions and the minimum principle of the temperature to Navier-Stokes-Fourier (NSF) system:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P = \mathrm{div}\mathbb{T}, \\ \big(\rho E\big)_t + \mathrm{div}\big((\rho E + P)u\big) = \mathrm{div}(u\mathbb{T}) + \mathrm{div}(\kappa \nabla \theta), \end{cases} \tag{1.1}$$

with the initial data

$$(\rho, u, \theta)|_{t=0} = (\rho_0(x), u_0(x), \theta_0(x)),\ x \in \mathbb{V}, \tag{1.2}$$

and the following boundary conditions: Dirichlet and thermo-insulated boundary conditions for (u, θ): $\mathbb{V} \subset \mathbb{R}^3$ is a bounded smooth domain and

$$u|_{\partial\mathbb{V}} = 0, \quad \nabla\theta \cdot n|_{\partial\mathbb{V}} = 0, \tag{1.3}$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal to $\partial\mathbb{V}$.

In system (1.1), $t \geq 0$ is the time variable; $x \in \mathbb{V}$ is the spatial coordinate; ρ is the density; u is the velocity of fluids; $E = \frac{1}{2}|u|^2 + e$ is the specific total energy and e is the specific internal energy; P is the pressure for polytropic flow satisfying:

$$P = R\rho\theta = (\gamma - 1)\rho e, \quad e = c_v\theta = \frac{R}{\gamma - 1}\theta, \tag{1.4}$$

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where R and c_v are both positive constants, $\gamma > 1$ is the adiabatic exponent and θ is the absolute temperature. $\mathbb{T}$ is the viscosity stress tensor given by

$$\mathbb{T} = 2\mu D(u) + \lambda \mathrm{div} u \mathbb{I}_3, \quad D(u) = \frac{\nabla u + (\nabla u)^\top}{2}, \tag{1.5}$$

where $D(u)$ is the deformation tensor, $\mathbb{I}_3$ is the 3×3 identity matrix, μ is the shear viscosity coefficient and $\lambda + \frac{2}{3}\mu$ is the bulk viscosity coefficient. κ is the heat conductivity coefficient.

For constant viscosities and heat conductivity, many related results have been studied. Now we are interested in the case when they depend on temperature in a physical way. In truth, considering the derivation of the Navier-Stokes equations from the Boltzmann equations through the Chapman-Enskog expansion to the second order, cf. Chapman-Cowling [7] and Li-Qin [20], the viscosities and heat conductivity are shown to be functions of θ in a power law:

$$\mu(\theta) = \alpha\theta^b, \quad \lambda(\theta) = \beta\theta^b, \quad \kappa(\theta) = \nu\theta^b \quad \text{for} \quad b \in (0, +\infty), \tag{1.6}$$

where (α, β, ν) are all constants satisfying

$$\alpha > 0, \quad 2\alpha + 3\beta \geq 0, \quad \nu > 0. \tag{1.7}$$

In the classical setting when (μ, λ, κ) are all constants, for the existence of 3D solutions with arbitrary data, the main breakthrough is due to Lions [21], where he established the global existence of weak solutions for isentropic flow in $\mathbb{R}^3$, periodic domains or bounded domains with homogenous Dirichlet boundary conditions provided $\gamma > 9/5$. The restriction on γ is improved to $\gamma > 3/2$ by Feireisl [11][12], and the global existence of variational solutions for non-isentropic flow has also been established by Feireisl [13]. The existence of unique local strong solutions with vacuum has been studied by many papers, and we refer the readers to Cho-Choe-Kim [8][9][10]. Recently, the global existence of classical solutions with small energy and vacuum to Cauchy problem has been obtained by Huang-Li-Xin [17] for isentropic flow, and by Huang-Li [18] for non-isentropic flow.

However, when (μ, λ, κ) are dependent of θ as in (1.6), there are very few results no matter on weak solutions or strong solutions because of the possible degeneracy caused by the initial vacuum and the strong nonlinearity in viscosities and heat diffusion. As a pioneer paper in this direction, Jenssen-Karper [19] proved the global existence of a weak solution in 1D space to (1.1) under the assumption

$$\mu = \alpha, \quad \kappa(\theta) = \nu\theta^b \quad \text{for} \quad b \in \Big[0, \frac{3}{2}\Big). \tag{1.8}$$

Under this simplified relation the viscosity coefficient μ is a constant, but the porous medium type energy equation $(1.1)_3$ still introduces significant difficulties. To overcome these difficulties then obtain a global existence of weak solution, some new arguments have been introduced in [19]. When $b \in (0, +\infty)$, Pan-Zhang [23] showed that (1.1) admits a unique global strong solution away from vacuum in 1D space. On any bounded domain, as a weaker conclusion of the minimum principle, Mellet and Vassuer [22] show that the temperature is uniformly positive for $t \geq t_0$ (for arbitrary $t_0 > 0$) for any solutions with finite initial entropy in the case that (κ, μ, λ) are all constants. Recently for the more general case when (1.6) is satisfied, also on any bounded domain of $\mathbb{R}^3$, Zhang in his Ph.D Thesis [28] showed the global existence of variational solutions with vacuum via

establishing a uniform positive lower bound (on time) for θ if the pressure is given as $P = R\rho\theta + \rho^\gamma$ when $\gamma > 3$.

Unfortunately, due to the high degeneracy for this system in vacuum domain, the existences of strong or classical solutions with vacuum and arbitrarily large data in 3D space are totally open until this paper. For example, we notice that the leading coefficient of u_t and θ_t vanish when $\rho = 0$, and this leads to infinitely many ways to define u and θ (if they exist) at vacuum points. Theoretically, this degeneracy leads to an essential difficulty in determining u and θ at vacuum point, since it is difficult to find a reasonable way to extend the definition of u and θ into vacuum region. Hence it is difficult to get uniform estimates for the velocity or temperature near vacuum.

The mission in this paper is to establish the existence of unique local-in-time strong solution with vacuum to the initial-boundary value problem (IBVP) (1.1)-(1.3) under the similar assumptions on viscosities and heat conductivity as in [19][23]:

$$\mu = \alpha, \quad \lambda = \beta, \quad \kappa(\theta) = \nu\theta^b \quad \text{for} \quad b \in (0, +\infty), \tag{1.9}$$

where (α, β, ν) are all constants satisfying (1.7). Moreover, a minimum principle of the temperature for our system is shown in bounded smooth domain when $P = R\rho\theta$, which is also available for the case $P = R\rho\theta + \rho^\gamma$ when $\gamma > 1$.

In order to show our main results, we first give the following definition of strong solutions to IBVP (1.1)-(1.3).

Definition 1.1. *Let $q \in (3, 6]$. Functions (ρ, u, θ) are called a strong solution on $[0, T] \times \mathbb{V}$ to IBVP(1.1)- (1.3), if*

(1) *(ρ, u, θ) satisfy the system (1.1) a.e. in $(0, T) \times \mathbb{V}$;*
(2) *(ρ, u, θ) belong to the following class Φ with some regularities:*

$$\begin{aligned}\Phi =&\{(\rho, u, \theta)|0 \le \rho,\ \rho \in C([0, T]; H^1 \cap W^{1,q}),\ \rho_t \in C([0, T]; L^2 \cap L^q),\\ &(\theta, u) \in C([0, T]; D^1 \cap D^2) \cap L^2([0, T]; D^{2,q}),\\ &(\theta_t, u_t) \in L^2([0, T]; D^1),\ (\sqrt{\rho}\theta_t, \sqrt{\rho}u_t) \in L^\infty([0, T]; L^2)\};\end{aligned} \tag{1.10}$$

(3) *(ρ, u, θ) satisfy the corresponding initial conditions a.e. on $\{t = 0\} \times \mathbb{V}$, and also satisfy the corresponding boundary conditions in the sense of traces.*

Here and throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev spaces:

$$\|f\|_s = \|f\|_{H^s(\mathbb{V})}, \quad |f|_p = \|f\|_{L^p(\mathbb{V})}, \quad \|f\|_{W^{m,r}} = \|f\|_{W^{m,r}(\mathbb{V})},$$

$$\|f\|_{L^pL^q} = \|f\|_{L^p([0,T];L^q(\mathbb{V}))}, \quad D^{k,r} = \{f \in L^1_{loc}(\mathbb{V}) : |\nabla^k f|_r < +\infty\},$$

$$D^k = D^{k,2}, \quad |f|_{D^k} = \|f\|_{D^k(\mathbb{V})}, \quad |f|_{D^{k,r}} = \|f\|_{D^{k,r}(\mathbb{V})},$$

$$D^1_0 = \{f \in L^6(\mathbb{V}) : |\nabla f|_2 < \infty, f|_{\partial\mathbb{V}} = 0\}, \quad |f|_{D^1_0} = \|f\|_{D^1_0(\mathbb{V})}, \quad \int_{\mathbb{V}} f \mathrm{d}x = \int f.$$

A detailed study of homogeneous Sobolev spaces could be found in [15].

Now we will state our main results in the following two subsections.

1.1. **Existence of the unique local strong solution with vacuum.** As has been observed in [10], the lack of a positive lower bound of ρ_0 should be compensated with some compatibility conditions on the initial data. Similarly, if we denote

$$\begin{aligned} &V=\{x\in\mathbb{V}|\ \rho_0(x)=0\},\quad P^0=R\rho_0\theta_0,\\ &Q(u)=\frac{\mu}{2}|\nabla u+(\nabla u)^\top|^2+\lambda|\mathrm{div}u|^2, \end{aligned}\tag{1.11}$$

then we have the following existence theorem:

Theorem 1.1. *Let $\underline{\theta}>0$ be a real constant and initial data (ρ_0,u_0,θ_0) satisfy the following regularities:*

$$\begin{aligned} &\rho_0\geq 0,\quad \rho_0\in H^1\cap W^{1,q},\quad q\in(3,6],\\ &u_0\in D^1\cap D^2,\quad \theta_0\in D^1\cap D^2,\quad \theta_0\geq\underline{\theta}, \end{aligned}\tag{1.12}$$

and the initial layer compatibility conditions

$$\begin{aligned} -div\mathbb{T}(u_0)+\nabla P^0=&\sqrt{\rho_0}g_1,\\ -\frac{1}{c_v}\big(\nu(b+1)^{-1}\triangle\theta_0^{b+1}+Q(u_0)\big)=&\sqrt{\rho_0}g_2 \end{aligned}\tag{1.13}$$

for some $(g_1,g_2)\in L^2$. Then there exists a sufficiently small constant $\epsilon_0>0$ only depending on (ρ_0,u_0,θ_0), R, c_v, γ, b, ν and q such that if

$$|V|\leq\epsilon_0,\tag{1.14}$$

there exists a positive time T_* and a unique strong solution (ρ,u,θ) on $[0,T_*]\times\mathbb{V}$ to IBVP(1.1)-(1.3).

Remark 1.1. *In principle, we will follow the basic framework laid out in* [10] *for constant heat conductivity with extra attention to two new significant difficulties from the porous medium type diffusion in the energy equation $(1.1)_3$: the possible degeneracy and strong nonlinearity in heat diffusion. Take the higher order terms of θ as an example, from the classical regularity theory for elliptic equation and $(1.1)_3$, we only have*

$$|\theta|_{D^2}\leq C\Big|\frac{1}{\theta^b}|\nabla\theta|^2\Big|_2+C||\nabla u|^2|_2+....$$

Then it is obvious that we need to study the positivity of θ and deal with the quadratic terms $|\nabla\theta|^2$ and $|\nabla u|^2$. This is one of essential reasons that why we need to assume the size of the initial vacuum domain is sufficiently small.

Our second result can be seen as an explanation for the compatibility between (1.12) and (1.13):

Theorem 1.2. *Let conditions supposed in Theorem 1.1 hold. We assume that one of the following two conditions hold:*

(1) *$|V|=0$, i.e., the initial vacuum set V has zero 3-D Lebesgue measure;*
(2) *$|V|>0$ and the elliptic system*

$$\begin{cases} -div\mathbb{T}(u)=0\quad in\ V,\\ -\frac{1}{c_v}\big(\nu(b+1)^{-1}\triangle\psi+Q(u)\big)=0\quad in\ V \end{cases}\tag{1.15}$$

only has a zero solution (u, ψ) *in* $D_0^1(V) \cap D^2(V)$.

Then there exists a unique (local) strong solution (ρ, u, θ) *such that*

$$\|\rho(t) - \rho_0\|_{H^1 \cap W^{1,q}(\mathbb{V})} + \|(u(t) - u_0, \theta(t) - \theta_0)\|_{D^1 \cap D^2(\mathbb{V})} \to 0, \quad \text{as } t \to 0, \tag{1.16}$$

if and only if initial data (ρ_0, u_0, θ_0) *satisfy the compatibility conditions (1.13).*

Remark 1.2. (ρ, u, θ) *satisfies (1.1)-(1.2) in the sense of distribution, so we only have*

$$\rho u(t = 0, x) = \rho_0 u_0, \ \rho\theta(t = 0, x) = \rho_0 \theta_0.$$

In the vacuum domain V*, the relations*

$$u(t = 0, x) = u_0, \quad \text{and} \quad \theta(t = 0, x) = \theta_0$$

maybe not hold. Theorem 1.2 tells us that if the initial vacuum domain V *has a sufficiently simple geometry, for instance, the Lipschitz continuous domain, we have*

$$u(t = 0, x) = u_0, \quad \text{and} \quad \theta(t = 0, x) = \theta_0 \quad \text{a.e. in } \mathbb{V}.$$

1.2. **Minimum principle on the absolute temperature.** It is well-known that the linear heat equation with thermo-insulated boundary condition

$$\begin{cases} \theta_t = \Delta\theta \quad \text{in } V, \\ \theta(t = 0, x) = \theta_0(x) \quad \text{in } V, \\ \nabla\theta \cdot n = 0 \quad \text{in } \partial V \end{cases} \tag{1.17}$$

satisfies the minimum principle. Particularly if the initial data

$$\theta_0(x) \geq \underline{\theta} > 0, \tag{1.18}$$

then for the solution θ we have,

$$\theta(t, x) \geq \underline{\theta} > 0. \tag{1.19}$$

This fundamental property could be generalized to some more complicated models. For system (1.1), there are several obstacles for obtaining the same result. First, the temperature is in a non-linear system instead of a scalar linear equation. Second, the pressure may do work so that the specific internal energy may decrease. At last, we notice that the leading coefficient of θ_t in $(1.1)_3$ vanishes when $\rho = 0$, and this degeneracy results in that the temperature equation loses its classical parabolic structure as that of the equation in (1.17), which has degenerated into a non-linear elliptic equations with some quadratic terms in vacuum domain.

However, based on some interpolation estimates and a new continuity argument, we show that under the thermo-insulated boundary conditions in bounded domain, the minimum of the temperature will not increase. This minimum principle verifies the necessity of the third law of thermodynamics in our nonlinear system.

Theorem 1.3. *Let* (ρ_0, u_0, θ_0) *satisfy (1.12)-(1.13). If* (ρ, u, θ) *on* $[0, T] \times \mathbb{V}$ *is the strong solution to IBVP (1.1)-(1.3), then*

$$\theta(t, x) \geq \underline{\theta}, \quad \text{for all } (t, x) \in [0, T] \times \mathbb{V}. \tag{1.20}$$

We now outline the organization of the rest of this paper. In Section 2, we list some important lemmas that will be used frequently in our proof. In Section 3, the existence of the unique local strong solutions with vacuum will be proved. We first reformulate our problem into a new form in terms of some new variable in Subsection 3.1. Next we give the proof of the local existence of strong solutions to this reformulated problem. This is achieved in four steps: 1) we construct approximate solutions for the linearized problem when initial density has positive lower bound; 2) we establish the a prior estimates independent of the lower bound of density for the linearized problem; 3) we then pass to the limit to recover the solution of this linearized problem allowing initial vacuum; 4) we prove the unique solvability of the reformulated problem through a standard iteration process. Section 4 is devoted to proving the necessity and sufficiency of the compatibility conditions shown in Theorem 1.2. The proof of the minimum principle in Theorem 1.3 is given in Section 5. We also give an appendix in order to show the proof for some important lemma shown in Section 2.

Finally, we remark that our framework is applicable to the case $P = R\rho\theta + C\rho^\gamma$ for $\gamma \geq 1$ with some minor modifications. We will not pursue this in this paper.

2. Preliminary

In this section, we give some important lemmas which will be used frequently in our proof. The first one comes from the well-known Gagliardo-Nirenberg inequality:

Lemma 2.1. [15] *For $q \in (3, 6]$, there exist a constant $C > 0$ (depend on q) such that for*

$$f \in D_0^1(\mathbb{V}), \quad g \in D_0^1 \cap D^2(\mathbb{V}), \quad h \in W^{1,q}(\mathbb{V}),$$

then we have the following inequalities:

$$|f|_6 \leq C|f|_{D_0^1}, \quad |g|_\infty \leq C|g|_6^{\frac{1}{2}}|\nabla g|_6^{\frac{1}{2}} \leq |g|_{D_0^1\cap D^2}, \quad |h|_\infty \leq C\|h\|_{W^{1,q}}. \tag{2.1}$$

Second we will introduce some Poincaré type inequality (see Chapter 8 in [21]):

Lemma 2.2. [21] *There exists a constant C depending only on $\mathbb{V}$, $|\rho|_r$ $(r \geq 1)$ ($\rho \geq 0$ is a real function satisfying $|\rho|_1 > 0$), such that for every $F \geq 0$ satisfying*

$$\rho F \in L^1(\mathbb{V}), \quad \sqrt{\rho}F \in L^2, \quad \nabla F \in L^2(\mathbb{V}),$$

we have

$$|F|_6 \leq C(|\rho F|_1 + (1 + |\rho|_2)|\nabla F|_2) \leq C(|\sqrt{\rho}F|_2 + (1 + |\rho|_2)|\nabla F|_2).$$

Proof. We first denote that

$$\overline{F} = \frac{1}{|\mathbb{V}|}\int F(y)\mathrm{d}y,$$

then via the classical Poincaré inequality, we quickly deduce that

$$\begin{aligned}\overline{F}\int \rho = &\int \rho(\overline{F} - F) + \int \rho F\\ \leq &C(|\rho F|_1 + |\rho|_2|\nabla F|_2).\end{aligned}$$

So, via the classical Sobolev imbedding theorem, we have

$$\begin{aligned}|F|_6 \leq &C\|F\|_1 \leq |\nabla F|_2 + |F - \overline{F}|_2 + \overline{F}|\mathbb{V}|^{\frac{1}{2}}\\ \leq &C(|\rho F|_1 + (1 + |\rho|_2)|\nabla F|_2) \leq C(|\sqrt{\rho}F|_2 + (1 + |\rho|_2)|\nabla F|_2).\end{aligned} \tag{2.2}$$

□

Next we consider the following homogenous Dirichlet boundary value problem: let $U = (U^1, U^2, U^3)$, $F = (F^1, F^2, F^3)$ and

$$\begin{cases} -\mu\triangle U - (\mu+\lambda)\nabla \mathrm{div} U = F \text{ in } \mathbb{V}, \\ U|_{\partial\mathbb{V}} = 0. \end{cases} \tag{2.3}$$

If $F \in W^{-1,2}(\mathbb{V})$, then there exists a unique weak solution $U \in H_0^1(\mathbb{V})$. We begin with recalling various estimates for this system in $L^l(\mathbb{V})$ spaces, which can be seen in [2].

Lemma 2.3. [2] *Let* (1.6) *hold,* $\mathbb{V}$ *be a bounded, smooth domain and* $l \in (1, +\infty)$*. There exists a constant* C *depending only on* λ*,* μ*,* l *and* $\mathbb{V}$ *such that: if* $F \in L^l(\mathbb{V})$*, then we have*

$$\|U\|_{W^{2,l}} \leq C|F|_l. \tag{2.4}$$

Moreover, if

$$\triangle U = F, \quad \nabla U(x)\cdot n|_{\partial\mathbb{V}} = 0,$$

then for any weak solution $U \in H^1$*, (2.4) also holds.*

Now we specify some recursion relation under which the sequence $\{U_k\}_{k=1}^{\infty}$ satisfying

$$U_k \to 0, \quad \text{as} \quad k \to +\infty.$$

Lemma 2.4. [28] *We assume that* U_0 *is a small enough positive number,* $\gamma_1, \gamma_3, \gamma_5 \geq 0$*,* $\gamma_2, \gamma_4 > 1$*,* $M > 0$ *and* $z > 1$ *are all real constants. If the following conditions hold:*

$$U_k \leq P_1\Big(\frac{z^k}{M}\Big)^{\gamma_1} U_{k-1}^{\gamma_2} M^{\gamma_5} + P_2\Big(\frac{1}{M}\Big)^{\gamma_3} U_{k-1}^{\gamma_4} M^{\gamma_5}, \tag{2.5}$$

where $P_1 > 0$ *and* $P_2 > 0$ *are both constants independent of* M*, then we have the limit*

$$\lim_{k\to\infty} U_k = 0.$$

The proof for this lemma could be seen in [28] or the appendix of this paper. Finally, using Aubin-Lions Lemma, one has (c.f. [24]),

Lemma 2.5. [24] *Let* X_0*,* X *and* X_1 *be three Banach spaces satisfying* $X_0 \subset X \subset X_1$*. Suppose that* X_0 *is compactly embedded in* X *and that* X *is continuously embedded in* X_1*.*

I) Let G *be bounded in* $L^p(0,T;X_0)$ *with* $1 \leq p < +\infty$*, and* $\frac{\partial G}{\partial t}$ *be bounded in* $L^1(0,T;X_1)$*. Then* G *is relatively compact in* $L^p(0,T;X)$*.*

II) Let F *be bounded in* $L^\infty(0,T;X_0)$ *and* $\frac{\partial F}{\partial t}$ *be bounded in* $L^p(0,T;X_1)$ *with* $p > 1$*. Then* F *is relatively compact in* $C(0,T;X)$*.*

3. Well-posedness of strong solutions with vacuum

In this section, we aim at proving Theorem 1.1. To this end, we first reformulated our system (1.1) into a new form.

3.1. **Reformulation.** Due to $E=\frac{1}{2}|u|^2+e$ and relation (1.4), the energy equation $(1.1)_3$ can be rewritten as

$$\rho\theta_t+\rho u\cdot\nabla\theta+\frac{1}{c_v}\big(R\rho\theta\mathrm{div}u-\nu\mathrm{div}(\theta^b\nabla\theta)\big)=\frac{1}{c_v}Q(u). \tag{3.1}$$

Then via introducing the new variable $\psi=\theta^{b+1}$, system (1.1) can be rewritten as

$$\begin{cases}\rho_t+\mathrm{div}(\rho u)=0,\\ \rho u_t+\rho u\cdot\nabla u+\nabla\Big(R\rho\psi^{\frac{1}{b+1}}\Big)=\mathrm{div}\mathbb{T},\\ \rho\psi_t+\rho u\cdot\nabla\psi+\dfrac{1}{c_v}R(b+1)\rho\psi\mathrm{div}u-\dfrac{1}{c_v}\nu\psi^{\frac{b}{b+1}}\triangle\psi=\dfrac{1}{c_v}(b+1)\psi^{\frac{b}{b+1}}Q(u).\end{cases} \tag{3.2}$$

The initial data is given by

$$(\rho,u,\psi)|_{t=0}=(\rho_0(x),u_0(x),\psi_0(x))=(\rho_0(x),u_0(x),\theta_0^{b+1}(x)),\ x\in\mathbb{V}, \tag{3.3}$$

and the boundary conditions can be shown as: Dirichlet and thermo-insulated boundary conditions for (u,ψ): $\mathbb{V}\subset\mathbb{R}^3$ is a bounded smooth domain and

$$u|_{\partial\mathbb{V}}=0,\quad \nabla\psi\cdot n|_{\partial\mathbb{V}}=0. \tag{3.4}$$

To prove Theorem 1.1, our first step is to establish the following existence result for the reformulated problem (3.2)-(3.4).

Theorem 3.1. *Let initial data (ρ_0,u_0,ψ_0) satisfy the following regularities:*

$$\begin{aligned}&\rho_0\geq 0,\quad \rho_0\in H^1\cap W^{1,q},\quad q\in(3,6],\\ &u_0\in D^1\cap D^2,\quad \psi_0\in D^1\cap D^2,\quad \psi_0\geq\underline{\psi}=\underline{\theta}^{b+1},\end{aligned} \tag{3.5}$$

and the initial layer compatibility conditions

$$\begin{aligned}-div\mathbb{T}(u_0)+\nabla\Big(R\rho_0(\psi_0)^{\frac{1}{b+1}}\Big)=&\sqrt{\rho_0}g_1,\\ -\frac{1}{c_v}(b+1)\psi_0^{\frac{b}{b+1}}\big(\nu(b+1)^{-1}\triangle\psi_0+Q(u_0)\big)=&\sqrt{\rho_0}g_2\end{aligned} \tag{3.6}$$

for some $(g_1,g_2)\in L^2$. Then there exists a sufficiently small constant $\epsilon_0>0$ only depending on (ρ_0,u_0,ψ_0), R, c_v, γ, b, ν and q such that if

$$|V|\leq\epsilon_0, \tag{3.7}$$

there exists a positive time T_* and a unique strong solution (ρ,u,ψ) on $[0,T_*]\times\mathbb{V}$ to IBVP(3.2)-(3.4) satisfying

$$\begin{aligned}&0\leq\rho,\ \rho\in C([0,T_*];H^1\cap W^{1,q}),\ \rho_t\in C([0,T_*];L^2\cap L^q),\\ &(\psi,u)\in C([0,T_*];D^1\cap D^2)\cap L^2([0,T_*];D^{2,q}),\\ &(\psi_t,u_t)\in L^2([0,T_*];D^1),\ (\sqrt{\rho}\psi_t,\sqrt{\rho}u_t)\in L^\infty([0,T_*];L^2).\end{aligned} \tag{3.8}$$

We will prove this existence theorem in the following Subsections 3.2-3.5, and at the end of this section we will show that this theorem indeed implies Theorem 1.1. For simplicity, in the following subsections, we denote

$$a_1=\frac{1}{c_v}R(b+1),\quad a_2=\frac{1}{c_v}\nu,\quad a_3=\frac{1}{c_v}(b+1),\quad P^*=R\rho\psi^{\frac{1}{b+1}}.$$

3.2. **Linearization.** Let $\mathbb{V}$ be a bounded smooth domain. Now we consider the following linearized problem:

$$\begin{cases} \rho_t + \mathrm{div}(\rho w) = 0, \\ \rho\psi_t + \rho w \cdot \nabla\psi + a_1\rho\psi \mathrm{div} w - a_2\phi^{\frac{b}{b+1}}\triangle\psi = a_3\phi^{\frac{b}{b+1}}Q(w), \\ \rho u_t + \rho w \cdot \nabla u + \nabla P^* - \mathrm{div}\mathbb{T} = 0, \\ (\rho, u, \psi)|_{t=0} = (\rho_0^\delta(x), u_0(x), \psi_0(x)), \\ u(t,x)|_{\partial\mathbb{V}} = 0, \quad \nabla\psi(t,x)\cdot n|_{\partial\mathbb{V}} = 0, \quad \text{for} \quad t \geq 0, \end{cases} \tag{3.9}$$

where $\rho_0^\delta = \rho_0 + \delta$ for some constant $\delta > 0$, $w(t,x) \in \mathbb{R}^3$ is a known vector, and $\phi(t,x)$ is a known function. Assume that

$$\begin{aligned} &(w,\phi) \in C([0,T];H^2) \cap L^2([0,T];W^{2,q}), \quad (w_t,\phi_t) \in L^2([0,T];H^1), \\ &\phi \geq \frac{1}{2}\underline{\psi} > 0, \quad w|_{\partial\mathbb{V}} = 0, \quad \nabla\phi\cdot n|_{\partial\mathbb{V}} = 0, \\ &(w(t=0,x), \phi(t=0,x)) = (u_0(x), \psi_0(x)), \quad \text{for} \quad x \in \mathbb{R}^3. \end{aligned} \tag{3.10}$$

We easily have the global existence of the unique strong solution $(\rho^\delta, u^\delta, \psi^\delta)$ to (3.9)-(3.10) by the standard methods for every $\delta > 0$.

Lemma 3.1. *Assume that (ρ_0, u_0, ψ_0) satisfies (3.5)-(3.6). Then there exists a unique strong solution $(\rho^\delta, u^\delta, \psi^\delta)$ to IBVP (3.9)-(3.10) for every $\delta > 0$ satisfying*

$$\begin{aligned} &\rho^\delta \in C([0,T];W^{1,q}), \ \rho_t^\delta \in C([0,T];L^q), \\ &(\psi^\delta, u^\delta) \in C([0,T];H^2) \cap L^2([0,T];W^{2,q}), \\ &(\psi_t^\delta, u_t^\delta) \in L^\infty([0,T];L^2) \cap L^2([0,T];H^1), \end{aligned} \tag{3.11}$$

and $\rho^\delta \geq \underline{\delta}$ for some positive constant $\underline{\delta} > 0$.

Proof. First, the existence of a unique solution ρ^δ to $(3.9)_1$ can be obtained by the standard theory of transport equation, and ρ^δ can be written as

$$\rho^\delta(t,x) = \rho_0^\delta(U(0;t,x)) \exp\Big(-\int_0^t \mathrm{div} w(s, U(s;t,x))\mathrm{d}s\Big), \tag{3.12}$$

where $U \in C([0,T]\times[0,T]\times\mathbb{V})$ is the solution to the initial value problem

$$\begin{cases} \dfrac{d}{ds}U(s;t,x) = w(s,U(s;t,x)), \quad 0 \leq s \leq T, \\ U(t;t,x) = x, \quad x \in \mathbb{V}, \quad 0 \leq t \leq T. \end{cases} \tag{3.13}$$

So we can get the lower bound of ρ^δ easily.

Second, from the regularity properties of ρ^δ, w and ϕ, it is not difficult to solve (ψ^δ, u^δ) from the linear parabolic equations

$$\begin{cases} \psi_t^\delta + w\cdot\nabla\psi^\delta + a_1\psi^\delta \mathrm{div} w - \dfrac{a_2}{\rho^\delta}\phi^{\frac{b}{b+1}}\triangle\psi^\delta = \dfrac{a_3}{\rho^\delta}\phi^{\frac{b}{b+1}}Q(w), \\ u_t^\delta + w\cdot\nabla u^\delta + (\rho^\delta)^{-1}\nabla\Big(R\rho^\delta(\psi^\delta)^{\frac{1}{b+1}}\Big) - (\rho^\delta)^{-1}\mathrm{div}\mathbb{T}(u^\delta) = 0, \end{cases} \tag{3.14}$$

to complete the proof of this lemma. Here we omit the details. □

3.3. **A prior estimates independent of δ.** In this subsection, we will get a prior estimates for the solution $(\rho^\delta, u^\delta, \psi^\delta)$ obtained in Lemma 3.1, which is independent of δ.

We first fix a positive constant c_0 that

$$2 + \|\rho_0\|_{W^{1,q}} + \|(\psi_0, u_0)\|_2 + |(g_1, g_2)|_2 \le c_0. \tag{3.15}$$

Then we rewrite (3.6) into

$$\begin{aligned} -\mathrm{div}\mathbb{T}(u_0) + \nabla\Big(R\rho_0^\delta \psi_0^{\frac{1}{b+1}}\Big) =& \sqrt{\rho_0^\delta} g_1^\delta, \\ -\frac{1}{c_v}(b+1)\psi_0^{\frac{b}{b+1}}\big(\nu(b+1)^{-1}\triangle\psi_0 + Q(u_0)\big) =& \sqrt{\rho_0^\delta} g_2^\delta, \end{aligned} \tag{3.16}$$

where

$$g_1^\delta = \Big(\frac{\rho_0}{\rho_0^\delta}\Big)^{\frac{1}{2}} g_1 + R\delta \frac{\nabla \psi_0^{\frac{1}{b+1}}}{(\rho_0^\delta)^{\frac{1}{2}}}, \quad g_2^\delta = \Big(\frac{\rho_0}{\rho_0^\delta}\Big)^{\frac{1}{2}} g_2.$$

Then according to (3.15), for any $\delta > 0$ small enough, we have

$$1 + \|\rho_0^\delta\|_{W^{1,q}} + \|(\psi_0, u_0)\|_2 + |(g_1^\delta, g_2^\delta)|_2 \le c_0. \tag{3.17}$$

Second, for w and ψ, let positive constants c_i $(i = 1, 2, 3, 4, 5)$ satisfy

$$\begin{aligned} \sup_{0\le t\le T^*} \|w(t)\|_1^2 + \int_0^{T^*} \Big(|w|_{D^{2,q}}^2 + \|w_t\|_1^2\Big)\mathrm{d}t \le& c_1^2 \\ \sup_{0\le t\le T^*} |w(t)|_{D^2}^2 \le c_2^2, \quad \sup_{0\le t\le T^*} \|\phi(t)\|_1^2 \le& c_3^2, \\ \int_0^{T^*} \Big(|\phi|_{D^{2,q}}^2 + \|\phi_t\|_1^2\Big)\mathrm{d}t \le c_4^2, \quad \sup_{0\le t\le T^*} |\phi(t)|_{D^2}^2 \le& c_5^2, \end{aligned} \tag{3.18}$$

for some time $T^* \in (0, T)$, where constants c_i $(i = 1, 2, 3, 4, 5)$ satisfy

$$2 < c_0 < c_1 < c_2 < c_3 < c_4 < c_5.$$

Constants c_i $(i = 1, 2, 3, 4, 5)$ and T^* will be determined later and depend only on c_0 and the fixed constants μ, λ, ν, q, R, c_v, $|\mathbb{V}|$ and T (see (3.69)). We denote $C \ge 1$ a generic constant depending only on fixed constants μ, λ, ν, q, R, c_v, $|\mathbb{V}|$ and T. For simplicity, we denote $a_4 = a_2\Big(\frac{1}{2}\underline{\psi}\Big)^{\frac{b}{b+1}}$.

For simplicity, in the rest of Subsection 3.3, we will denote $(\rho^\delta, u^\delta, \psi^\delta)$ by (ρ, u, ψ) if there is no confusion. Now we start with the a prior estimates for ρ.

Lemma 3.2.

$$1 + |\rho(t)|_\infty^2 + \|\rho(t)\|_{W^{1,q}}^2 \le Cc_0^2, \quad |\rho_t(t)|_q \le Cc_0c_2,$$

for $0 \le t \le T_1 = \min(T^*, (1 + c_2^2)^{-1})$.

Proof. From standard energy estimate for the transport equation, we have

$$\|\rho(t)\|_{W^{1,q}} \le \|\rho_0^\delta\|_{W^{1,q}} \exp\Big(C\int_0^t \|\nabla w(s)\|_{W^{1,q}}\mathrm{d}s\Big), \tag{3.19}$$

where we have used the fact $w \cdot n|_{\partial\mathbb{V}} = 0$. We observe that

$$\int_0^t \|\nabla w(s)\|_{W^{1,q}} \mathrm{d}s \le t^{\frac{1}{2}} \Big(\int_0^t \|\nabla w(s)\|^2_{W^{1,q}} \mathrm{d}s \Big)^{\frac{1}{2}} \le C(c_2 t + c_2 t^{\frac{1}{2}}) \le C, \tag{3.20}$$

for $0 \le t \le T_1 = \min(T^*, (1+c_2^2)^{-1})$, thus the desired estimate for ρ is available.

For the term ρ_t, from the continuity equation $(3.9)_1$, we get

$$|\rho_t|_q \le C(|\rho|_\infty |\nabla w|_q + |w|_\infty |\nabla \rho|_q) \le C c_0 ||w||_2 \le C c_0 c_2.$$

□

Next we will study the evolution of the initial vacuum domain for ρ_0. We denote $V_{R_0} \subset \mathbb{V}$ a neighborhood containing the initial vacuum region V:

$$V \subset V_{R_0} = \{x \in \mathbb{V} | \mathrm{dist}(x, V) \le R_0\},$$

where $R_0 > 0$ is a sufficiently small constant. We denote $X \in C([0,T_1] \times [0,T_1] \times \mathbb{V})$ is the solution to the initial value problem

$$\begin{cases} \dfrac{d}{dt} X(t;0,x_0) = w(t, X(t;0,x_0)), \quad 0 \le t \le T_1, \\ X(0;0,x_0) = x_0, \quad x_0 \in \mathbb{V}. \end{cases} \tag{3.21}$$

Then we can denote by $A(t,R_0)$, $B(t,R_0)$ closed regions that are the images of V_{R_0} and $(\mathbb{V}/V_{R_0})$ respectively under the flow map (3.21):

$$A(t,R_0) = \{X(t;0,x_0) | x_0 \in V_{R_0}\}, \quad B(t,R_0) = \{X(t;0,x_0) | x_0 \in (\mathbb{V}/V_{R_0})\}.$$

Lemma 3.3. *For every sufficiently small $R_0 << 1$, there exists a time $T_{R_0} \in (0,T^*)$ small enough and a constant $a_{R_0} > 0$ such that*

$$\rho(t,x) \ge a_{R_0} + \frac{1}{2}\delta > 0, \quad \forall\ (t,x) \in [0,T_{R_0}] \times (\mathbb{V}/V_{R_0}),$$

where $T_{R_0} = \min(T_1, (\ln 2)^2 (Cc_2)^{-2}, 2R_0(6c_2)^{-1})$ and a_{R_0} are both independent of δ.

Proof. Due to the definition of V and the continuity of ρ_0, it is obvious to see that for any sufficiently small $R' > 0$, there exists a constant $a_{R'}$ independent of δ such that

$$\rho_0^\delta(x) \ge a_{R'} + \delta > 0, \quad \forall\ x \in (\mathbb{V}/V_{R'}). \tag{3.22}$$

From the continuity equation $(3.9)_1$, we have

$$\rho(t,x) = \rho_0^\delta(X(0;0,x_0)) \exp\Big(- \int_0^t \mathrm{div} w(s; X(s;0,x_0)) \mathrm{d}s \Big). \tag{3.23}$$

According to (3.20), we have

$$\int_0^t |\mathrm{div} w(t, X(t;0,x_0))| \mathrm{d}s \le \int_0^t |\nabla w|_\infty \mathrm{d}s \le c_2 t^{1/2} \le \ln 2, \tag{3.24}$$

for $0 \le t \le T' = \min(T_1, (\ln 2)^2 (Cc_2)^{-2})$.

So via (3.22) and (3.24), we easily know that for $0 \le t \le T'$,

$$\rho(t,x) \ge \frac{1}{2}(a_{R'} + \delta) > 0, \quad \forall\ x \in B(t). \tag{3.25}$$

From the ODE problem (3.21), we get

$$|X(0;0,x_0)-x| = |X(0;0,x_0)-X(t;0,x_0)|$$
$$\leq \int_0^t |w(\tau, X(\tau;0,x_0))|\mathrm{d}\tau \leq c_2 t \leq R'/2,$$

for all $(t,x)\in[0,T_{R'}]\times\mathbb{V}$, and $T_{R'}=\min(T', R'(2c_2)^{-1})$, which means,

$$\mathbb{V}/V_{3R'/2}\subset B(t,R'). \tag{3.26}$$

Thus we can choose that $R_0=\frac{3}{2}R'$, $a_{R_0}=\frac{1}{2}a_{R'}$ and

$$T_{R_0}=\min\Big(T_1,(\ln 2)^2(Cc_2)^{-2}, 2R_0(6c_2)^{-1}\Big).$$

□

Before we give the estimates for ψ, first we need to study its positivity.

Lemma 3.4.

$$\psi(t,x)\geq \frac{1}{2}\underline{\psi}, \quad \textit{for} \quad (t,x) \in [0,T_2]\times\mathbb{V},$$

where $T_2=\min(T_1,(\ln 2)^2(Cc_2)^{-2})$.

Proof. From equation $(3.9)_2$, it is easy to have

$$\rho\psi_t+\rho w\cdot\nabla\psi - a_2\phi^{\frac{b}{b+1}}\triangle\psi \geq -a_1\rho\psi \mathrm{div} w, \tag{3.27}$$

where we have used the fact that $\phi^{\frac{b}{b+1}}Q(w)\geq 0$. We define

$$T''=\inf\{\ t\in(0,T]\ \big|\ \psi(t,x)=0, \ \text{ for some } \ x\in\mathbb{V}\}.$$

From Lemma 3.1, we know that $\rho\geq\underline{\delta}>0$. Thus (3.27) implies that

$$\psi_t+w\cdot\nabla\psi-\frac{a_2}{\rho}\phi^{\frac{b}{b+1}}\triangle\psi\geq -a_1\psi|\mathrm{div} w|_\infty \text{ for } \ (t,x) \ \in \ [0,T'']\times\mathbb{V}. \tag{3.28}$$

Denote

$$\psi^*=\psi\exp\Big(a_1\int_0^t|\mathrm{div} w(\tau,x)|_\infty \mathrm{d}\tau\Big),$$

then along curve $X(t;0,x_0)$, we have

$$\frac{d}{dt}\psi^*-\frac{a_2}{\rho}\phi^{\frac{b}{b+1}}\triangle\psi^*\geq 0. \tag{3.29}$$

Then, from $\psi_0(x)\geq\underline{\psi}$ and the classical minimum principle, we have

$$\psi(t,x)\geq\inf_{x\in\mathbb{V}}\psi_0(x)\exp\Big(-(\gamma-1)\int_0^t|\mathrm{div} w(\tau,x)|_\infty\mathrm{d}\tau\Big)>0, \tag{3.30}$$

for $t\in\ [0,T'']$, which is contradictory with the definition of T''. Thus we have (3.30) holds for $t\in[0,T_1]$.

Moreover, for

$$0\leq t\leq T_2=\min(T_1,(\ln 2)^2(Cc_2)^{-2}),$$

we have $\psi(t,x)\geq\frac{1}{2}\underline{\psi}$.

□

Next we will show the a prior estimates for ψ in the following lemma.

Lemma 3.5.

$$|\sqrt{\rho}\psi(t)|_2^2+\|\psi(t)\|_1^2\le Cc_0^3,\quad |\sqrt{\rho}\psi_t(t)|_2^2+\int_0^t|\psi_t(s)|_{D^1}^2\,ds\le M(c_3),$$
$$\int_0^t|\psi(s)|_{D^{2,q}}^2\,ds\le M(c_3),\quad |\psi(t)|_{D^2}\le qM(c_3)c_5^{\frac{b}{b+1}},$$

for $0\le t\le T_3=\min(T_2,T_{R_0},M^{-1}(c_5))$*, where* R_0 *satisfying*

$$|V_{R_0}|\le\big(a_4/\big(20Cc_5^K\big)\big)^3$$

for a sufficiently large constant $K\ge 18$*, and*

$$M=M(\cdot):[2,+\infty)\to[1,+\infty)$$

denotes a strictly increasing continuous function depending only on fixed constants μ*,* λ*,* ν*,* q*,* R*,* c_v*,* $|\mathbb{V}|$ *and* T*.*

Proof. Step 1. Multiplying $(3.9)_2$ by ψ and integrating over $\mathbb{V}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|\psi|^2+a_2\int_{\mathbb{V}}\phi^{\frac{b}{b+1}}|\nabla\psi|^2\\ \le&C\int\Big(|\nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi\psi|+\rho|\psi|^2|\mathrm{div}w|+\phi^{\frac{b}{b+1}}|\nabla w|^2|\psi|\Big)\\ \le&C\Lambda_1+C\Big(|\rho|_\infty^{\frac{1}{2}}|\sqrt{\rho}\psi|_2|\nabla w|_3+|\phi|_\infty^{\frac{b}{b+1}}|\nabla w|_2|\nabla w|_3\Big)|\psi|_6\\ \le&C\Lambda_1+\frac{a_4}{20c_0^2}(|\sqrt{\rho}\psi|_2^2+c_0^2|\nabla\psi|_2^2)+Cc_2^5|\sqrt{\rho}\psi|_2^2+Cc_5^8,\end{aligned}\tag{3.31}$$

where we have used the Poincaré type inequality for ψ in Lemma 2.2:

$$|\psi|_6\le C(|\sqrt{\rho}\psi|_2+(1+|\rho|_2)|\nabla\psi|_2)\le C(|\sqrt{\rho}\psi|_2+c_0|\nabla\psi|_2).\tag{3.32}$$

For the term Λ_1, from Lemmas 3.2-3.4, we have

$$\begin{aligned}\Lambda_1=&\int|\nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi\psi|\\ =&\int_{V_{R_0}}|\nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi\psi|\mathrm{d}x+\int_{\mathbb{V}/V_{R_0}}|\nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi\psi|\mathrm{d}x\\ \le&C|\nabla\psi|_2|\psi|_6|\nabla\phi|_6|V_{R_0}|^{\frac{1}{6}}+C|\nabla\psi|_2|\sqrt{\rho}\psi|_2^{\frac{1}{2}}|\sqrt{\rho}\psi|_6^{\frac{1}{2}}|\nabla\phi|_6\\ \le&\frac{a_4}{20}|\nabla\psi|_2^2+Cc_5^2|V_{R_0}|^{\frac{1}{3}}(|\sqrt{\rho}\psi|_2^2+c_0^2|\nabla\psi|_2^2)+Cc_5^{\frac{5}{2}}|\sqrt{\rho}\psi|_2|\psi|_6\\ \le&\frac{a_4}{10}|\nabla\psi|_2^2+C(c_5^2|V_{R_0}|^{\frac{1}{3}}+c_5^7)|\sqrt{\rho}\psi|_2^2+Cc_5^4|V_{R_0}|^{\frac{1}{3}}|\nabla\psi|_2^2,\end{aligned}\tag{3.33}$$

which, along with (3.31), from Gronwall's inequality, implies

$$|\sqrt{\rho}\psi|_2^2+\int_0^t|\nabla\psi|_2^2\mathrm{d}s\le C(c_0^3+c_5^8t)\exp(Cc_5^7t)\le Cc_0^3\tag{3.34}$$

for $0\le t\le T'''=\min(T_2,T_{R_0},(1+c_5^8)^{-1})$ and R_0 satisfying

$$|V_{R_0}|\le\big(a_4/(20Cc_5^4)\big)^3.$$

Step:2. Differentiating $(3.9)_2$ with respect to t, we have

$$\begin{aligned}&\rho\psi_{tt}-a_2\phi^{\frac{b}{b+1}}\triangle\psi_t-a_2\phi_t^{\frac{b}{b+1}}\triangle\psi\\&=-\rho_t\psi_t-(\rho w\cdot\nabla\psi)_t-a_1(\rho\psi\mathrm{div}w)_t+a_3\big(\phi^{\frac{b}{b+1}}Q(w)\big)_t.\end{aligned}\tag{3.35}$$

Multiplying (3.35) by ψ_t and integrating over $\mathbb{V}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|\psi_t|^2+a_2\int\phi^{\frac{b}{b+1}}|\nabla\psi_t|^2\\&\leq C\int\Big(|\nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi_t\psi_t|+|\phi_t^{\frac{b}{b+1}}\triangle\psi\psi_t|+|\rho_t w\cdot\nabla\psi\psi_t|+|\rho w_t\cdot\nabla\psi\psi_t|\Big)\\&\quad+C\int\Big(|\rho w\cdot\nabla\psi_t\psi_t|+|(\rho\psi)_t\mathrm{div}w\psi_t|+|\rho\psi\mathrm{div}w_t\psi_t|\Big)\\&\quad+C\int\Big(\big|\phi^{\frac{b}{b+1}}Q(w)_t\psi_t\big|+\big|\phi_t^{\frac{b}{b+1}}Q(w)\psi_t\big|\Big)\equiv:C\sum_{i=1}^{9}I_i.\end{aligned}\tag{3.36}$$

For terms I_1-I_9, according to Lemmas 3.2-3.4, 2.1-2.2 and Young's inequality, we have

$$\begin{aligned}I_1=&\int|\nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi_t\psi_t|=\int_{V_{R_0}}|\nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi_t\psi_t|\mathrm{d}x+\int_{\mathbb{V}/V_{R_0}}|\nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi_t\psi_t|\mathrm{d}x\\&\leq C|\nabla\psi_t|_2|\psi_t|_6|\nabla\phi|_6|V_{R_0}|^{\frac{1}{6}}+C|\nabla\psi_t|_2|\sqrt{\rho}\psi_t|_2^{\frac{1}{2}}|\sqrt{\rho}\psi_t|_6^{\frac{1}{2}}|\nabla\phi|_6\\&\leq\frac{a_4}{20}|\nabla\psi_t|_2^2+Cc_5^2|V_{R_0}|^{\frac{1}{3}}(|\sqrt{\rho}\psi_t|_2^2+c_0^2|\nabla\psi_t|_2^2)+Cc_5^{\frac{5}{2}}|\sqrt{\rho}\psi_t|_2|\psi_t|_6\\&\leq\frac{a_4}{10}|\nabla\psi_t|_2^2+C(c_5^2|V_{R_0}|^{\frac{1}{3}}+c_5^7)|\sqrt{\rho}\psi_t|_2^2+Cc_5^4|V_{R_0}|^{\frac{1}{3}}|\nabla\psi_t|_2^2,\\I_2=&\int|\phi_t^{\frac{b}{b+1}}\triangle\psi\psi_t|=\int_{V_{R_0}}|\phi_t^{\frac{b}{b+1}}\triangle\psi\psi_t|\mathrm{d}x+\int_{\mathbb{V}/V_{R_0}}|\phi_t^{\frac{b}{b+1}}\triangle\psi\psi_t|\mathrm{d}x\\&\leq C|\triangle\psi|_2|\psi_t|_6|\phi_t|_6|V_{R_0}|^{\frac{1}{6}}+C|\triangle\psi|_2|\sqrt{\rho}\psi_t|_2^{\frac{1}{2}}|\sqrt{\rho}\psi_t|_6^{\frac{1}{2}}|\phi_t|_6\\&\leq\frac{a_4}{20c_0^2}(|\sqrt{\rho}\psi_t|_2^2+c_0^2|\nabla\psi_t|_2^2)+C|V_{R_0}|^{\frac{1}{3}}|\phi_t|_6^2|\triangle\psi|_2^2+\eta|\phi_t|_6^2\\&\quad+\eta^{-1}|\sqrt{\rho}\psi_t|_2|\rho|_\infty^{\frac{1}{2}}|\psi_t|_6|\triangle\psi|_2^2\\&\leq\frac{a_4}{10c_0^2}(|\sqrt{\rho}\psi_t|_2^2+c_0^2|\nabla\psi_t|_2^2)+C|V_{R_0}|^{\frac{1}{3}}|\phi_t|_6^2|\triangle\psi|_2^2+\eta|\phi_t|_6^2\\&\quad+\eta^{-2}c_0^3|\sqrt{\rho}\psi_t|_2^2|\triangle\psi|_2^4,\\I_3=&\int|\rho_t w\cdot\nabla\psi\psi_t|\\&\leq C|\rho_t|_3|w|_\infty|\nabla\psi|_2|\psi_t|_6\leq Cc_2^8|\nabla\psi|_2^2+\frac{a_4}{20c_0^2}(|\sqrt{\rho}\psi_t|_2^2+c_0^2|\nabla\psi_t|_2^2),\end{aligned}\tag{3.37}$$

where $\eta > 0$ is a constant. Similarly, we have

$$
\begin{aligned}
I_4 + I_7 =& \int |\rho w_t \cdot \nabla\psi\psi_t| + \int |\rho\psi \mathrm{div} w_t \psi_t| \\
\leq& C|\rho|_\infty^{\frac{1}{2}}(|w_t|_6|\nabla\psi|_2 + |\nabla w_t|_2|\psi|_6)|\sqrt{\rho}\psi_t|_2^{\frac{1}{2}}|\sqrt{\rho}\psi_t|_6^{\frac{1}{2}} \\
\leq& C|\nabla w_t|_2^2 + C(c_0^{11} + c_0^9|\nabla\psi|_2^4)|\sqrt{\rho}\psi_t|_2^2 + \frac{a_4}{20c_0^2}(|\sqrt{\rho}\psi_t|_2^2 + c_0^2|\nabla\psi_t|_2^2), \\
I_5 =& \int |\rho w \cdot \nabla\psi_t\psi_t| \leq C|\rho|_\infty^{\frac{1}{2}}|w|_\infty|\nabla\psi_t|_2|\sqrt{\rho}\psi_t|_2 \leq Cc_2^3|\sqrt{\rho}\psi_t|_2^2 + \frac{a_4}{20}|\nabla\psi_t|_2^2, \\
I_6 =& \int |P_t^* \mathrm{div} w\psi_t| \leq C(|\rho_t|_2|\psi|_6|\nabla w|_6|\psi_t|_6 + |\rho|_\infty^{\frac{1}{2}}|\sqrt{\rho}\psi_t|_2|\nabla w|_3|\psi_t|_6) \\
\leq& Cc_2^5|\sqrt{\rho}\psi_t|_2^2 + \frac{a_4}{20c_0^2}(|\sqrt{\rho}\psi_t|_2^2 + c_0^2|\nabla\psi_t|_2^2) + Cc_2^{10}|\nabla\psi|_2^2 + Cc_2^{11}, \\
I_8 =& \int \left|\phi^{\frac{b}{b+1}} Q(w)_t \psi_t\right| \\
\leq& C|\phi|_6|\psi_t|_6|\nabla w_t|_2|\nabla w|_6 \leq \frac{a_4}{20c_0^2}(|\sqrt{\rho}\psi_t|_2^2 + c_0^2|\nabla\psi_t|_2^2) + Cc_3^6|\nabla w_t|_2^2, \\
I_9 =& \int \left|\phi_t^{\frac{b}{b+1}} Q(w)\psi_t\right| = \int_{V_{R_0}} \left|\phi_t^{\frac{b}{b+1}} Q(w)\psi_t\right| \mathrm{d}x + \int_{\mathbb{V}/V_{R_0}} \left|\phi_t^{\frac{b}{b+1}} Q(w)\psi_t\right| \mathrm{d}x \\
\leq& C|\nabla w|_3|\nabla w|_6|\ \psi_t|_6|\phi_t|_6|V_{R_0}|^{\frac{1}{6}} + C|\nabla w|_3|\nabla w|_6|\phi_t|_6|\sqrt{\rho}\psi_t|_2^{\frac{1}{2}}|\sqrt{\rho}\psi_t|_6^{\frac{1}{2}} \\
\leq& \frac{a_4}{20c_0^2}(|\sqrt{\rho}\psi_t|_2^2 + c_0^2|\nabla\psi_t|_2^2) + Cc_2^6|V_{R_0}|^{\frac{1}{3}}|\phi_t|_6^2 \\
&+ \eta|\phi_t|_6^2 + Cc_2^4\eta^{-1}|\sqrt{\rho}\psi_t|_2|\sqrt{\rho}\psi_t|_6 \\
\leq& \frac{a_4}{10c_0^2}(|\sqrt{\rho}\psi_t|_2^2 + c_0^2|\nabla\psi_t|_2^2) + Cc_2^6|V_{R_0}|^{\frac{1}{3}}|\phi_t|_6^2 + \eta|\phi_t|_6^2 + Cc_2^{11}\eta^{-2}|\sqrt{\rho}\psi_t|_2^2,
\end{aligned}
\tag{3.38}
$$

where we have used the Poincaré type inequality (see Lemma 2.2):

$$
\begin{aligned}
|\psi|_6 \leq& C(|\sqrt{\rho}\psi|_2 + (1 + |\rho|_2)|\nabla\psi|_2) \leq C(c_0^{\frac{3}{2}} + c_0|\nabla\psi|_2), \\
|\psi_t|_6 \leq& C(|\sqrt{\rho}\psi_t|_2 + (1 + |\rho|_2)|\nabla\psi_t|_2) \leq C(|\sqrt{\rho}\psi_t|_2 + c_0|\nabla\psi_t|_2).
\end{aligned}
\tag{3.39}
$$

Then combining (3.36)-(3.38), we quickly have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|\psi_t|^2 + a_2\int \phi^{\frac{b}{b+1}}|\nabla\psi_t|^2 \\
\leq& \Big(\frac{a_4}{2} + Cc_5^4|V_{R_0}|^{\frac{1}{3}}\Big)|\nabla\psi_t|_2^2 + Cc_2^{11} + Cc_3^6|\nabla w_t|_2^2 \\
&+ C\big(|V_{R_0}|^{\frac{1}{3}}(|\triangle\psi|_2^2 + c_2^6) + \eta\big)|\phi_t|_6^2 \\
&+ C\big(c_5^{11} + c_5^2|V_{R_0}|^{\frac{1}{3}} + \eta^{-2}c_2^{11} + c_0^9|\nabla\psi|_2^4 + \eta^{-2}c_0^3|\triangle\psi|_2^2\big)L(t).
\end{aligned}
\tag{3.40}
$$

Now we need to consider the term $|\psi|_{D^2}$. From equation $(3.9)_2$, we know:

$$
-a_2\phi^{\frac{b}{b+1}}\triangle\psi = -(\rho\psi_t + \rho w \cdot \nabla\psi + a_1\rho\psi \mathrm{div} w) + a_3\phi^{\frac{b}{b+1}}Q(w). \tag{3.41}
$$

Then from Lemmas 2.1 and 2.3, we have

$$\begin{aligned}|\psi|_{D^2} \leq & C\big(|\rho\psi_t|_2 + |\rho w\cdot\nabla\psi|_2 + |\rho\psi \mathrm{div} w|_2 + |\phi|_\infty^{\frac{b}{b+1}}|Q(w)|_2\big)\\ \leq & C\Big(|\rho|_\infty^{\frac{1}{2}}|\sqrt{\rho}\psi_t|_2 + |\rho|_\infty|w|_\infty|\nabla\psi|_2 + |\rho|_6|\psi|_6|\mathrm{div} w|_6 + c_5^{\frac{b}{b+1}}|\nabla w|_6|\nabla w|_3\Big)\\ \leq & Cc_2^3(|\nabla\psi|_2 + |\sqrt{\rho}\psi_t|_2) + Cc_3^{\frac{7}{2}}c_5^{\frac{b}{b+1}},\end{aligned} \tag{3.42}$$

where we have used the fact that

$$|\psi|_6 \leq C(|\sqrt{\rho}\psi|_2 + c_0|\nabla\psi|_2) \leq Cc_0^{\frac{3}{2}} + Cc_0|\nabla\psi|_2.$$

On the other hand, due to (3.39) and $|\psi_t|_2 \leq C|\psi_t|_6$, we have

$$\frac{d}{dt}|\nabla\psi(t)|_2^2 \leq C|\nabla\psi|_2|\nabla\psi_t|_2 \leq C|\nabla\psi|_2^2 + \frac{a_4}{20}|\nabla\psi_t|_2^2. \tag{3.43}$$

Then letting R_0 sufficiently small such that

$$|V_{R_0}| \leq \min\Big(\frac{a_4}{20Cc_5^K}\Big)^3$$

for a sufficiently large constant $K \geq 18$, and letting

$$\eta = c_5^{-K}, \quad L(t) = 1 + |\nabla\psi|_2^2 + |\sqrt{\rho}\psi_t|_2^2,$$

from (3.40) and (3.42)-(3.43) we have

$$\frac{d}{dt}L(t) + a_4\int|\nabla\psi_t|^2 \leq Cc_3^6|\nabla w_t|_2^2L(t) + C(c_5^{2K+12} + c_5^{9-K}|\phi_t|_6^2)L^3(t). \tag{3.44}$$

Denote

$$H(t) = L(t)\exp\Big(-\int_0^t Cc_3^6|\nabla w_t|_2^2\mathrm{d}s\Big),$$

then from (3.44), we have

$$\frac{d}{dt}H(t) \leq C(c_5^{2K+12} + c_5^{9-K}|\phi_t|_6^2)H^3(t). \tag{3.45}$$

Next we need to solve this inequality. From $(3.9)_2$, we have

$$|\sqrt{\rho}\psi_t|_2^2 \leq |\rho|_\infty\|\nabla w\|_1^2|\nabla\psi|_2^2 + \int|\Phi|^2/\rho, \tag{3.46}$$

where

$$\Phi = -\frac{1}{c_v}(b+1)\psi_0^{\frac{b}{b+1}}\big(\nu(b+1)^{-1}\triangle\psi_0 + Q(u_0)\big).$$

Via Lemma 3.1, we easily have

$$\lim_{t\mapsto 0}\int\Big(\frac{|\Phi(t)|^2}{\rho} - \frac{|\Phi(0)|^2}{\rho_0}\Big) \leq \lim_{t\mapsto 0}\Big(\frac{1}{\underline{\delta}}\int|\Phi(t)-\Phi(0)|^2 + \frac{1}{\delta\underline{\delta}}|\rho(t)-\rho_0|_\infty\int|\Phi(0)|^2\Big) = 0.$$

According to the compatibility conditions (3.6) and equation $(3.9)_2$, we have

$$\limsup_{\tau\to 0}|\sqrt{\rho}\psi_t(\tau)|_2^2 \leq |\rho_0|_\infty\|\nabla w_0\|_1^2|\nabla\psi_0|_2^2 + |g_2|_2^2 \leq Cc_0^5, \tag{3.47}$$

which implies that

$$\limsup_{\tau\to 0}H(\tau) \leq Cc_0^5.$$

Now we integrate the inequality (3.45) over $[\tau, t]$ for any $\tau \in (0, t)$, we have,

$$H(t) \leq H(\tau) + \int_\tau^t C(c_5^{2K+12} + c_5^{9-K}|\phi_t|_6^2)H^3(s)\mathrm{d}s. \tag{3.48}$$

Letting $\tau \to 0$ in (3.48), we have

$$H(t) \leq Cc_0^5 + \int_0^t C(c_5^{2K+12} + c_5^{9-K}|\phi_t|_6^2)H^3(s)\mathrm{d}s. \tag{3.49}$$

Then solving this inequality directly, we have

$$H(t) \leq Cc_0^5 + \frac{Cc_0^5}{\left(1 - Cc_0^5(c_5^{2K+12}t + c_5^{11-K})\right)^{\frac{1}{2}}} \leq Cc_0^5,$$

when $0 < t \leq T'''' = \min\left(T_2, T_{R_0}, (1 + Cc_5)^{-4K}\right)$ for constant K sufficiently large. Then we immediately have

$$L(t) \leq M(c_3), \quad \text{for} \quad 0 \leq t \leq T'''', \tag{3.50}$$

where $M = M(\cdot) : [2, +\infty) \to [1, +\infty)$ denotes a strictly increasing continuous function, and depends only on fixed constants μ, λ, ν, q, R, c_v, $|\mathbb{V}|$ and T.

Therefore, from (3.44) and (3.50), we quickly have

$$L(t) + a_4 \int_0^t |\psi_t|_{D^1}^2 \mathrm{d}s \leq M(c_3), \quad \text{for} \quad 0 \leq t \leq T''''. \tag{3.51}$$

From (3.42), we quickly have

$$|\psi|_{D^2} \leq Cc_2^2(|\nabla\psi|_2 + |\sqrt{\rho}\psi_t|_2) + Cc_3^3 c_5^{\frac{b}{b+1}} \leq M(c_3)c_5^{\frac{b}{b+1}}. \tag{3.52}$$

For the term $|\psi|_{D^{2,q}}$, similarly, via Lemma 2.3 and (3.41), we also have

$$\int_0^t |\psi|_{D^{2,q}}^2 \mathrm{d}s \leq C\int_0^t \left|\rho\psi_t + \rho w \cdot \nabla\psi + a_1\rho\psi \mathrm{div} w - a_3\phi^{\frac{b}{b+1}}Q(w)\right|_q^2 \mathrm{d}s \leq M(c_3) \tag{3.53}$$

for $0 \leq t \leq T_3 = \min\left(T'''', M^{-1}(c_5)\right)$. According to $P^* = R\rho\psi^{\frac{1}{b+1}}$ and Lemma 3.4, for $0 \leq t \leq T_3$, we easily obtain that

$$|\nabla P^*|_2 \leq M(c_3), \quad |\nabla P^*|_q \leq M(c_3)c_5^{\frac{b}{b+1}}, \quad |P_t^*|_2 \leq M(c_3). \tag{3.54}$$

<u>Step 3</u>. Multiplying $(3.9)_2$ by ψ_t and integrating over $\mathbb{V}$, we have

$$\begin{aligned}
&\frac{a_2}{2}\frac{d}{dt}\int \phi^{\frac{b}{b+1}}|\nabla\psi|^2 + \int \rho|\psi_t|^2 \\
\leq& C\int \left(|\phi_t^{\frac{b}{b+1}}||\nabla\psi|^2 + \rho|\psi||\mathrm{div} w||\psi_t| + \rho|w||\nabla\psi||\psi_t| + \phi^{\frac{b}{b+1}}Q(w)\psi_t\right) \\
\leq& C|\phi_t|_6|\nabla\psi|_2|\nabla\psi|_3 + C|\sqrt{\rho}\psi_t|_2|\rho|_\infty^{\frac{1}{2}}|\psi|_6|\nabla w|_3 \\
&+ C|\sqrt{\rho}\psi_t|_2|\rho|_\infty^{\frac{1}{2}}|\nabla\psi|_2|w|_\infty + C|\phi|_\infty^{\frac{b}{b+1}}|\nabla w|_2|\nabla w|_3|\psi_t|_6 \\
\leq& M(c_5)(|\phi_t|_6 + |\sqrt{\rho}\psi_t|_2 + |\psi_t|_6) \\
\leq& \eta(|\phi_t|_6^2 + |\psi_t|_6^2) + \frac{1}{2}|\sqrt{\rho}\psi_t|_2^2 + \eta^{-1}M(c_5),
\end{aligned} \tag{3.55}$$

where $\eta = M^{-1}(c_5)$. Then integrating (3.55) over $(0,t)$, we have

$$|\nabla\psi|_2^2 + \int_0^t |\sqrt{\rho}\psi_t|_2^2 \mathrm{ds} \le Cc_0^3, \quad \text{for} \quad 0 \le t \le T_3 = \min\big(T'''', M^{-1}(c_5)\big). \tag{3.56}$$

□

Next we give the a prior estimates for the velocity u.

Lemma 3.6.

$$\|u(t)\|_1^2 + |\sqrt{\rho}u_t(t)|_2^2 + \int_0^t |u_t(s)|_{D^1}^2 ds \le Cc_0^5,$$
$$\int_0^t |u(s)|_{D^{2,q}}^2 ds \le Cc_0^7, \quad |u(t)|_{D^2}^2 \le Cc_1^{13}, \quad \textit{for } 0 \le t \le T_3.$$

Proof. Differentiating $(3.9)_3$ with respect to t, we have

$$\rho u_{tt} - \mathrm{div}\mathbb{T}_t = -\rho_t u_t - (\rho w \cdot \nabla u)_t - \nabla P_t^*. \tag{3.57}$$

Then multiplying (3.57) by u_t and integrating over $\mathbb{V}$, via (3.4) we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2 + \int \big((\mu+\lambda)|\mathrm{div}u_t|^2 + \mu(\nabla u_t)^2\big) \\ \le& C\int \Big(|\rho_t w\cdot\nabla u\cdot u_t| + |\rho w\cdot\nabla u_t\cdot u_t| + |\rho w_t\cdot\nabla u\cdot u_t| + |\ P_t^*\mathrm{div}u_t|\Big) \equiv: C\sum_{i=10}^{13} I_i. \end{aligned} \tag{3.58}$$

According to Lemmas 3.1-3.5, Lemma 2.1 and Young's inequality, we have

$$\begin{aligned} I_{10} =& \int |\rho_t w\cdot\nabla u\cdot u_t| \le C|\rho_t|_3|w|_\infty|\nabla u|_2|u_t|_6 \le Cc_2^6|\nabla u|_2^2 + \frac{\mu}{20}|\nabla u_t|_2^2, \\ I_{11} =& \int |\rho w\cdot\nabla u_t\cdot u_t| \le C|\rho|_\infty^{\frac{1}{2}}|w|_\infty|\nabla u_t|_2|\sqrt{\rho}u_t|_2 \le Cc_2^3|\sqrt{\rho}u_t|_2^2 + \frac{\mu}{20}|\nabla u_t|_2^2, \\ I_{12} =& \int |\rho w_t\cdot\nabla u\cdot u_t| \le C|\rho|_\infty^{\frac{1}{2}}|w_t|_6|\nabla u|_3|\sqrt{\rho}u_t|_2 \\ \le& C\eta^{-1}c_0\|\nabla u\|_1^2 + C\eta|\nabla w_t|_2^2|\sqrt{\rho}u_t|_2^2, \\ I_{13} =& \int |\ P_t^*\mathrm{div}u_t| \le C|P_t^*|_2|\nabla u_t|_2 \le M(c_3) + \frac{\mu}{20}|\nabla u_t|_2^2. \end{aligned} \tag{3.59}$$

Then combining (3.58)-(3.59), we have

$$\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2 + \int |\nabla u_t|^2 \le M(c_3) + C(c_2^6 + \eta|\nabla w_t|_2^2)|\sqrt{\rho}u_t|_2^2 + C\eta^{-1}c_0\|\nabla u\|_1^2. \tag{3.60}$$

Now we have to estimate $|u|_{D^2}$, due to

$$\mathrm{div}\mathbb{T} = \rho u_t + \rho w\cdot\nabla u + \nabla P^*, \tag{3.61}$$

and Lemma 2.3, we have

$$\begin{aligned} |u|_{D^2} \le& C\big(|\rho u_t|_2 + |\rho w\cdot\nabla u|_2 + |\nabla P^*|_2\big) \\ \le& C(|\rho|_\infty^{\frac{1}{2}}|\sqrt{\rho}u_t|_2 + |\rho|_2|w|_6|\nabla u|_2^{\frac{1}{2}}|\nabla u|_3^{\frac{1}{2}} + |\rho|_\infty|\nabla\psi|_2 + |\psi|_6|\nabla\rho|_3) \\ \le& Cc_1^4(|\sqrt{\rho}u_t|_2 + |\nabla u|_2) + \frac{1}{2}|u|_{D^2} + Cc_0^{\frac{5}{2}}. \end{aligned} \tag{3.62}$$

On the other hand, we also have

$$\frac{d}{dt}|\nabla u(t)|_2^2 \le 2|\nabla u|_2|\nabla u_t|_2 \le C|\nabla u|_2^2 + \frac{\mu}{20}|\nabla u_t|_2^2, \tag{3.63}$$

which, along with (3.60)-(3.62), implies that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|\sqrt{\rho}u_t|_2^2 + |\nabla u|_2^2) + \frac{\mu}{2}\int |\nabla u_t|^2\\ \le& M(c_3) + \eta^{-1}c_0^6 + C(c_2^6 + \eta|\nabla w_t|_2^2 + \eta^{-1}c_1^9)\big(|\sqrt{\rho}u_t|_2^2 + |\nabla u|_2^2\big).\end{aligned} \tag{3.64}$$

Similarly to the proof of (3.47), via (3.6) and equations $(3.9)_2$, we have

$$\limsup_{\tau\to 0}|\sqrt{\rho}u_t(\tau)|_2^2 \le |\rho_0|_\infty\|\nabla w_0\|_1^2|\nabla u_0|_2^2 + |g_1|_2^2 \le Cc_0^5. \tag{3.65}$$

Then from Gronwall's inequality and (3.65), via letting $\eta = (c_2^2)^{-1}$, we have

$$\begin{aligned}&|\nabla u(t)|_2^2 + |\sqrt{\rho}u_t(t)|_2^2 + \int_0^t |u_t|_{D^1}^2 \mathrm{d}s\\ \le&(Cc_0^5 + M(c_3)t)\exp\Big(\int_0^t (c_2^{11} + c_2^{-2}|\nabla w_t|_2^2)\mathrm{d}s\Big) \le Cc_0^5\end{aligned} \tag{3.66}$$

for $0 \le t \le \min\big(T_3, M^{-1}(c_5)\big)$, which together with (3.62), implies that

$$|u|_{D^2} \le Cc_1^{6\frac{1}{2}}, \quad \text{for } 0 \le t \le \min\big(T_3, M^{-1}(c_5)\big).$$

Then similarly, we have

$$\int_0^t |u|_{D^{2,q}}^2 \mathrm{d}s \le \int_0^t \big(|\rho u_t + \rho w\cdot\nabla u + \nabla P|_q^2\big)\mathrm{d}s \le Cc_0^7,$$

for $0 \le t \le T_4 = \big(T_3, M^{-1}(c_5)\big)$. □

Then based on Lemmas 3.2-3.6, we have choosen

$$M = M(\cdot) : [2, +\infty) \to [1, +\infty)$$

as a strictly increasing continuous function depending only on fixed constants μ, λ, ν, q, R, c_v, $|\mathbb{V}|$ and T. When for $0 \le t \le T_3 = \min(T_2, T_{R_0}, M^{-1}(c_5))$, where R_0 satisfying

$$|V_{R_0}| \le \big(a_4/(20Cc_5^K)\big)^3, \tag{3.67}$$

for a sufficiently large constant $K \ge 18$, the following a prior estimates hold

$$\begin{aligned}&|\rho(t)|_\infty^2 + \|\rho(t)\|_{W^{1,q}}^2 \le Cc_0^2, \quad |\rho_t(t)|_q \le Cc_0c_2, \quad \psi(t,x) \ge \frac{1}{2}\underline{\psi}, \quad \|\psi(t)\|_1^2 \le Cc_0^3,\\ &|\sqrt{\rho}\psi_t(t)|_2^2 + \int_0^t (|\psi_t(s)|_{D^1}^2 + |\psi(s)|_{D^{2,q}}^2)\mathrm{d}s \le M(c_3), \quad |\psi(t)|_{D^2}^2 \le M(c_3)c_5^{\frac{2b}{b+1}},\\ &\|u(t)\|_1^2 + |\sqrt{\rho}u_t(t)|_2^2 + \int_0^t \Big(|u(s)|_{D^{2,q}}^2 + |u_t(s)|_{D^1}^2\Big)\mathrm{d}s \le Cc_0^7, \quad |u(t)|_{D^2}^2 \le Cc_1^{13}.\end{aligned} \tag{3.68}$$

Therefore, if we define the constants c_i $(i=1,2,3,4,5)$ and T^* by

$$\begin{aligned}
&c_1 = C^{\frac{1}{2}} c_0^{\frac{7}{2}}, \quad c_2 = C^{\frac{1}{2}} c_1^{\frac{13}{2}} = C^{\frac{15}{4}} c_0^{\frac{91}{4}}, \quad c_3 = c_2 = C^{\frac{15}{4}} c_0^{\frac{91}{4}}, \\
&c_4 = \sqrt{M}(c_3) = \sqrt{M}(C^{\frac{15}{4}} c_0^{\frac{91}{4}}), \quad c_5 = M^{\frac{b+1}{2}}(c_3) = M^{\frac{b+1}{2}}(C^{\frac{15}{4}} c_0^{\frac{91}{4}}), \\
&T^* = \min(T, (12R_0c_2)^{-1}, M^{-1}(c_5)),
\end{aligned} \tag{3.69}$$

then we deduce that

$$\begin{aligned}
&\sup_{0\le t\le T^*} \|u(t)\|_1^2 + \operatorname{ess}\sup_{0\le t\le T^*} |\sqrt{\rho}u_t(t)|_2^2 + \int_0^{T^*} \Big(|u(s)|_{D^{2,q}}^2 + |u_t(s)|_{D^1}^2\Big)\mathrm{d}s \le c_1^2, \\
&\sup_{0\le t\le T^*} |u(t)|_{D^2}^2 \le c_2^2, \quad \sup_{0\le t\le T^*} |\sqrt{\rho}\psi(t)|_2^2 + \sup_{0\le t\le T^*} \|\psi(t)\|_1^2 \le c_3^2, \\
&\psi(t,x) \ge \frac{1}{2}\underline{\psi}, \quad \operatorname{ess}\sup_{0\le t\le T^*} |\sqrt{\rho}\psi_t(t)|_2^2 + \int_0^{T^*} (|\psi_t(s)|_{D^1}^2 + |\psi(s)|_{D^{2,q}}^2)\mathrm{d}s \le c_4^2, \\
&\sup_{0\le t\le T^*} |\psi(t)|_{D^2} \le c_5^2, \quad \sup_{0\le t\le T^*} \big(|\rho(t)|_\infty^2 + \|\rho(t)\|_{W^{1,q}}^2 + |\rho_t(t)|_q\big) \le c_2^2.
\end{aligned} \tag{3.70}$$

Moreover, for sufficiently small $R_0 > 0$ satisfying (3.67), we also have

$$\rho(t,x) \ge a_{R_0} + \frac{1}{2}\delta > 0, \quad \forall\ (t,x) \in [0,T^*] \times (\mathbb{V}/V_{R_0}), \tag{3.71}$$

where a_{R_0} is a positive constant independent of δ.

3.4. **Passing limit from nonvacuum to vacuum.** In this subsection, we will give the existence of the strong solution with vacuum to our linear problem:

$$\begin{cases}
\rho_t + \operatorname{div}(\rho w) = 0, \\
\rho\psi_t + \rho w\cdot\nabla\psi + a_1\rho\psi\operatorname{div}w - a_2\phi^{\frac{b}{b+1}}\triangle\psi = a_3\phi^{\frac{b}{b+1}}Q(w), \\
\rho u_t + \rho w\cdot\nabla u + \nabla P^* - \operatorname{div}\mathbb{T} = 0, \\
(\rho,u,\psi)|_{t=0} = (\rho_0(x), u_0(x), \psi_0(x)), \\
u|_{\partial\mathbb{V}} = 0, \quad \nabla\psi\cdot n|_{\partial\mathbb{V}} = 0.
\end{cases} \tag{3.72}$$

Lemma 3.7. *Assume (3.5)-(3.6) hold. Then there exists a unique strong solution (ρ,u,ψ) on $[0,T^*]\times\mathbb{V}$ to IBVP (3.72) satisfying*

$$\begin{aligned}
&\rho \in C([0,T^*];W^{1,q}),\ \ \rho_t \in C([0,T^*];L^q), \\
&(\psi,u) \in C([0,T^*];H^2) \cap L^2([0,T^*];W^{2,q}), \quad \psi \ge \frac{1}{2}\underline{\psi}, \\
&(\sqrt{\rho}\psi_t, \sqrt{\rho}u_t) \in L^\infty([0,T^*];L^2), \quad (\psi_t,u_t) \in L^2([0,T^*];H^1).
\end{aligned} \tag{3.73}$$

Moreover, the a prior estimates (3.70) also hold for our solution (ρ,u,ψ), and for sufficient small $R_0>0$ there exists a constant a_{R_0} independent of δ such that

$$\rho(t,x) \ge a_{R_0} > 0, \quad \forall\ (t,x) \in [0,T^*] \times (\mathbb{V}/V_{R_0}). \tag{3.74}$$

Proof. We divide the proof into three steps.

Step 1 Existence. For $\rho_0^\delta = \rho_0 + \delta$ with $\delta \in (0,1)$, from Lemma 3.1, there exists a unique strong solution $(\rho^\delta, u^\delta, \psi^\delta)$ on $[0,T^*] \times \mathbb{V}$ satisfying (3.70)-(3.71), where the constants c_1-c_5, C, R_0, T^* and a_{R_0} are independent of δ. Then there exists a subsequence of solutions $(\rho^\delta, u^\delta, \psi^\delta)$ converges to a limit (ρ, u, ψ) in weak or weak* sense:

$$\begin{aligned} \rho^\delta &\rightharpoonup \rho \quad \text{weakly*} \ \text{ in } \ L^\infty([0,T^*]; W^{1,q}(\mathbb{R}^3)), \\ (\psi^\delta, u^\delta) &\rightharpoonup (\psi, u) \quad \text{weakly*} \ \text{ in } \ L^\infty([0,T^*]; H^2(\mathbb{R}^3)), \\ (\psi_t^\delta, u_t^\delta) &\rightharpoonup (\psi_t, u_t) \quad \text{weakly} \ \text{ in } \ L^2([0,T^*]; H^1(\mathbb{R}^3)). \end{aligned} \tag{3.75}$$

Moreover, due to the compact property in [24], there exists a subsequence of solutions $(\rho^\delta, u^\delta, \psi^\delta)$ satisfying:

$$(\rho^\delta, u^\delta, \psi^\delta) \to (\rho, u, \psi) \text{ in } \ C([0,T^*]; H^1(\mathbb{K})), \tag{3.76}$$

where $\mathbb{K}$ is any compact subset of $\mathbb{V}$.

From the lower semi-continuity of norms, we know that (ρ, u, ψ) also satisfies the estimates (3.70)-(3.71). Then it is easy to show (ρ, u, ψ) is a weak solution in the sense of distribution and satisfies:

$$\begin{aligned} &\rho \in L^\infty([0,T^*]; W^{1,q}), \ \rho_t \in L^\infty([0,T^*]; L^q), \\ &(\psi, u) \in L^\infty([0,T^*]; H^2) \cap L^2([0,T^*]; W^{2,q}), \quad \psi \geq \frac{1}{2}\underline{\psi}, \\ &(\sqrt{\rho}\psi_t, \sqrt{\rho}u_t) \in L^\infty([0,T^*]; L^2), \quad (\psi_t, u_t) \in L^2([0,T^*]; H^1). \end{aligned} \tag{3.77}$$

Step 2 Uniqueness. Let (ρ_1, u_1, ψ_1) and (ρ_2, u_2, ψ_2) be two solutions obtained above. Then $\rho_1 = \rho_2$ can be obtained by the same method used in Lemma 3.1. Letting $\overline{\psi} = \psi_1 - \psi_2$, from equation $(3.72)_2$, we have

$$\rho\overline{\psi}_t + \rho w \cdot \nabla\overline{\psi} + a_1\rho\overline{\psi}\mathrm{div}w - a_2\phi^{\frac{b}{b+1}}\triangle\overline{\psi} = 0.$$

Then multiplying the above equations by $\overline{\psi}$ and integrating over $\mathbb{V}$, we have

$$\frac{1}{2}\frac{d}{dt}|\overline{\psi}|_2^2 + a_4\int|\nabla\overline{\psi}|^2 \leq C|\overline{\psi}|_2^2, \tag{3.78}$$

which, along with $\nabla\overline{\psi} \cdot n|_{\partial\mathbb{V}} = 0$, immediately means that $\psi_1 = \psi_2$. Via the similar argument, we can show that $u_1 = u_2$.

Step 3 The time-continuity. The continuity of ρ can be obtained via the same method as in Lemma 3.1. Similarly, from (3.77), we have

$$(\psi, u) \in C([0,T^*]; H^1) \cap C([0,T^*]; D^2 - \text{weak}).$$

From equations (3.72) and(3.77), we know that

$$(\rho\psi_t, \rho u_t) \in L^2([0,T^*]; L^2), \text{ and } ((\rho\psi_t)_t, (\rho u_t)_t) \in L^2([0,T^*]; H^{-1}).$$

Thus from Aubin-Lions lemma, we have $(\rho\psi_t, \rho u_t) \in C([0,T^*]; L^2)$. Due to

$$\begin{cases} -a_2\phi^{\frac{b}{b+1}}\triangle\psi = -(\rho\psi_t + \rho w \cdot \nabla\psi + a_1\rho\psi\mathrm{div}w) + a_3\phi^{\frac{b}{b+1}}Q(w), \\ \mathrm{div}\mathbb{T} = \rho u_t + \rho w \cdot \nabla u + \nabla\Big(R\rho\psi^{\frac{1}{b+1}}\Big), \end{cases} \tag{3.79}$$

and the elliptic regularity estimates in Lemma 2.3, we have $(u,\psi)\in C([0,T^*];D^2)$. □

3.5. **Strong convergence in L^2 space.** In this subsection, we will give the proof for Theorem 3.1, which is based on some classical iteration scheme. Let us denote as in Section 3.3:

$$2+\|\rho_0\|_{W^{1,q}}+\|(u_0,\psi_0)\|_2+|(g_1,g_2)|_2\le c_0.$$

Next, let (u^0,ψ^0) be the solutions to the following linear problems

$$\begin{cases} u_t^0-\triangle u^0=0;\ u^0(0)=u_0 \text{ in } \mathbb{V};\ u|_{\partial\mathbb{V}}=0,\\ \psi_t^0-\triangle\psi^0=0;\ \psi^0(0)=\psi_0 \text{ in } \mathbb{V};\ \nabla\psi\cdot n|_{\partial\mathbb{V}}=0.\end{cases}$$

Then we can choose a time $T^{**}\in(0,T^*)$ such that

$$\begin{aligned}\sup_{0\le t\le T^{**}}\|u^0(t)\|_1^2+\int_0^{T^{**}}\Big(|u^0|_{D^{2,q}}^2+\|u_t^0\|_1^2\Big)\mathrm{d}t\le& c_1^2,\\ \sup_{0\le t\le T^{**}}|u^0(t)|_{D^2}^2\le c_2^2,\quad \sup_{0\le t\le T^{**}}\|\psi^0(t)\|_1^2\le& c_3^2,\\ \int_0^{T^{**}}\Big(|\psi^0|_{D^{2,q}}^2+\|\psi_t^0\|_1^2\Big)\mathrm{d}t\le c_4^2,\quad \sup_{0\le t\le T^{**}}|\psi^0(t)|_{D^2}^2\le& c_5^2.\end{aligned}$$

Proof. Step 1. Existence. Letting $(w,\phi)=(u^0,\psi^0)$, we can get (ρ^1,u^1,ψ^1) as a strong solution to (3.72). Then we construct approximate solutions $(\rho^{k+1},u^{k+1},\psi^{k+1})$ inductively as follows: assuming (u^k,ψ^k) was defined for $k\ge1$, let $(\rho^{k+1},u^{k+1},\psi^{k+1})$ be the solution to (3.72) with (w,ϕ) replaced by (u^k,ψ^k) as following:

$$\begin{cases} \rho_t^{k+1}+\mathrm{div}(\rho^{k+1}u^k)=0,\\ \rho^{k+1}\psi_t^{k+1}+\rho^{k+1}u^k\cdot\nabla\psi^{k+1}+a_1\rho^{k+1}\psi^{k+1}\mathrm{div}u^k\\ =a_2(\psi^k)^{\frac{b}{b+1}}\triangle\psi^{k+1}+a_3(\psi^k)^{\frac{b}{b+1}}Q(u^k),\\ \rho^{k+1}u_t^{k+1}+\rho^{k+1}u^k\cdot\nabla u^{k+1}+\nabla(P^*)^{k+1}-\mathrm{div}\mathbb{T}(u^{k+1})=0,\end{cases}\tag{3.80}$$

where $(P^*)^{k+1}=R\rho^{k+1}(\psi^{k+1})^{\frac{1}{b+1}}$. The initial date is given by

$$(\rho^{k+1},u^{k+1},\psi^{k+1})|_{t=0}=(\rho_0(x),u_0(x),\psi_0(x)),\quad x\in\mathbb{R}^3.$$

Then from Subsection 3.4, we know that the solution sequences (ρ^k,u^k,ψ^k) also satisfy the a prior estimates (3.70) and (3.74).

Next, we show that (ρ^k,u^k,ψ^k) converges to a limit in a strong sense. Denote

$$\overline{\rho}^{k+1}=\rho^{k+1}-\rho^k,\ \overline{u}^{k+1}=u^{k+1}-u^k,\ \overline{\psi}^{k+1}=\psi^{k+1}-\psi^k,$$

then we have

$$
\begin{aligned}
&\overline{\rho}_t^{k+1}+\mathrm{div}(\overline{\rho}^{k+1}u^k)+\mathrm{div}(\rho^k\overline{u}^k)=0,\\
&\rho^{k+1}\overline{\psi}_t^{k+1}+\rho^{k+1}u^k\cdot\nabla\overline{\psi}^{k+1}-a_2(\psi^k)^{\frac{b}{b+1}}\triangle\overline{\psi}^{k+1}\\
=&a_2\big((\psi^k)^{\frac{b}{b+1}}-(\psi^{k-1})^{\frac{b}{b+1}}\big)\triangle\psi^k+a_3\big((\psi^k)^{\frac{b}{b+1}}-(\psi^{k-1})^{\frac{b}{b+1}}\big)Q(u^k)\\
&+a_3(\psi^{k-1})^{\frac{b}{b+1}}(Q(u^k)-Q(u^{k-1}))-\overline{\rho}^{k+1}(\psi_t^k+u^{k-1}\cdot\nabla\psi^k)\\
&-a_1\overline{\rho}^{k+1}\psi^k\mathrm{div}u^{k-1}-\rho^{k+1}\Big(\overline{u}^k\cdot\nabla\psi^k+a_1\overline{\psi}^{k+1}\mathrm{div}u^k+a_1\psi^k\mathrm{div}\overline{u}^k\Big),\\
&\rho^{k+1}\overline{u}_t^{k+1}+\rho^{k+1}u^k\cdot\nabla\overline{u}^{k+1}-\mathrm{div}\mathbb{T}(\overline{u}^{k+1})\\
=&\overline{\rho}^{k+1}(-u_t^k-u^{k-1}\cdot\nabla u^k)-\rho^{k+1}\overline{u}^k\cdot\nabla u^k\\
&-R\nabla\Big(\rho^{k+1}\big((\psi^{k+1})^{\frac{1}{b+1}}-(\psi^k)^{\frac{1}{b+1}}\big)+\overline{\rho}^{k+1}(\psi^k)^{\frac{1}{b+1}}\Big).
\end{aligned}
\tag{3.81}
$$

First, multiplying $(3.81)_1$ by $\overline{\rho}^{k+1}$ and integrating over $\mathbb{V}$, for constant $0<\eta\le\frac{1}{10}$, we have

$$
\frac{d}{dt}|\overline{\rho}^{k+1}|_2^2\le C\Big(|u^k|_{D^{2,q}}^2+\eta^{-1}+1\Big)|\overline{\rho}^{k+1}|_2^2+\eta|\nabla\overline{u}^k|_2^2,
\tag{3.82}
$$

Second, multiplying $(3.81)_2$ by $\overline{\psi}^{k+1}$ and integrating over $\mathbb{V}$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2+a_2\int(\psi^k)^{\frac{b}{b+1}}|\nabla\overline{\psi}^{k+1}|^2\\
=&\int\Big(-a_2\nabla(\psi^k)^{\frac{b}{b+1}}\cdot\nabla\overline{\psi}^{k+1}+a_2\big((\psi^k)^{\frac{b}{b+1}}-(\psi^{k-1})^{\frac{b}{b+1}}\big)\triangle\psi^k\Big)\overline{\psi}^{k+1}\\
&+\int a_3\Big(\big((\psi^k)^{\frac{b}{b+1}}-(\psi^{k-1})^{\frac{b}{b+1}}\big)Q(u^k)+(\psi^{k-1})^{\frac{b}{b+1}}\big(Q(u^k)-Q(u^{k-1})\big)\Big)\overline{\psi}^{k+1}\\
&-\int\overline{\rho}^{k+1}\Big(\psi_t^k+u^{k-1}\cdot\nabla\psi^k+a_1\psi^k\mathrm{div}u^{k-1}\Big)\overline{\psi}^{k+1}\\
&-\int\rho^{k+1}\Big(\overline{u}^k\cdot\nabla\psi^k+a_1\overline{\psi}^{k+1}\mathrm{div}u^k+a_1\psi^k\mathrm{div}\overline{u}^k\Big)\overline{\psi}^{k+1}\equiv:\sum_{i=14}^{23}I_i.
\end{aligned}
$$

According to Hölder's inequality, Lemma 2.1 and Young's inequality we have

$$
\begin{aligned}
I_{14}=&-a_2\int\nabla(\psi^k)^{\frac{b}{b+1}}\cdot\nabla\overline{\psi}^{k+1}\overline{\psi}^{k+1}\\
\le&C\int_{V_{R_0}}|\nabla\psi^k||\nabla\overline{\psi}^{k+1}||\overline{\psi}^{k+1}|\mathrm{d}x+C\int_{\mathbb{V}/V_{R_0}}|\nabla\psi^k||\nabla\overline{\psi}^{k+1}||\overline{\psi}^{k+1}|\mathrm{d}x\\
\le&C|\nabla\psi^k|_6|\nabla\overline{\psi}^{k+1}|_2|\overline{\psi}^{k+1}|_6|V_{R_0}|^{\frac{1}{6}}\\
&+C|\nabla\psi^k|_6|\nabla\overline{\psi}^{k+1}|_2|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^{\frac{1}{2}}|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_6^{\frac{1}{2}}\\
\le&\frac{a_4}{20}|\nabla\overline{\psi}^{k+1}|_2^2+C|\nabla\psi^k|_6^2|V_{R_0}|^{\frac{1}{3}}\big(|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2+(1+|\rho^{k+1}|_2)^2|\nabla\overline{\psi}^{k+1}|_2^2\big)\\
&+C|\nabla\psi^k|_6^2|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_6\\
\le&\Big(\frac{a_4}{10}+Cc_5^4|V_{R_0}|^{\frac{1}{3}})|\nabla\overline{\psi}^{k+1}|_2^2+C(c_5^2|V_{R_0}|^{\frac{1}{3}}+c_5^7)|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2,
\end{aligned}
\tag{3.83}
$$

where we have used the Poincaré type inequality (see Lemma 2.2) for $|\overline{\psi}^{k+1}|_6$. Similarly,

$$
\begin{aligned}
I_{15} =& \int a_2\big((\psi^k)^{\frac{b}{b+1}} - (\psi^{k-1})^{\frac{b}{b+1}}\big)\triangle\psi^k\overline{\psi}^{k+1} \\
\leq& C\int_{V_{R_0}} |\triangle\psi^k||\overline{\psi}^k||\overline{\psi}^{k+1}|\mathrm{d}x + C\int_{\mathbb{V}/V_{R_0}} |\triangle\psi^k||\overline{\psi}^k||\overline{\psi}^{k+1}|\mathrm{d}x \\
\leq& C|\triangle\psi^k|_2|\overline{\psi}^k|_6|\overline{\psi}^{k+1}|_6|V_{R_0}|^{\frac{1}{6}} + C|\triangle\psi^k|_2|\overline{\psi}^k|_6|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^{\frac{1}{2}}|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_6^{\frac{1}{2}} \\
\leq& \eta|\overline{\psi}^k|_6^2 + C\eta^{-1}|\triangle\psi^k|_2^2|V_{R_0}|^{\frac{1}{3}}\big(|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2 + (1+|\rho^{k+1}|_2)^2|\nabla\overline{\psi}^{k+1}|_2^2\big) \\
&+ C\eta^{-1}|\triangle\psi^k|_2^2|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_6 \\
\leq& \eta|\overline{\psi}^k|_6^2 + \Big(\frac{a_4}{10} + C\eta^{-1}c_5^4|V_{R_0}|^{\frac{1}{3}}\Big)|\nabla\overline{\psi}^{k+1}|_2^2 \\
&+ C(\eta^{-1}c_5^2|V_{R_0}|^{\frac{1}{3}} + c_5^7\eta^{-2} + 1)|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2, \\
I_{16} =& a_3\int\Big((\psi^k)^{\frac{b}{b+1}} - (\psi^{k-1})^{\frac{b}{b+1}}\Big)Q(u^k)\overline{\psi}^{k+1} \\
\leq& C\int_{V_{R_0}} |\overline{\psi}^k||\overline{\psi}^{k+1}||\nabla u^k|^2\mathrm{d}x + C\int_{\mathbb{V}/V_{R_0}} |\overline{\psi}^k||\overline{\psi}^{k+1}||\nabla u^k|^2\mathrm{d}x \\
\leq& C|\nabla u^k|_3|\nabla u^k|_6|\overline{\psi}^k|_6|\overline{\psi}^{k+1}|_6|V_{R_0}|^{\frac{1}{6}} \\
&+ C|\overline{\psi}^k|_6|\nabla u^k|_3|\nabla u^k|_6|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^{\frac{1}{2}}|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_6^{\frac{1}{2}} \\
\leq& \eta|\overline{\psi}^k|_6^2 + C\eta^{-1}|\nabla u^k|_3^2|\nabla u^k|_6^2|V_{R_0}|^{\frac{1}{3}}\big(|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2 + (1+|\rho^{k+1}|_2)^2|\nabla\overline{\psi}^{k+1}|_2^2\big) \\
&+ C\eta^{-1}|\nabla u^k|_3^2|\nabla u^k|_6^2|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_6 \\
\leq& \eta|\overline{\psi}^k|_6^2 + \Big(\frac{a_4}{10} + Cc_2^6\eta^{-1}|V_{R_0}|^{\frac{1}{3}}\Big)|\nabla\overline{\psi}^{k+1}|_2^2 \\
&+ C(c_2^4\eta^{-1}|V_{R_0}|^{\frac{1}{3}} + c_2^{11}\eta^{-2} + 1)|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2, \\
I_{17} =& a_3\int(\psi^{k-1})^{\frac{b}{b+1}}\Big(Q(u^k) - Q(u^{k-1})\Big)\overline{\psi}^{k+1} \\
\leq& C|\psi^{k-1}|_\infty^{\frac{b}{b+1}}|\nabla u^k + \nabla u^{k-1}|_3|\nabla\overline{u}^k|_2|\overline{\psi}^{k+1}|_6 \leq C|\nabla\overline{u}^k|_6^2 + \frac{a_4}{10}|\nabla\overline{\psi}^{k+1}|_2^2, \\
I_{18}+I_{19}+I_{20} =& -\int\overline{\rho}^{k+1}\Big(\psi_t^k + u^{k-1}\cdot\nabla\psi^k + a_1\psi^k\mathrm{div}u^{k-1}\Big)\overline{\psi}^{k+1} \\
\leq& C|\overline{\rho}^{k+1}|_2|\psi_t^k|_3|\overline{\psi}^{k+1}|_6 + C|\overline{\rho}^{k+1}|_2|\overline{\psi}^{k+1}|_6\|\psi^k\|_2\|u^{k-1}\|_2 \\
\leq& \frac{a_4}{20}(|\nabla\overline{\psi}^{k+1}|_2^2 + |\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2) + C(1+|\psi_t^k|_3^2)|\overline{\rho}^{k+1}|_2^2, \\
I_{21} =& -\int\rho^{k+1}\overline{u}^k\cdot\nabla\psi^k\overline{\psi}^{k+1} \\
\leq& C|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2|\rho^{k+1}|_\infty^{\frac{1}{2}}|\overline{u}^k|_6\|\nabla\psi^k\|_1 \leq C|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2 + C|\nabla\overline{u}^k|_2^2,
\end{aligned}
\tag{3.84}
$$

$$
\begin{aligned}
I_{22}+I_{23} =& -a_1\int \rho^{k+1}\Big(\overline{\psi}^{k+1}\mathrm{div}u^k+\psi^k\mathrm{div}\overline{u}^k\Big)\overline{\psi}^{k+1}\\
\leq& C|\mathrm{div}u^k|_{W^{1,q}}|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2+C|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2|\rho^{k+1}|_\infty^{\frac{1}{2}}|\nabla\overline{u}^k|_2|\psi^k|_\infty\\
\leq& C\Big(1+\|\mathrm{div}u^k\|_{W^{1,q}}\Big)|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2+C|\nabla\overline{u}^k|_2^2.
\end{aligned}
\tag{3.85}
$$

Then combining the estimates for I_i $(i=14,...,23)$, for $t\in[0,T^{**}]$, we have

$$
\begin{cases}
\dfrac{d}{dt}|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2+a_2|(\psi^k)^{\frac{b}{2(b+1)}}\nabla\overline{\psi}^{k+1}|_2^2\\
\leq E_{1\eta}^k(t)|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2+E_2^k(t)|\overline{\rho}^{k+1}|_2^2+\eta(|\nabla\overline{\psi}^k|_2^2+|\sqrt{\rho}^k\overline{\psi}^k|_2^2)\\
\quad+\Big(\dfrac{a_4}{2}+C\eta^{-1}c_5^6|V_{R_0}|^{\frac{1}{3}})|\nabla\overline{\psi}^{k+1}|_2^2+C|\nabla\overline{u}^k|_2^2,\\
E_{1\eta}^k(t)=C\Big(1+\eta^{-2}+|u^k|_{D^{2,q}}+\eta^{-1}|V_{R_0}|^{\frac{1}{3}}\Big),\quad E_2^k(t)=C(1+|\psi_t^k|_3^2).
\end{cases}
\tag{3.86}
$$

Finally, multiplying $(3.81)_3$ by $\overline{u}^{k+1}$ and integrating over $\mathbb{V}$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\sqrt{\rho}^{k+1}\overline{u}^{k+1}|_2^2+\int\Big(\mu|\nabla\overline{u}^{k+1}|^2+(\lambda+\mu)|\mathrm{div}\overline{u}^{k+1}|_2^2\Big)\\
=&\int\Big(-\overline{\rho}^{k+1}(u_t^k+u^{k-1}\cdot\nabla u^k)\cdot\overline{u}^{k+1}-\rho^{k+1}(\overline{u}^k\cdot\nabla u^k)\cdot\overline{u}^{k+1}\Big)\\
&-\int R\nabla\Big(\rho^{k+1}\big((\psi^{k+1})^{\frac{1}{b+1}}-(\psi^k)^{\frac{1}{b+1}}\big)+\overline{\rho}^{k+1}(\psi^k)^{\frac{1}{b+1}}\Big)\cdot\overline{u}^{k+1}\equiv:\sum_{i=24}^{28}I_i.
\end{aligned}
$$

According to the Lemma 2.1, Hölder's inequality and Young's inequality, we have

$$
\begin{aligned}
I_{24}+I_{25}=&\int-\overline{\rho}^{k+1}(u_t^k+u^{k-1}\cdot\nabla u^k)\cdot\overline{u}^{k+1}\\
\leq& C|\overline{\rho}^{k+1}|_2|\nabla\overline{u}^{k+1}|_2(|u_t^k|_3+|\nabla u^k|_6|\nabla u^{k-1}|_6)\\
\leq&\frac{\mu}{20}|\nabla\overline{u}^{k+1}|_2^2+C(1+|u_t^k|_3^2)|\overline{\rho}^{k+1}|_2^2,\\
I_{26}=&-\int\rho^{k+1}(\overline{u}^k\cdot\nabla u^k)\cdot\overline{u}^{k+1}\\
\leq& C|\rho^{k+1}|_\infty^{\frac{1}{2}}|\sqrt{\rho}^{k+1}\overline{u}^{k+1}|_2|\nabla u^k|_3|\nabla\overline{u}^k|_2\\
\leq& C\eta^{-1}|\sqrt{\rho}^{k+1}\overline{u}^{k+1}|_2^2+\eta|\nabla\overline{u}^k|_2^2,\\
I_{27}+I_{28}=&-\int R\nabla\Big(\rho^{k+1}\big((\psi^{k+1})^{\frac{1}{b+1}}-(\psi^k)^{\frac{1}{b+1}}\big)+\overline{\rho}^{k+1}(\psi^k)^{\frac{1}{b+1}}\Big)\cdot\overline{u}^{k+1}\\
\leq& C|\nabla\overline{u}^{k+1}|_2(|\rho^{k+1}|_\infty^{\frac{1}{2}}|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2+|\overline{\rho}^{k+1}|_2|\psi^k|_\infty)\\
\leq&\frac{\mu}{20}|\nabla\overline{u}^{k+1}|_2^2+C(|\overline{\rho}^{k+1}|_2^2+|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2).
\end{aligned}
\tag{3.87}
$$

Then combining the above estimates for I_i $(i=24,...,28)$, we have

$$\begin{aligned}&\frac{d}{dt}|\sqrt{\rho}^{k+1}\overline{u}^{k+1}|_2^2+\frac{\mu}{2}|\nabla\overline{u}^{k+1}|_2^2\\ \leq& C\big(1+\eta^{-1}\big)(|\sqrt{\rho}^{k+1}\overline{u}^{k+1}|_2^2+|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2)+F^k(t)|\overline{\rho}^{k+1}|_2^2+\eta|\nabla\overline{u}^k|_2^2,\end{aligned}\tag{3.88}$$

where the term $F^k(t)=F^k(t)=C(1+|u_t^k|_3^2)$.

Now, let $\epsilon>0$ be a sufficiently small constant and denote

$$\begin{aligned}\Lambda^{k+1}(T^{**},\eta,\epsilon)=&\sup_{0\leq t\leq T^{**}}|\overline{\rho}^{k+1}(t)|_2^2+\epsilon\sup_{0\leq t\leq T^{**}}|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}(t)|_2^2\\ &+\sup_{0\leq t\leq T^{**}}|\sqrt{\rho}^{k+1}\overline{u}^{k+1}(t)|_2^2,\end{aligned}$$

then letting $|V_{R_0}|^{\frac{1}{3}}\leq a_4\eta(Cc_5^6)^{-1}$, from (3.82)-(3.88) and Gronwall's inequality, we have

$$\begin{aligned}&\Lambda^{k+1}(T^{**},\eta,\epsilon)+\int_0^{T^{**}}\Big(\frac{a_4}{4}\epsilon|\nabla\overline{\psi}^{k+1}|_2^2+\frac{\mu}{2}|\nabla\overline{u}^{k+1}|_2^2\Big)\mathrm{d}t\\ \leq&\int_0^{T^{**}}G_{\eta,\epsilon}^k\Lambda^{k+1}(s,\eta,\epsilon)\mathrm{d}t+\int_0^{T^{**}}\Big(\eta\epsilon|\nabla\overline{\psi}^k|_2^2+\eta\epsilon|\sqrt{\rho}^k\overline{\psi}^k|_2^2+(\eta+C\epsilon)|\nabla\overline{u}^k|_2^2\Big)\mathrm{d}t\end{aligned}$$

for some G_η^k such that

$$\int_0^t G_\eta^k(s)\mathrm{d}s\leq C\big(1+\epsilon+\eta^{-2}t+\eta^{-1}|V_{R_0}|^{\frac{1}{3}}t\big)=f(C,t,\epsilon,\eta,R_0),\quad\text{for}\quad 0\leq t\leq T^{**}.$$

Then from Gronwall's inequality, we have

$$\begin{aligned}&\Lambda^{k+1}(T^{**},\eta,\epsilon)+\int_0^{T^{**}}\Big(\frac{a_4}{4}\epsilon|\nabla\overline{\psi}^{k+1}|_2^2+\frac{\mu}{2}|\nabla\overline{u}^{k+1}|_2^2\Big)\mathrm{d}t\\ \leq&\Big(\eta\epsilon T^{**}\sup_{0\leq t\leq T^{**}}|\sqrt{\rho}^k\overline{\psi}^k(t)|_2^2+\int_0^{T^{**}}\Big(\eta\epsilon|\nabla\overline{\psi}^k|_2^2+(\eta+C\epsilon)|\nabla\overline{u}^k|_2^2\Big)\mathrm{d}t\Big)\exp f(C,t,\epsilon,\eta,R_0).\end{aligned}$$

First, we can choose $0<\epsilon=\epsilon_0<1$ small enough such that

$$(1+C)\epsilon_0\exp(C+C\epsilon_0)\leq\min\Big(\frac{\mu}{32},\frac{1}{32}\Big);$$

second, we can choose $0<\eta=\eta_0$ small enough such that

$$(1+C)(\eta_0+\eta_0\epsilon_0)\exp\big(C+C\epsilon_0\big)\leq\Big(\frac{a_4\epsilon_0}{32},\frac{\mu}{32},\frac{\epsilon_0}{32},\frac{1}{32}\Big);$$

third, we can choose $T^{**}=T_*$ small enough such that

$$(1+\eta_0\epsilon_0T_*)\exp\big(C\eta_0^{-2}T_*\big)\leq 2;$$

at last, we can choose R_0 sufficiently small such that

$$\exp\big(C\eta_0^{-1}|V_{R_0}|^{\frac{1}{3}}T_*\big)\leq 2.$$

So, when $\Lambda^{k+1}=\Lambda^{k+1}(T_*,\eta_0,\epsilon_0)$, we have

$$\sum_{k=1}^{\infty}\Big(\frac{1}{2}\Lambda^{k+1}+\int_0^{T_*}\Big(\frac{a_4\epsilon_0}{8}|\nabla\overline{\psi}^{k+1}|_2^2+\frac{\mu}{8}|\nabla\overline{u}^{k+1}|_2^2\Big)\mathrm{d}s\Big)\leq C<+\infty.$$

Thus we know that the full consequence (ρ^k, u^k, ψ^k) converges to a limit (ρ, u, ψ) in the following strong sense:

$$\rho^k \to \rho \text{ in } L^\infty([0,T_*]; L^2(\mathbb{V})),\ (\psi^k, u^k) \to (\psi, u) \text{ in } L^2([0,T_*]; D^1(\mathbb{V})). \tag{3.89}$$

Due to the local uniform estimates (3.70) and (3.74), and the strong convergence in (3.89), it is easy to see that (ρ, u, ψ) is a weak solution in the sense of distribution. Via the lower semi-continuity of norms, we also have that (ρ, u, ψ) has the regularities (3.77).

Step 2. The uniqueness. Let (ρ_1, u_1, ψ_1) and (ρ_2, u_2, ψ_2) be two strong solutions to IBVP(3.2)-(3.3) with (3.4) satisfying the regularity (3.77). We denote that

$$\overline{\rho} = \rho_1 - \rho_2, \quad \overline{u} = u_1 - u_2, \quad \overline{\psi} = \psi_1 - \psi_2.$$

Similarly to the derivations of (3.82)-(3.88), let

$$\Lambda(t) = |\overline{\rho}|_2^2 + |\sqrt{\rho_1}\overline{\psi}|_2^2 + \overline{C}|\sqrt{\rho_1}\overline{u}|_2^2,$$

where $\overline{C} > 0$ is a sufficiently large constant, then

$$\frac{d}{dt}\Lambda(t) + \frac{1}{2}\overline{C}\mu|\nabla\overline{u}|_2^2 + |\nabla\overline{\psi}|_2^2 \le \Psi(t)\Lambda(t), \tag{3.90}$$

where

$$\int_0^t \Psi(s)ds \le C \quad \text{for} \quad t \in [0, T_*].$$

Then from the Gronwall's inequality and

$$\nabla\psi \cdot n|_{\partial\mathbb{V}} = 0, \quad u \cdot n = 0,$$

we deduce that $\overline{\rho} = \overline{u} = \overline{\psi} = 0$.

Step 3. The time-continuity can be obtained by the same method as in the proof of Lemma 3.7. Here we omit the details. □

3.6. **Proof of Theorem 1.1.** Now we give the proof for Theorem 1.1.

Proof. First, from (3.1), IBVP (1.1)-(1.3) can be written into

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P = \mathrm{div}\mathbb{T}, \\ \rho\theta_t + \rho u \cdot \nabla\theta + \dfrac{1}{c_v}\big(R\rho\theta \mathrm{div} u - \nu \mathrm{div}(\theta^b \nabla\theta)\big) = \dfrac{1}{c_v}Q(u), \\ (\rho, u, \theta)|_{t=0} = (\rho_0(x), u_0(x), \theta_0(x)),\ x \in \mathbb{V}, \\ u|_{\partial\mathbb{V}} = 0, \quad \nabla\theta \cdot n|_{\partial\mathbb{V}} = 0. \end{cases} \tag{3.91}$$

Second, from Theorem 3.1, we know that IBVP

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla\Big(R\rho\psi^{\frac{1}{1+b}}\Big) = \mathrm{div}\mathbb{T}, \\ \rho\psi_t^{\frac{1}{1+b}} + \rho u \cdot \nabla\psi^{\frac{1}{1+b}} + \dfrac{1}{c_v}\big(R\rho\psi^{\frac{1}{1+b}}\mathrm{div}u - \nu\mathrm{div}(\psi^{\frac{b}{1+b}}\nabla\psi^{\frac{1}{1+b}})\big) = \dfrac{1}{c_v}Q(u), \\ (\rho, u, \psi^{\frac{1}{1+b}})|_{t=0} = (\rho_0(x), u_0(x), \psi_0^{\frac{1}{1+b}}(x) = \theta_0(x)),\ x \in \mathbb{V}, \\ u|_{\partial\mathbb{V}} = 0, \quad \nabla\psi^{\frac{1}{1+b}} \cdot n|_{\partial\mathbb{V}} = 0, \end{cases} \tag{3.92}$$

has an unique strong solution $(\rho, u, \psi^{\frac{1}{1+b}})$ belonging to the space Φ. From the uniqueness of the above IBVP (3.92) (see Section 3.5), we quickly have

$$(\rho, u, \theta) = (\rho, u, \psi^{\frac{1}{1+b}})$$

is our desired unique strong solution for (3.91), also IBVP(1.1)-(1.3). □

4. NECESSITY AND SUFFICIENCY OF THE COMPATIBILITY CONDITION

In this section, we need to prove Theorem 1.2.

Proof. Step 1. Necessity. Let (ρ, u, θ) be a strong solution on $[0, T_*] \times \mathbb{R}^3$ to (1.1)-(1.3) with the regularities shown in Definition 1.1. Then due to (1.1), we have

$$\begin{aligned} -\mathrm{div}\mathbb{T}(u) + \nabla P =& \sqrt{\rho}G_1, \\ -\frac{1}{c_v}\big(\nu(b+1)^{-1}\triangle\theta^{b+1} + Q(u)\big) =& \sqrt{\rho}G_2 \end{aligned} \tag{4.1}$$

for $0 \le t \le T_*$, where

$$G^1(t) = \sqrt{\rho}(-u_t - u \cdot \nabla u),\ G^2(t) = \sqrt{\rho}(-\theta_t - u \cdot \nabla\theta - R\theta\mathrm{div}u).$$

Since

$$(\sqrt{\rho}u_t, \sqrt{\rho}\theta_t, \sqrt{\rho}u \cdot \nabla u, \sqrt{\rho}u \cdot \nabla\theta, \sqrt{\rho}\theta\mathrm{div}u) \in L^\infty([0, T_*]; L^2),$$

we have

$$(G_1, G_2) \in L^\infty([0, T_*]; L^2).$$

So there exists a sequence $\{t_k\}$ $(t_k \to 0)$ such that

$$(G_(t_k), G_2(t_k)) \rightharpoonup (f, g) \quad \text{weakly*} \ \text{ in } \ L^2 \quad \text{for some} \quad (f, g) \in L^2.$$

So, let $t = t_k \to 0$ in (4.1), we obtain

$$\begin{aligned} \mathrm{div}\mathbb{T}(u(0, x)) + \nabla P(0, x) =& \sqrt{\rho}(0, x)f, \\ -\frac{1}{c_v}\big(\nu(b+1)^{-1}\triangle\theta^{b+1}(0, x) + Q(u(0, x))\big) =& \sqrt{\rho}(0, x)g. \end{aligned} \tag{4.2}$$

Combining with the strong convergence (1.16) and (4.2), we know that the necessity of the compatibility conditions are obtained. Moreover, from the construction of our strong solutions in Section 4, we easily deduce that

$$f = g_1, \quad g = g_2.$$

Step 2. To prove the sufficiency. Let (ρ_0, u_0, θ_0) be the initial data satisfying (1.12)-(1.13). Then there exists a unique solution (ρ, u, θ) to (1.1)- (1.3):

$$\rho(t,x) \in C([0,T_*]; W^{1,q}),\ (\theta, u)(t,x) \in C([0,T_*]; H^2(\mathbb{V})).$$

Then we only need to make sure that

$$\rho(0,x) = \rho_0,\ u(0,x) = u_0,\ \theta(0,x) = \theta_0,\ x \in \mathbb{V}.$$

From the weak formulation of the strong solution, we easily have

$$\rho(0,x) = \rho_0,\ \rho(0,x)u(0,x) = \rho_0 u_0,\ \rho(0,x)\theta(0,x) = \rho_0\theta_0,\ x \in \mathbb{V}.$$

It remains to prove that

$$u(0,x) = u_0(x), \quad \text{and} \quad \theta(0,x) = \theta_0(x)$$

when $x \in V$. Let

$$\overline{u}_0 = u_0 - u(0,x), \quad \overline{\psi}_0 = \theta_0^{b+1} - \theta^{b+1}(0,x).$$

According to the proof of the necessity, we know that $(\rho(0,x), u(0,x), \theta(0,x))$ also satisfies the relation (1.13) for $(g_1, g_2) \in L^2$. Then we quickly know that

$$(\overline{\psi}_0, \overline{u}_0) \in D_0^1(V) \cap D^2(V)$$

is the unique solution of the elliptic problem (1.15), and thus

$$\overline{u}_0 = 0, \quad \overline{\psi}_0 = 0, \quad \text{in} \quad V,$$

which implies that

$$u(0,x) = u_0(x),\ \theta(0,x) = \theta_0(x),\ x \in V.$$

□

5. Minimum principle.

In this section, we will give the proof for the minimum principle of θ shown in Theorem 1.3. Let (ρ, u, θ) on $[0,T] \times \mathbb{V}$ be the strong solution to IBVP (1.1)-(1.3), then we have

$$\begin{aligned}
\sup_{0\le t\le T} \|u(t)\|_2^2 + \operatorname{ess}\sup_{0\le t\le T} |\sqrt{\rho}u_t(t)|_2^2 + \int_0^{T^*} \Big(|u(s)|_{D^{2,q}}^2 + |u_t(s)|_{D^1}^2\Big)\mathrm{d}s \le & C,\\
\sup_{0\le t\le T} \|\theta(t)\|_2^2 + \operatorname{ess}\sup_{0\le t\le T} |\sqrt{\rho}\theta_t(t)|_2^2 + \int_0^{T^*} \Big(|\theta(s)|_{D^{2,q}}^2 + |\theta_t(s)|_{D^1}^2\Big)\mathrm{d}s \le & C,\\
\sup_{0\le t\le T} \big(|\rho|_\infty^2 + \|\rho(t)\|_{W^{1,q}}^2 + |\rho_t(t)|_q\big) \le & C,
\end{aligned} \tag{5.1}$$

where C is a generic constant depending only on (ρ_0, u_0, θ_0) and fixed constants μ, λ, ν, q, R, c_v, $|\mathbb{V}|$ and T.

Due to $\theta_0 \ge \underline{\theta} > 0$ and the continuity of θ, we know that there exists a time $T^o \in (0,T]$ and a constant $k_0 \in (0,1]$ such that

$$\theta(t,x) \ge k_0\underline{\theta} \quad \text{for} \quad [0,T^o] \times \Omega.$$

Next we will show that actually

$$k_0 = 1, \quad \text{and} \quad T^0 = T.$$

Otherwise, we assume that there exists some time $T^m \in (0,T]$ such that

$$T^m = \inf\{\ t \in (0,T]\ \big|\theta(t,x) = 0, \quad \text{for some} \quad x \in \mathbb{V}\},$$

which means that

$$\theta(t,x) > 0 \quad \text{for} \quad [0,T^m) \times \Omega. \tag{5.2}$$

Then based on (5.2), we state that

$$\theta(t,x) \geq \underline{\theta} \quad \text{for} \quad [0,T^m] \times \Omega, \tag{5.3}$$

which is impossible due to the defintion of T^m. So we quickly know that

$$\theta(t,x) > 0 \quad \text{for} \quad [0,T] \times \Omega. \tag{5.4}$$

Then via the completely same arguments used in the proof of (5.3), we will get the desired conclusions.

Next we will give the proof for the statement (5.3), which will be divided into four steps:

5.1. **Definition of functional U_k.** Let $m > b$ be a constant and $M^o \geq 1$ be a constant large enough. The size of M^o and the relation between m and b will be specified later. We define the ratio

$$K = 2m/(m-b) > 2. \tag{5.5}$$

Without lose of generality, we consider the case of $0 < \underline{\theta} \leq 1$ small enough, and the other cases could be obtained by scaling arguments. For the iteration technique used in our following proof, we need to introduce a sequence N_k:

$$N_0 = M^o, \quad N_{k+1} = N_k + \frac{M^o}{z^{k+1}}, \quad z \geq M^o, \tag{5.6}$$

which implies that

$$M \leq N_k \leq \frac{1}{1-\frac{1}{z}} M^o, \quad N_\infty = \lim_{k\to\infty} N_k = \frac{1}{1-\frac{1}{z}} M^o.$$

We also define

$$\begin{cases} \xi_k(\theta) = \left[\dfrac{1}{\theta^{\frac{m}{K}}} - N_k\right]_+, \\ \varphi_k(\theta) = \left[\dfrac{1}{\theta^m} - N_k^K\right]_+. \end{cases}$$

We can see that $\xi_k(\theta)$ and $\varphi_k(\theta)$ have the same supports, which can be denoted as

$$\mathbb{V}_k = \overline{\left\{(x,t)\big|\theta^m < 1/N_k^K\right\}}. \tag{5.7}$$

Here we have some relations between ξ_k and φ_k,

$$\xi_k(\theta) \leq \varphi_k(\theta), \quad \xi_k(\theta)^K \leq \varphi_k(\theta), \tag{5.8}$$

when $\underline{\theta}$ is sufficiently small. On the other hand, we have

$$\xi_{k-1} \geq M/z^k, \quad \text{in the region} \quad \mathbb{V}_k. \tag{5.9}$$

Now we define

$$U_k = \int_{\mathbb{V}} \rho\varphi_k(\theta)(t,x)\mathrm{d}x + \frac{1}{c_v}\int_0^t \int_{\mathbb{V}} m\left(\frac{Q(u)}{\theta^{m+1}} + \frac{(m+1)\kappa(\theta)|\nabla\theta|^2}{\theta^{m+2}}\right)\mathbf{1}_{\mathbb{V}_k}\mathrm{d}x\mathrm{d}s. \tag{5.10}$$

Then from the temperature equation $(1.1)_3$, via letting

$$\Big(\frac{1}{M^o}\Big)^{\frac{2}{m-b}}=\underline{\theta},$$

we quickly have

$$\begin{aligned}U_k=&-\frac{1}{c_v}\int_0^t\int_{\mathbb{V}}\frac{m}{\theta^{m+1}}R\rho\theta\mathbf{1}_{\mathbb{V}_k}\mathrm{div}u\mathrm{d}x\mathrm{d}s\\ \leq& C\int_0^t\int_{\mathbb{V}}\frac{\rho\theta}{\theta^{m+1}}|\mathrm{div}u|\mathbf{1}_{\mathbb{V}_k}\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{5.11}$$

We have the following bounds from the definition of U_k:

$$\|\rho\varphi_k\|_{L^\infty L^1}\leq CU_k,\quad \|\nabla\xi_k\|_{L^2L^2}\leq CU_k^{\frac{1}{2}}.\tag{5.12}$$

Next we need to control the right hand side term of (5.11) by some power of U_{k-1} such that we can use the De Giorgi iteration technique to get

$$\lim_{k\to\infty}U_k=0.$$

So the main task is the interpolation estimates of the right hand-side terms.

5.2. **Estimates for Iterations I.** In this section, we will give the estimates needed for Lemma 2.4 in the case when $b<1$.

The term in equation (5.11) could be controlled in this way:

$$\int_0^t\int_{\mathbb{V}}\frac{\rho\theta}{\theta^{m+1}}|\mathrm{div}u|\mathbf{1}_{\mathbb{V}_k}\mathrm{d}x\mathrm{d}s\leq\int_0^t\Big(\int_{\mathbb{V}}\Big(\delta_1\frac{|\nabla u|^2}{\theta^{m+1}}+C(\delta_1)\rho^2\theta^{-m+1}\mathbf{1}_{\mathbb{V}_k}\mathrm{d}x\Big)\mathrm{d}s,\tag{5.13}$$

where $\delta_1>0$ is a constant small enough and $C(\delta_1)>0$ is constant depending on δ_1 and the parameters of C. Then we just need to control the second term.

First using Hölder's inequality, we have

$$\begin{aligned}&\Big(\int_0^t\Big(\int_{\mathbb{V}}(\rho^{\mu_1}\xi_{k-1}^{\lambda_1K})^{q_1}\mathrm{d}x\Big)^{\frac{p_1}{q_1}}\mathrm{d}s\Big)^{\frac{1}{p_1}}\\ =&\Big(\int_0^t\Big(\int_{\mathbb{V}}\rho^{\mu_1q_1}\xi_{k-1}^{\mu_1q_1K}\cdot\xi_{k-1}^{(\lambda_1-\mu_1)q_1K}\mathrm{d}x\Big)^{\frac{p_1}{q_1}}\mathrm{d}s\Big)^{\frac{1}{p_1}}\\ \leq&\|\rho\xi_{k-1}^K\|_{L^\infty L^1}^{\mu_1}\|\xi_{k-1}\|_{L^2L^6}^{(\lambda_1-\mu_1)K},\end{aligned}\tag{5.14}$$

where the parameters satisfy

$$\frac{(\lambda_1-\mu_1)q_1K}{6}+\mu_1q_1=1,\quad \frac{(\lambda_1-\mu_1)p_1K}{2}=1.\tag{5.15}$$

Now we need to control the term $\|\xi_{k-1}\|_{L^2L^6}^{(\lambda_1-\mu_1)K}$. From Lemma 2.2, we have

$$|\xi_{k-1}|_6\leq C\big(|\rho\xi_{k-1}|_1+|\nabla\xi_{k-1}|_2\big),\tag{5.16}$$

which, along with (5.12), means that

$$\begin{aligned}\|\xi_{k-1}\|_{L^2L^6}^{(\lambda_1-\mu_1)K}\leq&C\big(\|\rho\xi_{k-1}\|_{L^2L^1}^{(\lambda_1-\mu_1)K}+\|\nabla\xi_{k-1}\|_{L^2L^2}^{(\lambda_1-\mu_1)K}\big)\\ \leq&C\Big(U_{k-1}^{(\lambda_1-\mu_1)K}+U_{k-1}^{\frac{(\lambda_1-\mu_1)K}{2}}\Big).\end{aligned}\tag{5.17}$$

Second we can choose m satisfying $b < m \leq 1$. Thus we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{V}} \rho^2 \theta^{-m+1} \mathbf{1}_{\mathbb{V}_k} \mathrm{d}x\mathrm{d}s \leq C \int_0^t \int_{\mathbb{V}} \rho^2 \frac{\xi_{k-1}^{\lambda_1 K}}{(M^o/z^k)^{\lambda_1 K}} \mathrm{d}x\mathrm{d}s \\
\leq & C \frac{1}{(M^o/z^k)^{\lambda_1 K}} \|\rho^{2-\mu_1}\|_{L^{p_2}L^{q_2}} \Big(\int_0^t \Big(\int_{\mathbb{V}} (\rho^{\mu_1} \xi_{k-1}^{\lambda_1 K})^{q_1} \mathrm{d}x \Big)^{\frac{p_1}{q_1}} \mathrm{d}s \Big)^{\frac{1}{p_1}} \\
\leq & C \frac{1}{(M^o/z^k)^{\lambda_1 K}} \|\rho^{2-\mu_1}\|_{L^{p_2}L^{q_2}} \|\rho \xi_{k-1}^K\|_{L^\infty L^1}^{\mu_1} \|\xi_{k-1}\|_{L^2L^6}^{(\lambda_1-\mu_1)K} \\
\leq & C \frac{1}{(M^o/z^k)^{\lambda_1 K}} \|\rho^{2-\mu_1}\|_{L^{p_2}L^{q_2}} \Big(U_{k-1}^{\mu_1+(\lambda_1-\mu_1)K} + U_{k-1}^{\mu_1+\frac{(\lambda_1-\mu_1)K}{2}} \Big),
\end{aligned} \tag{5.18}$$

where the positive indices also need to satisfy

$$\begin{aligned}
&\frac{(\lambda_1-\mu_1)q_1 K}{6} + \mu_1 q_1 = 1, \quad \frac{(\lambda_1-\mu_1)p_1 K}{2} = 1, \\
&\lambda_1 > \mu_1, \quad \mu_1 < 2, \quad \frac{1}{p_1} + \frac{1}{p_2} = 1, \quad \frac{1}{q_1} + \frac{1}{q_2} = 1, \\
&\lambda_1 K > 0, \quad \mu_1 + \frac{(\lambda_1-\mu_1)K}{2} > 1.
\end{aligned} \tag{5.19}$$

Denoting

$$x = \frac{(\lambda_1-\mu_1)K_1}{6}, \quad y = \mu_1, \tag{5.20}$$

due to (5.15) and (5.19), we need to consider the following stronger requirements:

$$0 < x + y < 1, \quad 3x + y > 1, \quad 3x < 1. \tag{5.21}$$

We just need to choose $x < \frac{1}{3}$ but very close to $\frac{1}{3}$, and y a positive number small enough. Thus, we have

$$U_k \leq C(z^k/M)^{\lambda_1 K} \Big(U_{k-1}^{\mu_1+(\lambda_1-\mu_1)K} + U_{k-1}^{\mu_1+\frac{(\lambda_1-\mu_1)K}{2}} \Big) \quad \text{for} \quad b \in (0,1). \tag{5.22}$$

Moreover, from the defintion of the functional U_k, $b < m \leq 1$ and (5.13), we also have the following estimate on U_k:

$$U_k \leq C(\delta_1) \int_0^t \Big(\int_{\mathbb{V}} \rho^2 \theta^{-m+1} \mathbf{1}_{\mathbb{V}_k} \mathrm{d}x \Big) \mathrm{d}s \leq C(\delta_1) \int_0^t \int_{\mathbb{V}} \rho^2 \mathrm{d}x\mathrm{d}s, \tag{5.23}$$

where $\delta_1 > 0$ is a constant small enough.

5.3. **Estimates for Iterations II.** In this section, we will finish the estimates needed for Lemma 2.4 in the case when $1 \leq b < \infty$.

The negative power of θ could be bounded in the following way:

$$\frac{1}{\theta^{m-1}} \mathbf{1}_{\mathbb{V}_k} \leq C \Big(\xi_{k-1}^{\frac{m-1}{\frac{m}{K}}} + (M^o)^{\frac{m-1}{\frac{m}{K}}} \Big). \tag{5.24}$$

So we have the estimates

$$\begin{aligned}&\int_0^t\int_{\mathbb{V}}\rho^2\theta^{-m+1}\mathbf{1}_{\mathbb{V}_k}\mathrm{d}x\mathrm{d}s\\ \leq& C\int_0^t\int_{\mathbb{V}}\rho^2\left(\frac{\xi_{k-1}^{\lambda_1K+\frac{m-1}{\frac{m}{K}}}}{(M^o/z^k)^{\lambda_1K}}+\frac{\xi_{k-1}^{\lambda_2K}}{(M^o/z^k)^{\lambda_2K}(M^o)^{-\frac{m-1}{\frac{m}{K}}}}\right)\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{5.25}$$

Let's define the constant K_1 as:

$$K_1=K+\frac{(m-1)}{\lambda_1m/K}.$$

Then the first term in inequality (5.25) could be estimated as

$$\begin{aligned}&\int_0^t\int_{\mathbb{V}}\rho^2\frac{\xi_{k-1}^{\lambda_1K+\frac{m-1}{\frac{m}{K}}}}{(M^o/z^k)^{\lambda_1K}}\mathrm{d}x\mathrm{d}s=\int_0^t\int_{\mathbb{V}}\rho^2\frac{\xi_{k-1}^{\lambda_1K_1}}{(M^o/z^k)^{\lambda_1K}}\mathrm{d}x\mathrm{d}s\\ \leq& C\frac{1}{(M^o/z^k)^{\lambda_1K}}\|\rho^{2-\mu_1}\|_{L^{p_2}L^{q_2}}\Big(\int_0^t\Big(\int_{\mathbb{V}}(\rho^{\mu_1}\xi_{k-1}^{\lambda_1K_1})^{q_1}\mathrm{d}x\Big)^{\frac{p_1}{q_1}}\mathrm{d}s\Big)^{\frac{1}{p_1}}\\ \leq& C\frac{1}{(M^o/z^k)^{\lambda_1K}}\|\rho^{2-\mu_1}\|_{L^{p_2}L^{q_2}}\|\rho\xi_1^K\|_{L^\infty L^1}^{\mu_1}\|\xi_{k-1}\|_{L^2L^6}^{(\lambda_1-\mu_1)K_1}\\ \leq& C\frac{1}{(M^o/z^k)^{\lambda_1K}}\|\rho^{2-\mu_1}\|_{L^{p_2}L^{q_2}}\Big(U_{k-1}^{\mu_1+(\lambda_1-\mu_1)K_1}+U_{k-1}^{\mu_1+\frac{(\lambda_1-\mu_1)K_1}{2}}\Big),\end{aligned}\tag{5.26}$$

where the positive index need to satisfy (5.19). While we can choose the parameters similarly as in Section 5.2. If (x,y) is still defined in (5.20), then we can also choose $x<\frac{1}{3}$ but x is very close to $\frac{1}{3}$, and y a positive number small enough.

Next we need more conditions on m and b later as we need to keep use the same m,K and b for the second term of this case,

$$\begin{aligned}&\int_0^t\int_{\mathbb{V}}\rho^2\frac{\xi_{k-1}^{\lambda_2K}}{(M^o/z^k)^{\lambda_2K}(M^o)^{-\frac{m-1}{\frac{m}{K}}}}\mathrm{d}x\mathrm{d}s\\ \leq& C\frac{1}{(M^o/z^k)^{\lambda_2K}(M^o)^{-\frac{m-1}{\frac{m}{K}}}}\|\rho^{2-\mu_2}\|_{L^{p_4}L^{q_4}}\Big(\int_0^t\Big(\int_{\mathbb{V}}(\rho^{\mu_2}\xi_{k-1}^{\lambda_2K})^{q_3}\mathrm{d}x\Big)^{\frac{p_3}{q_3}}\mathrm{d}s\Big)^{\frac{1}{p_3}}\\ \leq& C\frac{1}{(M^o/z^k)^{\lambda_2K}(M^o)^{-\frac{m-1}{\frac{m}{K}}}}\|\rho^{2-\mu_2}\|_{L^{p_4}L^{q_4}}\|\rho\xi^K\|_{L^\infty L^1}^{\mu_2}\|\xi_{k-1}\|_{L^2L^6}^{(\lambda_2-\mu_2)K}\\ \leq& C\frac{1}{(M^o/z^k)^{\lambda_2K}(M^o)^{-\frac{m-1}{\frac{m}{K}}}}\|\rho^{2-\mu_2}\|_{L^{p_4}L^{q_4}}\Big(U_{k-1}^{\mu_2+(\lambda_2-\mu_2)K}+U_{k-1}^{\mu_2+\frac{(\lambda_2-\mu_2)K}{2}}\Big),\end{aligned}\tag{5.27}$$

where the positive index need to satisfy

$$\begin{aligned}
&\frac{(\lambda_2-\mu_2)q_4K}{6}+\mu_1 q_2=1, \quad \frac{(\lambda_2-\mu_2)p_4K}{2}=1,\\
&\lambda_2>\mu_2, \quad \mu_2<2, \quad \frac{1}{p_3}+\frac{1}{p_4}=1, \quad \frac{1}{q_3}+\frac{1}{q_4}=1,\\
&\lambda_2 K>0, \quad \mu_2+\frac{(\lambda_2-\mu_2)K}{2}>1.
\end{aligned} \tag{5.28}$$

While we can choose the parameters similarly as in the above case. If

$$x=\frac{(\lambda_2-\mu_2)K}{6}, \quad y=\mu_2,$$

we need to choose $x<\frac{1}{3}$ but x is very close to $\frac{1}{3}$, and y a positive number small enough.

In summary, we have

$$\begin{aligned}
U_k \leq & C(z^k/M^o)^{s_1}\Big(U_{k-1}^{\mu_1+(\lambda_1-\mu_1)K_1}+U_{k-1}^{\mu_1+\frac{(\lambda_1-\mu_1)K_1}{2}}\\
&+U_{k-1}^{\mu_2+(\lambda_2-\mu_2)K}+U_{k-1}^{\mu_2+\frac{(\lambda_2-\mu_2)K}{2}}\Big)(M^o)^{\frac{(m-1)}{m/K}} \quad \text{for} \quad b\in[1,\infty),
\end{aligned} \tag{5.29}$$

where $s_1=\max\big(\lambda_1 K,\lambda_2 K\big)$.

5.4. **Continuity argument.** In this section, we will fix T, and show

$$\lim_{k\to\infty} U_k(t)=0. \tag{5.30}$$

When z is chosen properly, (5.30) will imply that

$$\theta\geq\Big(1-\frac{1}{z}\Big)^{\frac{K}{m}}\underline{\theta}. \tag{5.31}$$

Now we consider the following function

$$LG(t)=-\sup_{0<\tau<t}\Big|\frac{1}{\theta(\tau)}\Big|_{\infty}^{\frac{m}{K}}. \tag{5.32}$$

Notice that LG is monotone decreasing, and define

$$\begin{cases} t_1=\sup\limits_{LG(t)>-N_\infty}\{t\},\\ t_2=\inf\limits_{LG(t)<-N_\infty}\{t\},\end{cases}$$

i.e. for any $\epsilon>0$ small enough, there exists a $\delta(\epsilon)$, such that

$$\begin{cases} LG(t_2+\epsilon)<-N_\infty-\delta(\epsilon),\\ LG(t_1-\epsilon)>-N_\infty+\delta(\epsilon).\end{cases}$$

Obviously, $t_1\leq t_2$, as a fact, we could choose z so that they are the same:

Lemma 5.1. *We can also choose a $z_i\in(s,s+1)$ $(s\geq M)$ such that $t_2=t_1$.*

Proof. If for any z_i, there exists $t_1(z_i) \le t_2(z_i)$, then by the monotonicity of LG,

$$LG(t) = -\Big(1 - \frac{1}{z_i}\Big)M^o, \quad \text{for any} \quad t \in (t_1, t_2). \tag{5.33}$$

We want to show that $(t_1(z_a), t_2(z_a))$ and $(t_1(z_b), t_2(z_b))$ are two non-intersecting intervals for $z_a \neq z_b$. As a fact, if $z_a \neq z_b$, $\tau \in (t_1(z_a), t_2(z_a))$, and $\tau \in (t_1(z_b), t_2(z_b))$,

$$LG(\tau) = -\Big(1 - \frac{1}{z_a}\Big)M^o = -\Big(1 - \frac{1}{z_b}\Big)M^o, \tag{5.34}$$

which is impossible, so $(t_1(z_a), t_2(z_a))$ and $(t_1(z_b), t_2(z_b))$ are two non-intersecting intervals. So the total length of all these non-intersecting intervals is not larger than T^m,

$$\sum_{2<z_i<3} (t_2(z_i) - t_1(z_i)) \le T^m. \tag{5.35}$$

As a fact z_i is from uncountable set $(2, 3)$, so we have the existence of z_i. □

Next we divide the following proof into two parts:

5.4.1. *Case* $0 < b \le 1$. Now we first consider the case that $0 < b \le 1$, and we are ready to use the continuity argument to show the uniform bound for θ. First, we want to show $t_1 > 0$. Go back to the estimate (5.23):

$$U_k \le C(\delta_1) \int_0^t \int_{\mathbb{V}} \rho^2 \mathrm{d}x\mathrm{d}s. \tag{5.36}$$

So if $t \le \frac{1}{C_1}$ with

$$C_1 = \frac{1}{C(\delta_1)C^2C_2}.$$

then we have

$$U_k \le \frac{1}{C_2}, \quad \text{for} \quad C_2 \quad \text{large enough.}$$

Thus from Lemma 2.4 and (5.22), we have

$$\lim_{k\to\infty} U_k = 0,$$

which implies that $t_1 \ge \frac{1}{C_1}$.

Second, we want show that $t_1 = T_m$, i.e.

$$\theta \ge \Big(1 - \frac{1}{z}\Big)^{\frac{K}{m}} \underline{\theta}, \quad \text{for } t \in [0, T^m). \tag{5.37}$$

We can define $s_2 = t_1 - \frac{1}{2C_1}$. By the definition of t_1, we have $LG(s_2) > -N_\infty$. So when k large enough, say when $k > k_1(s_2)$, we have

$$LG(s_2) > -N_{k_1(s_2)} \ge -N_k > -N_\infty, \tag{5.38}$$

which means that when $s \in (0, s_2)$:

$$\theta \ge \Big(1 - \frac{1}{z}\Big)^{\frac{K}{m}} \underline{\theta}.$$

Then we have

$$U_k \leq C(\delta_1) \int_{s_2}^{t} \int_{\mathbb{V}} \rho^2 \mathbf{1}_{\mathbb{V}_k} \mathrm{d}x \mathrm{d}\tau. \tag{5.39}$$

As long as $t < s_2 + \frac{1}{C_1}$, we have $\lim_{k\to\infty} U_k = 0$. Thus

$$t_2 > s_2 + \frac{1}{C_1}, \tag{5.40}$$

which is contradicted to $t_2 = t_1$. Replacing $(s, s+1)$ in Lemma 5.1 with $(n, n+1)$ for sufficiently large constant n, we can also choose an increasing sequence of z_i such that $z_n \in (n, n+1)$. So we have

$$\lim_{n\to\infty} z_n = \infty, \quad \text{as} \quad n \to \infty.$$

So the desired minimum principle can be obtained via letting $z_n \to \infty$ in (5.31).

5.4.2. *Case* $b > 1$. From (5.11), we know

$$\begin{aligned} U_k \leq & C \int_0^t \int_{\mathbb{V}} \frac{\rho\theta}{\theta^{m+1}} |\mathrm{div} u| \mathbf{1}_{\mathbb{V}_k} \mathrm{d}x \mathrm{d}s \\ \leq & \frac{m}{2c_v} \int_0^t \int_{\mathbb{V}} \frac{|\mathrm{div} u|^2}{\theta^{m+1}} \mathbf{1}_{\mathbb{V}_k} \mathrm{d}x \mathrm{d}s + C \int_0^t \int_{\mathbb{V}} \rho^2 \theta (\varphi^k + N_k^K) \mathrm{d}x \mathrm{d}s \\ \leq & \frac{m}{2c_v} \int_0^t \int_{\mathbb{V}} \frac{|\mathrm{div} u|^2}{\theta^{m+1}} \mathbf{1}_{\mathbb{V}_k} \mathrm{d}x \mathrm{d}s + C \int_0^t U_k(s) \mathrm{d}s + CtN_k^K |\mathbb{V}|, \end{aligned} \tag{5.41}$$

which, along with (5.10), implies that

$$\begin{aligned} & \int_{\mathbb{V}} \rho \varphi_k(\theta)(t,x) \mathrm{d}x + \frac{1}{2c_v} \int_0^t \int_{\mathbb{V}} m \left(\frac{Q(u)}{\theta^{m+1}} + \frac{(m+1)\kappa(\theta)|\nabla\theta|^2}{\theta^{m+2}} \right) \mathbf{1}_{\mathbb{V}_k} \mathrm{d}x \mathrm{d}s \\ \leq & C \int_0^t U_k(s) \mathrm{d}s + CtN_k^K |\mathbb{V}|. \end{aligned} \tag{5.42}$$

Then from Gronwall's inequality, we have

$$U_k(t) \leq (CN_k^K |\mathbb{V}| t) \exp(Ct). \tag{5.43}$$

So if $t \leq \frac{1}{C_3}$ with

$$C_3 = \frac{1}{C(N_k^K |\mathbb{V}| C_4 + 1)},$$

then we have

$$U_k \leq \frac{2}{C_4}, \quad \text{for} \quad C_4 \quad \text{large enough.}$$

Thus from Lemma 2.4 and (5.29), we have

$$\lim_{k\to\infty} U_k = 0,$$

which implies that $t_1 \geq \frac{1}{C_3}$. Via the similar argument used in Subsection 5.4.1, we will obtain the desired conclusions.

6. Appendix

In this section, we specify some recursion relations, and the conditions for the sequence of functionals U_k to converge to 0. Moreover, we will give the proof for Lemma 2.4, which also can be seen in [28]. First, we have

Lemma 6.1. *If U_k is a nonnegative sequence such that*

$$U_{k+1} \leq P_3 z^{\beta_1 k} U_k^{\beta_2}, \tag{6.1}$$

where P_3 is a positive constant independent of β_1, β_2, k, z. $\beta_2 > 1$, $\beta_1 > 0$ and $z > 1$ are all constants. If U_0 is small enough (specified in equation (6.6)), we have $\lim_{k\to\infty} U_k = 0$.

Proof. Let's define

$$x_k = \log U_k. \tag{6.2}$$

Then we have the recursion relation

$$x_{k+1} \leq \log P_3 + \beta_1 k \log z + \beta_2 x_k. \tag{6.3}$$

We want to choose x_k going to negative infinity by geometric speed, so we want to show

$$x_k \leq x_0 \Big(\frac{\beta_2+1}{2}\Big)^k. \tag{6.4}$$

We will prove equation (6.4) by induction, for $k = 0$, the equation (6.4) is obvious, then if

$$x_l \leq x_0 \Big(\frac{\beta_2+1}{2}\Big)^l. \tag{6.5}$$

And we specify the x_0 by assuming

$$x_0 < -P_4 \leq -\max_{k\geq 0} \frac{\log P_3 + \beta_1 k \log z}{\frac{\beta_2-1}{2}(\frac{\beta_2+1}{2})^k}. \tag{6.6}$$

The constant P_4 depends only P_3, β_1, z and β_2. Then

$$\begin{aligned} x_{l+1} &\leq x_0 \Big(\frac{\beta_2+1}{2}\Big)^l \beta_2 + \log P_3 + \beta_1 l \log z \\ &\leq x_0 \Big(\frac{\beta_2+1}{2}\Big)^{l+1} + \frac{\beta_2-1}{2} x_0 \Big(\frac{\beta_2+1}{2}\Big)^l + \log P_3 + \beta_1 l \log z \\ &\leq x_0 \Big(\frac{\beta_2+1}{2}\Big)^{l+1}. \end{aligned} \tag{6.7}$$

So equation (6.4) is true for any positive integer k, and by letting $k \to \infty$ we finish the proof of lemma 6.1. One can notice that the constant $P_4(P_3, \beta_1, \beta_2, z)$ in the requiment of U_0 could be chosen continuously depending on z. □

Then we have the following variant:

Lemma 6.2. *If U_k is a nonnegative sequence such that*

$$U_{k+1} \leq P_5 z^{\beta_1 k} U_k^{\beta_2} M^{\beta_3}, \tag{6.8}$$

where P_5 is a positive constant independent of $\beta_1, \beta_2, \beta_3, k, M, z$. $z > 1$, $M > 1$, $\beta_2 > 1$, $\beta_1 > 0$ and $\beta_3 > 0$ are all constants. Then there exists a constant P_6 which is independent of M, and if

$$U_0 \leq P_6 \Big(\frac{1}{M}\Big)^{\frac{\beta_3}{\beta_2-1}}, \tag{6.9}$$

we have $\lim_{k\to\infty} U_k = 0$.

Proof. We can define $y_k = M^{\frac{\beta_3}{\beta_2-1}} U_k$, then we have $y_{k+1} \le P_5 z^{\beta_1 k} y_k^{\beta_2}$. □

As a direct consequence of the above lemma, we have

Lemma 6.3. *Let U_0 is a small enough positive number. $\gamma_1, \gamma_3 \ge 0$, $\gamma_2, \gamma_4 > 1$, and $z > 1$ are all constants. P_7 and P_8 are two constants independent of M. If*

$$U_k \le P_7 \Big(\frac{z^k}{M}\Big)^{\gamma_1} U_{k-1}^{\gamma_2} + P_8 \Big(\frac{1}{M}\Big)^{\gamma_3} U_{k-1}^{\gamma_4}, \tag{6.10}$$

then $\lim_{k\to\infty} U_k = 0$.

Finally, it is easy to see that Lemma 2.4 is a direct consequence of Lemmas 6.2-6.3.

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