UNIQUENESS AND REGULARITY OF CONSERVATIVE SOLUTION TO A WAVE SYSTEM MODELING NEMATIC LIQUID CRYSTAL

HONG CAI, GENG CHEN, AND YI DU

Abstract. In this paper, a system of wave equations modeling nematic liquid crystal is considered. The solution of this system in general has gradient blowup. The global-in-time existence of Hölder continuous energy conservative solution for the initial value problem was proved in [13] by the second author, P. Zhang & Y. Zheng. In this paper, we prove the uniqueness and generic regularity of energy conservative solution.

1. Introduction

In this paper, we study the uniqueness and generic regularity for energy conservative Hölder continuous solution to the system of wave equations

$$\partial_{tt}n_i - \partial_x(c^2(n_1)\partial_x n_i) = \big(-|\mathbf{n}_t|^2 + (2c^2-\lambda_i)|\mathbf{n}_x|^2\big)n_i, \qquad i=1,2,3, \tag{1.1}$$

on $\mathbf{n} = (n_1, n_2, n_3)$ with

$$|\mathbf{n}| = 1. \tag{1.2}$$

Here, the time t and space variables x belong to $\mathbb{R}^+$ and $\mathbb{R}$, respectively. The constants

$$\lambda_1 = \gamma > 0 \quad \text{and} \quad \lambda_2 = \lambda_3 = \alpha > 0.$$

The (positive) wave speed c depends on n_1 with

$$c^2(n_1) = \alpha + (\gamma - \alpha)n_1^2. \tag{1.3}$$

The initial data are

$$n_i|_{t=0} = n_{i0} \in H^1, \quad (n_i)_t|_{t=0} = n_{i1} \in L^2, \quad i = 1,2,3. \tag{1.4}$$

We briefly introduce the origin of system (1.1) from nematic liquid crystal. Liquid crystal is often viewed as an intermediate state between liquid and solid. It possesses none or partial positional order but displays an orientational order at the same time. For the nematic phase, the molecules float around as in a liquid phase, but have the tendency of aligning along a preferred direction due to their orientation. The mean orientation of the long molecules in a nematic liquid crystal is described by a director field of unit vectors, $\mathbf{n} \in \mathbb{S}^2$, the unit sphere. Associated with the director field $\mathbf{n}$, there is the well-known Oseen-Franck potential energy density W given by

$$W(\mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}\alpha(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}\beta(\mathbf{n}\cdot\nabla\times\mathbf{n})^2 + \frac{1}{2}\gamma\,|\mathbf{n}\times(\nabla\times\mathbf{n})|^2. \tag{1.5}$$

The positive constants α, β, and γ are elastic constants of the liquid crystal, corresponding to splay, twist, and bend, respectively.

There are many studies on the constrained elliptic system of equations for $\mathbf{n}$ derived through variational principles from the potential (1.5), and on the parabolic flow associated with it, see [2, 14, 17, 20, 22, 26] and references therein.

In the regime in which inertia effects dominate viscosity, the propagation of the orientation waves in the director field may then be modelled by the least action principle ([1, 23])

$$\frac{\delta}{\delta \mathbf{n}} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \left\{ \frac{1}{2} \partial_t \mathbf{n} \cdot \partial_t \mathbf{n} - W(\mathbf{n}, \nabla \mathbf{n}) \right\} d\mathbf{x}\, dt = 0, \qquad \mathbf{n} \cdot \mathbf{n} = 1. \tag{1.6}$$

When the space dimension is one (1-d), i.e. $x \in \mathbb{R}$, and when $\alpha = \beta$, system (1.6) exactly gives (1.1), on which we focus in this paper. There is a simpler case when $\mathbf{n} = (\cos u(x,t), \sin u(x,t), 0)$ (planar deformation). In this case, the function u satisfies

$$u_{tt} - c(u)(c(u)\, u_x)_x = 0, \tag{1.7}$$

with $c^2(u) = \gamma \cos^2 u + \alpha \sin^2 u$. See [13] and [11] for the derivations of (1.1) and (1.7).

Because of the strong nonlinearity, the solution for (1.6) fails to be Lipschitz continuous even for 1-d solution with C^∞ initial data, such as for solutions of (1.1) and (1.7). See [18] for an example with finite time gradient blowup. More precisely, the 1-d solution in general includes cusp singularity, which means that solution is only Hölder continuous, due to the energy concentration. This causes the following major difficulties in studying the existence, uniqueness and Lipschitz continuous dependence of global weak solution respectively:

- Classical solution in general does not exist. One has to study weak solutions.
- Solution in general is not unique. To select a unique solution, one needs a physical admissible condition, such as the energy conservation used in this paper. However, the energy conservation laws are only in the weak form.

  Another type of solutions are called dissipative solutions. See existence of dissipative solution for (1.7) with monotonic wave speed $c(\cdot)$ in [9, 27, 28].
- The energy conservative solution fails to be Lipschitz continuous in the natural Sobolev space from energy law, such as $(u, u_t) \in H^1 \times L^2$ for (1.7) and $(\mathbf{n}, \mathbf{n}_t) \in (H^1 \times L^2)^3$ for (1.1). To prove the Lipschitz continuous dependence, one needs to introduce a new metric measuring the distance of two solutions.

The first difficulty is obvious due to the singularity formation. The second and third difficulties, for the uniqueness and Lipschitz continuous dependence, can be clearly seen from the characteristic equations of (1.7) and (1.1) (c is expressed in terms of n_1)

$$\frac{d}{dt} x(t) = \pm c\big(u\big(t, x(t)\big)\big).$$

Here u is a Hölder continuous function with exponent $1/2$, although $c(\cdot)$ is a smooth function. So very loosely speaking, the characteristic equation behaves like the ODE

$$x'(t) = \lambda\, x^{\frac{1}{2}}. \tag{1.8}$$

When $x(0) = 0$, the solutions flows of (1.8) are not unique and the nonzero solution flow is not Lipschitz. For PDEs, this is corresponding to the situation when cusp singularity forms.

Another major difficulty for all well-posedness problems is that energy in one wave direction may transfer to another wave direction in wave interactions.

The research on global well-posedness of weak solutions for wave equations including cusp singularity was initiated from the variational wave equation (1.7). Currently, the global well-posedness of conservative solution for (1.7) has been fairly well understood after a sequence of papers: Global existence [11, 21], Uniqueness [8], Lipschitz continuous dependence under Finsler type transport metric [6], and generic regularity [5]. Here the generic regularity result, besides its own interest, serves as a key part in the Lipschitz continuous dependence result, because it shows that the regular enough transport plane exists. Ideas used in these works will be introduced in Subsection 2.1. We also refer the reader to a survey paper [4].

Now, one major open problem is how to study system of wave equations whose solutions include cusp singularities. In this paper, we will address this open problem by starting from the wave system (1.1). In [13], the global existence for Hölder continuous conservative solution for system (1.1), was established by the second author, Zhang and Zheng. In the current paper, we prove the following main results

- The uniqueness of conservation solution in Theorem 2.2.
- A generic regularity result in Theorem 2.3, which very roughly speaking tells that most of weak solutions (or generic solutions) are piecewise smooth for a.e. t.

The second result itself gives concrete generic structure of weak solutions, and will also serve as a key part in the future study on the Lipschitz continuous dependence of weak solution under some transport metric.

It is worth mentioning one key difficulty when extending the well-posedness theory from a scalar wave equation to a system of wave equations. In fact, for systems, one also needs to consider the possible transfer of energy between different components in the same wave direction. More precisely, the Riemann variables change from scalar functions R, S to vector valued functions $\mathbf{R}, \mathbf{S}$, so one needs to control both the exchange of energies between waves in two directions such as between $\mathbf{R}$ and $\mathbf{S}$, and between two components in the same direction, such as between R_1 and R_2. This issue will be treated in the proofs of Theorems 2.2 and 2.3.

The analysis of system (1.1) is also much more complex than the scalar wave equation (1.7). We consider this paper as a trumpet in studying the wave model of liquid crystal with more genuine liquid crystal structure, comparing to (1.7). In fact, equation (1.7), which can be derived from the least action principle:

$$\frac{\delta}{\delta u}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\frac{1}{2}\Big(u_t^2-c^2(u)\,u_x^2\Big)\,dx\,dt=0,$$

is a natural model from elasticity.

For 1-d solution of (1.6) with $\alpha\neq\beta$, there is currently no existence result. In [30], Zhang and Zheng proved a global existence result for a system, which can apply to the 1-d energy conservative weak solutions of (1.6) when $\alpha>\beta$, under an additional global-in-time assumption that $|n_1|$ is uniformly away from 1 for any $(t,x)\in(\mathbb{R}^+,\mathbb{R})$.

This paper is divided into 4 sections. In Section 2, after a survey of the key analytical ideas and existence result, main results in this paper will be introduced. Section 3 is on the uniqueness of conservative solution. Section 4 is on the generic regularity of weak solution.

2. Main ideas and results

2.1. **Main analytical ideas.** For readers' convenience, we summarize the main ideas used for the global well-posedness, which include ideas used in this paper toward the uniqueness and generic regularity for (1.1).

Existence. One first constructs and solves a semi-linear system. Then, after a reverse transformation, one obtains a solution for the original system. However, this method cannot rule out the possibility that solution can be constructed in another way. So it cannot give the uniqueness of original system, neither does the Lipschitz continuous dependence.

Uniqueness. The general idea is to use the energy conservation law in the weak form to first select a unique characteristic then reconstruct a semi-linear system, which has a unique solution.

So this is essentially an inverse process of the existence proof. However, the techniques needed are completely different from the existence proof.

In the first step, to select the unique characteristic, we need to introduce a pair of variables corresponding to the forward and backward energies, respectively, then some weighted distances, to control the increase of energy in each direction due to wave interaction. This idea was first introduced by Bressan-Chen-Zhang in [8] for (1.7). The proof also relies on a modified generalized characteristic idea originally used by Dafermos for Hunter-Saxton equation in [15] then by Bressan-Chen-Zhang for Camassa-Holm equation in [7]. In this step, the analysis for (1.1) is analog to [8].

The major difference in this paper comparing to [8] is in the second step: how to prove that all weak solutions satisfy a semi-linear system. In fact, the Riemann variables change from scalar functions R, S to vector valued functions $\mathbf{R}, \mathbf{S}$. One needs to control both the exchange of energies between waves in two directions such as between $\mathbf{R}$ and $\mathbf{S}$, and between two components in the same direction, such as between R_1 and R_2. We choose some dependent variables in (3.23) different from [8] to track the propagation of each component of $\mathbf{R}$ and $\mathbf{S}$. Accordingly, many new estimates are given in Subsection 3.2 comparing to [8].

Generic regularity and Lipschitz continuous dependence. In [6] and [5], Bressan and the second author established a Finsler type optimal transport metric and proved the Lipschitz continuous dependence of weak solution under this metric, although the solution flow fails to be Lipschitz continuous under standard Sobolev metric. To make the new transport metric well defined, or in another word to prove that regular enough transport path between two solutions exists, a generic regularity result was established in [5]. This result shows that the generic solutions, very loosely speaking piecewise smooth solutions, are dense in the space $(u, u_t) \in H^1 \times L^2$. We extend the generic regularity result to (1.1) in Theorem 2.3. The difficulty we meet and overcome in this project is similar to the uniqueness project.

There are some other generic regularity results, such as [16, 24] for hyperbolic conservation laws. Our proof is based on the analysis of solutions along characteristics, using the semi-linear equations, which is quite different from the results in [16, 24]. In the proof, we also use the Thom's transversality theorem [3, 5, 19].

This result gives concrete generic structure of weak solutions, i.e. there are only three types of generic singularities, for generic solutions: Starting and ending point of the singular curve, inner point on the singular curve and intersection point of singularity curves in two directions. And these singularities are all transversal.

This result will also serve as a key part in the future study on the Lipschitz continuous dependence of weak solutions.

2.2. **Existence results for (1.1).** In this part, we review the global existence result in [13]. We denote Riemann variables as

$$\begin{cases} \mathbf{R} = (R_1, R_2, R_3) \doteq \mathbf{n}_t + c\mathbf{n}_x, \\ \mathbf{S} = (S_1, S_2, S_3) \doteq \mathbf{n}_t - c\mathbf{n}_x, \end{cases} \tag{2.1}$$

and use the following notations

$$\mathbf{R}^2 = \mathbf{R} \cdot \mathbf{R}, \qquad \mathbf{S}^2 = \mathbf{S} \cdot \mathbf{S}.$$

Then (1.1) can be reformulated as:

$$
\left\{
\begin{aligned}
\partial_t R_i - c\partial_x R_i &= \frac{1}{4c^2}\big\{(c^2-\lambda_i)(\mathbf{R}^2+\mathbf{S}^2) - 2(3c^2-\lambda_i)\mathbf{R}\cdot\mathbf{S}\big\}n_1 \\
&\qquad + \frac{c'(n_1)}{2c(n_1)}(R_i - S_i)R_1, \\
\partial_t S_i + c\partial_x S_i &= \frac{1}{4c^2}\big\{(c^2-\lambda_i)(\mathbf{R}^2+\mathbf{S}^2) - 2(3c^2-\lambda_i)\mathbf{R}\cdot\mathbf{S}\big\}n_1 \\
&\qquad - \frac{c'(n_1)}{2c(n_1)}(R_i - S_i)S_1, \\
\mathbf{n}_x &= \frac{\mathbf{R}-\mathbf{S}}{2c(n_1)} \quad \text{or} \quad \mathbf{n}_t = \frac{\mathbf{R}+\mathbf{S}}{2},
\end{aligned}
\right.
\tag{2.2}
$$

with

$$\lambda_1 = \gamma \quad \text{and} \quad \lambda_2 = \lambda_3 = \alpha.$$

System (2.2) has the following form of energy conservation law:

$$\frac{1}{4}\partial_t\big(\mathbf{R}^2+\mathbf{S}^2\big) - \frac{1}{4}\partial_x\big(c(n_1)(\mathbf{R}^2-\mathbf{S}^2)\big) = 0,$$

and two balance laws for energy densities in two directions, respectively,

$$
\left\{
\begin{aligned}
(\mathbf{R}^2)_t - (c\mathbf{R}^2)_x &= \tfrac{c'}{2c}(n_1)(\mathbf{R}^2 S_1 - R_1\mathbf{S}^2)\,, \\
(\mathbf{S}^2)_t + (c\mathbf{S}^2)_x &= -\tfrac{c'}{2c}(n_1)(\mathbf{R}^2 S_1 - R_1\mathbf{S}^2)\,.
\end{aligned}
\right.
\tag{2.3}
$$

Note all equations from (2.2) to (2.3) only hold for smooth solutions.

The analysis in [13] shows that the problem (1.1)-(1.2) has a weak solution which conserves the total energy, where the main existence result can be summarized as follows.

Definition 2.1 (Weak solution [13]). *The vector function $\mathbf{n}(t,x)$, defined for all $(t,x)\in\mathbb{R}^+\times\mathbb{R}$, is a* **weak solution** *to the Cauchy problem (1.1)–(1.4) if it satisfies*

(i) *In the t-x plane, the functions (n_1,n_2,n_3) are locally Hölder continuous with exponent $1/2$. This solution $t\mapsto(n_1,n_2,n_3)(t,\cdot)$ is continuously differentiable as a map with values in L^p_{loc}, for all $1\le p<2$. Moreover, it is Lipschitz continuous with respect to (w.r.t.) the L^2 distance, i.e.*

$$\big\|n_i(t,\cdot)-n_i(s,\cdot)\big\|_{L^2} \le L\,|t-s|, \quad i=1,2,3,$$

for all $t,s\in\mathbb{R}^+$.

(ii) *The functions (n_1,n_2,n_3) take on the initial conditions in (1.4) pointwise, while their temporal derivatives hold in L^p_{loc} for $p\in[1,2)$.*

(iii) *The equations (1.1) hold in distributional sense for test function $\varphi\in C^1_c(\mathbb{R}^+\times\mathbb{R})$.*

Definition 2.2 (Energy conservation [13]). *Under the previous assumptions, a solution $\mathbf{n}=\mathbf{n}(t,x)$ can be constructed which is* conservative *in the following sense.*

There exist two families of positive Radon measures on the real line: $\{\mu^t_-\}$ and $\{\mu^t_+\}$, depending continuously on t in the weak topology of measures, with the following properties.

(i) *At every time t one has*

$$\mu^t_-(\mathbb{R}) + \mu^t_+(\mathbb{R}) \;=\; E_0 \;\doteq\; 2\int_{-\infty}^{\infty}\Big[|\mathbf{n}_1|^2(x) + c^2(n_{10}(x))|\mathbf{n}_{0,x}(x)|^2\Big]\,dx\,,$$

where we denote the initial data

$$\mathbf{n}_0 = (n_{10},n_{20},n_{30}) = \mathbf{n}|_{t=0}, \qquad \mathbf{n}_1 = (n_{11},n_{21},n_{31}) = \mathbf{n}_t|_{t=0}.$$

(ii) *For each t, the absolutely continuous parts of μ_-^t and μ_+^t w.r.t. the Lebesgue measure have densities respectively given by*

$$\left|\mathbf{n}_t + c(n_1)\mathbf{n}_x\right|^2 = \mathbf{R}^2, \qquad \left|\mathbf{n}_t - c(n_1)\mathbf{n}_x\right|^2 = \mathbf{S}^2.$$

(iii) *For almost every $t \in \mathbb{R}^+$, the singular parts of μ_-^t and μ_+^t are concentrated on the set where $c'(n_1) = 0$.*

(iv) *The measures μ_-^t and μ_+^t provide measure-valued solutions respectively to the balance laws*

$$\begin{cases} w_t - (cw)_x &= \frac{c'}{2c}(n_1)(\mathbf{R}^2 S_1 - R_1\mathbf{S}^2)\,, \\ z_t + (cz)_x &= -\frac{c'}{2c}(n_1)(\mathbf{R}^2 S_1 - R_1\mathbf{S}^2)\,. \end{cases} \tag{2.4}$$

Then, one has the following theorem.

Theorem 2.1 (Existence [13])**.** *The problem (1.1)–(1.4) has a global weak energy conservative solution defined for all $(t,x) \in [0,\infty)\times\mathbb{R}$.*

Remark 2.1. *In principle, the equations (2.4) should be written as*

$$\begin{cases} w_t - (cw)_x &= \frac{c'}{2c}(n_1)(S_1 w - R_1 z)\,, \\ z_t + (cz)_x &= -\frac{c'}{2c}(n_1)(S_1 w - R_1 z)\,. \end{cases} \tag{2.5}$$

Here $w = w^a + w^s$ is a measure with an absolutely continuous part and a singular part. Because of the item (iii) in Definition 2.2, the product $c'(n_1)w^s = 0$ for a.e. time t. For this reason, on the right hand side of (2.5) we can replace w with the measure w^a having density $\mathbf{R}^2$ w.r.t. Lebesgue measure. Similarly, we can replace z with the measure z^a having density $\mathbf{S}^2$ w.r.t. Lebesgue measure.

The total energy represented by the sum $\mu_- + \mu_+$ is conserved in time. However, occasionally, some of this energy is concentrated on a set of measure zero. At the times τ when this happens, μ_τ has a non-trivial singular part and

$$E(\tau) \;\doteq\; \int_{-\infty}^{\infty}\Big[|\mathbf{n}_t|^2(\tau,x) + c^2(n_1(\tau,x))|\mathbf{n}_x|^2(\tau,x)\Big]\,dx \;<\; E_0\,.$$

The condition (iii) puts some restrictions on the set of such times τ. In particular, if $c'(n_1) \neq 0$ for all $\mathbf{n}$, then this set has measure zero.

2.3. **Main results of this paper.**

The first main result of our paper is the uniqueness of conservative solution.

Theorem 2.2. *For any initial data $n_{i0} \in H^1(\mathbb{R})$, $n_{i1} \in \mathbf{L}^2(\mathbb{R}), i = 1,2,3$, the energy conservative solution to Cauchy problem (1.1)-(1.4) is unique.*

The result for the structure of conservative solutions to (1.1)–(1.4) reads

Theorem 2.3. *Assume the generic condition $\alpha \neq \gamma$ is satisfied. Let $T > 0$ be given, then there exists an open dense set*

$$\mathcal{D} \subset \Big(\mathcal{C}^3(\mathbb{R}) \cap H^1(\mathbb{R})\Big) \times \Big(\mathcal{C}^2(\mathbb{R}) \cap L^2(\mathbb{R})\Big),$$

such that, for $(n_{i0}, n_{i1}) \in \mathcal{D}, i = 1,2,3$, the conservative solution $\mathbf{n} = (n_1, n_2, n_3)$ of (1.1) is twice continuously differentiable in the complement of finitely many characteristic curves, within the domain $[0,T]\times\mathbb{R}$.

3. Uniqueness of conservative solutions

The uniqueness approach in this paper is in some sense an inverse process of [13]. Given a conservative solution $\mathbf{n} = \mathbf{n}(t,x)$, we define a set of independent variables X, Y and dependent variables $\mathbf{n}, p, q, \nu, \eta, \boldsymbol{\xi}, \boldsymbol{\zeta}$, and show that these variables satisfy a semi-linear system of equations. By proving that this semi-linear system has unique solutions, we eventually obtain the uniqueness of solutions to the original equation (1.1).

We divide the proof into two major steps. Given an energy conservative solution $\mathbf{n}(t,x)$, we

1. Prove the existence and uniqueness of characteristic in each direction. (Subsection 3.1)
2. Reconstruct the semi-linear system, then prove Theorem 2.2. (Subsection 3.2)

3.1. **A new coordinate, and the existence and uniqueness of characteristic.** This part is analog to the corresponding part in [8], because the energy laws in (2.3) on $\mathbf{R}^2$ and $\mathbf{S}^2$ are very similar to those used in [8], and also in this part we do not have to estimate the propagation of each component R_i, S_i separately. We omit some details, and also leave the proof of a Lemma (Lemma 3.2) in the appendix in order to make the paper self-contained.

Step 1 is crucial because essentially the semilinear system, which will be constructed in Step 2, describes the evolution of $\mathbf{n}$ along characteristic curves, i.e. curves $t \mapsto x^{\pm}(t)$ which satisfy

$$\dot{x}^-(t) \;=\; -c(n_1(t, x^-(t))), \qquad \dot{x}^+(t) \;=\; c(n_1(t, x^+(t))), \tag{3.1}$$

with initial data

$$x^-(0) \;=\; \bar{y}, \qquad x^+(0) \;=\; \bar{y}. \tag{3.2}$$

Note, as explained in the introduction, this system might have multiple solutions, since $n_1(t,x)$ is only Hölder continuous. So we need to find a way to use the energy conservation laws in the weak form to select a unique characteristic. The first idea is to introduce a pair of energy related variables.

Let $\mathbf{n} = \mathbf{n}(t,x)$ be an energy conservative solution of (1.1). In view of (1.2) and (1.3), there are constants c_0 and M, such that

$$0 \;<\; c_0 \;\leq\; c(n_1) \;<\; M, \quad |c'(n_1)| \;=\; \frac{\gamma-\alpha}{c(n_1)}|n_1| \;<\; M. \tag{3.3}$$

For simplicity, we introduce a constant to be used throughout this section.

$$C_0 \;\doteq\; \Big\|\frac{c'(n_1)}{2c(n_1)}\Big\|_{L^\infty} \;\leq\; \frac{M}{2c_0}. \tag{3.4}$$

For any time t and any $\alpha, \beta \in \mathbb{R}$, let us define the points $x(t,\alpha)$ and $y(t,\beta)$ by

$$x(t,\alpha) \;\doteq\; \sup\Big\{x\,;\; x + \mu_-^t\big(\,]-\infty, x]\,\big) \;<\; \alpha\Big\}, \tag{3.5}$$

$$y(t,\beta) \;\doteq\; \sup\Big\{x\,;\; x + \mu_+^t\big(\,]-\infty, x]\,\big) \;<\; \beta\Big\}. \tag{3.6}$$

Notice that the above holds if and only if, for some $\theta, \theta' \in [0,1]$, one has

$$x(t,\alpha) + \mu_-^t\Big(\,]-\infty\,,\; x(t,\alpha)[\,\Big) + \theta\cdot\mu_-^t\Big(\big\{x(t,\alpha)\big\}\Big) \;=\; \alpha\,, \tag{3.7}$$

$$y(t,\beta) + \mu_+^t\Big(\,]-\infty\,,\; y(t,\beta)[\,\Big) + \theta'\cdot\mu_-^t\Big(\big\{y(t,\beta)\big\}\Big) \;=\; \beta\,. \tag{3.8}$$

Since the measures μ_-^t, μ_+^t are both positive and bounded, it is clear that these points are well defined. Actually for smooth enough solutions

$$x(t,\alpha) + \int_{-\infty}^{x(t,\alpha)} \mathbf{R}^2(t,\xi)\, d\xi \;=\; \alpha\,, \tag{3.9}$$

$$y(t,\beta) + \int_{-\infty}^{y(t,\beta)} \mathbf{S}^2(t,\xi)\, d\xi \;=\; \beta\,. \tag{3.10}$$

Here α denotes an energy related parameter of the backward characteristic, while β the forward characteristic. Then because of equations

$$\frac{d}{dt}\int_{-\infty}^{x^-(t)} \mathbf{R}^2(t,x)\, dx \;=\; \int_{-\infty}^{x^-(t)} \frac{c'}{2c}(\mathbf{R}^2 S_1 - R_1\mathbf{S}^2)\, dx\,, \tag{3.11}$$

$$\frac{d}{dt}\int_{-\infty}^{x^+(t)} \mathbf{S}^2(t,x)\, dx \;=\; -\int_{-\infty}^{x^+(t)} \frac{c'}{2c}(\mathbf{R}^2 S_1 - R_1\mathbf{S}^2)\, dx\,, \tag{3.12}$$

it is convenient to work with an adapted set of variables $x(t,\alpha)$, $y(t,\beta)$, instead of the variables (t,x).

We can prove the following lemma, using very similar method as in [8]. We omit the proof for brievity.

Lemma 3.1. *For every fixed t, the maps $\alpha \mapsto x(t,\alpha)$ and $\beta \mapsto y(t,\beta)$ are both Lipschitz continuous with constant 1. Moreover, for fixed α,β, the maps $t \mapsto x(t,\alpha)$ and $t \mapsto y(t,\beta)$ are absolutely continuous, locally Hölder continuous with exponent $1/2$, and have locally bounded variation.*

Then, we are ready to recover the characteristic from the solution $\mathbf{n}(t,x)$. The next lemma, which plays a crucial role in our analysis, shows that for a conservative solution the characteristic curves can be uniquely determined. The proof is given in the Appendix. One main spirit in the proof is to introduce a weighted distance, including some wave interaction potential, in order to control the possible increase of forward or backward energy.

Lemma 3.2. *Let $\mathbf{n}$ be an energy conservative solution of (1.1). Then, for any $\bar{y} \in \mathbb{R}$, there exists unique Lipschitz continuous maps $t \mapsto x^\pm(t)$ which satisfy (3.1)-(3.2) together with (3.11)-(3.12).*

For future use, we introduce some characteristic coordinates (X,Y) and discuss their properties. For any couple $(X,Y) \in \mathbb{R}^2$, a unique point (t,x) can be determined as follows. Choose points $\bar{x}$ and $\bar{y}$ such that

$$X \;=\; \bar{x} + \int_{-\infty}^{\bar{x}} \mathbf{R}^2(0,x)dx, \qquad\qquad Y \;=\; \bar{y} + \int_{-\infty}^{\bar{y}} \mathbf{S}^2(0,x)dx. \tag{3.13}$$

In view of Lemma 3.2, there exists a unique backward characteristic $t \mapsto x^-(t,\bar{x})$ starting at $\bar{x}$, and a unique forward characteristic $t \mapsto x^+(t,\bar{y})$ starting at $\bar{y}$. Without loss of generality, we set $\bar{x} \geq \bar{y}$, then denote $(t(X,Y), x(X,Y))$ be the unique point where these two characteristics cross. That means

$$x^-(t(X,Y),\, \bar{x}) \;=\; x^+(t(X,Y),\, \bar{y}) \;=\; x(X,Y). \tag{3.14}$$

Next, we define

$$\mathbf{n}(X,Y) \;\doteq\; \mathbf{n}\big(t(X,Y), x(X,Y)\big). \tag{3.15}$$

The following lemma implies that the maps $t, x, \mathbf{n}$ defined above are Lipschitz continuous w.r.t. X, Y. The details can be found in [8], we omit it here for brevity.

Lemma 3.3. *The map $(X,Y)\to(t,x,\mathbf{n})(X,Y)$ is locally Lipschitz continuous.*

In addition, one has the following remark.

Remark 3.1. *By Rademacher's theorem and the above results, it is easy to see that the map*

$$\Lambda:(X,Y)\mapsto(t(X,Y),\,x(X,Y))$$

is a.e. differentiable. From this, we can set

$$\Omega \doteq \Big\{(X,Y)\,;\ \text{either } D\Lambda(X,Y) \text{ does not exists, or else } \det D\Lambda(X,Y)=0\Big\}, \tag{3.16}$$

and

$$V \doteq \Big\{\Lambda(X,Y)\,;\ (X,Y)\in\Omega\Big\}.$$

By the area formula [31], the 2-dimensional measure of V is zero. Notice that, the map $\Lambda:\mathbb{R}^2\mapsto\mathbb{R}^2$ is onto but not one-to-one. However, for each $(t_0,x_0)\notin V$, there exist a unique point (X,Y) such that $\Lambda(X,Y)=(t_0,x_0)$.

Next, for any function $f(t,x)$, with $f\in\mathbf{L}^1(\mathbb{R}^2)$, the composition $\tilde f(X,Y)=f(\Lambda(X,Y))$ is well defined at a.e. point $(X,Y)\in\mathbb{R}^2\setminus\Omega$. One thus has

$$\int_{\mathbb{R}^2} f(t,x)\,dxdt \;=\; \int_{\mathbb{R}^2\setminus\Omega}\tilde f(X,Y)\cdot|\det D\Lambda(X,Y)|\,dXdY, \tag{3.17}$$

where the determinant of the Jacobian matrix $D\Lambda$ can be computed as

$$x_X = c(n_1)\,t_X\,,\qquad x_Y=-c(n_1)\,t_Y\,,$$

$$D\Lambda=\begin{pmatrix} t_X & t_Y\\ x_X & x_Y\end{pmatrix}=\begin{pmatrix}\frac{x_X}{c(n_1)} & -\frac{x_Y}{c(n_1)}\\ x_X & x_Y\end{pmatrix}.$$

Hence

$$\big|\det D\Lambda\big| = \frac{2}{c(n_1)}x_Xx_Y\,.$$

Finally, in the X-Y plane we denote a set

$$\mathcal{G}\doteq\mathbb{R}^2\setminus\Omega\,. \tag{3.18}$$

3.2. **An equivalent semi-linear system.** In this subsection, we first introduce some variables in the X-Y plane, then we are devoted to showing that these variables satisfy a semi-linear system with smooth coefficients. Moreover, their values are uniquely determined by the initial data. At last, by showing that the map $(X,Y)\mapsto(t,x,\mathbf{n})(X,Y)$ is uniquely determined, we can prove the uniqueness of conservative solutions $\mathbf{n}(t,x)$ of the Cauchy problem (1.1)-(1.4).

Let the initial values $\bar\alpha,\bar\beta$ be given, a careful look at the proof of Lemma 3.2 (in the Appendix) shows that we can denote $t\mapsto\alpha(t,\bar\alpha)$ and $t\mapsto\beta(t,\bar\beta)$ be the unique solutions to

$$\begin{aligned}\alpha(t)=&\;\bar\alpha+\int_0^t\left(-c(0)+\int_{-\infty}^{x(t,\alpha(t))}\frac{c'(S_1-R_1+\mathbf{R}^2S_1-R_1\mathbf{S}^2)}{2c}\,dx\right)dt\,,\\ \beta(t)=&\;\bar\beta+\int_0^t\left(c(0)-\int_{-\infty}^{y(t,\beta(t))}\frac{c'(S_1-R_1+\mathbf{R}^2S_1-R_1\mathbf{S}^2)}{2c}\,dx\right)dt\,.\end{aligned}\tag{3.19}$$

Set the new dependent variables $p(X,Y)$ and $q(X,Y)$ by

$$p(X,Y)=\frac{\partial}{\partial\bar\alpha}\alpha(\tau,\bar\alpha)\Big|_{\bar\alpha=X,\,\tau=t(X,Y)},\qquad q(X,Y)=\frac{\partial}{\partial\bar\beta}\beta(\tau,\bar\beta)\Big|_{\bar\beta=Y,\,\tau=t(X,Y)}. \tag{3.20}$$

On the other hand, note that $t \mapsto x^-(t) = x(t, \alpha(t))$ and $t \mapsto x^+(t) = y(t, \beta(t))$ are the unique backward and forward characteristics starting from the points $x(0, \bar{\alpha})$ and $y(0, \bar{\beta})$, respectively. Also, recall the definitions of the maps $\alpha \mapsto x(t, \alpha)$ and $\beta \mapsto y(t, \beta)$ in (3.7)-(3.8), we further define the feature of characteristics by

$$\nu(X,Y) \doteq \frac{\partial x}{\partial \alpha}(t(X,Y), \alpha(t, x(X,Y))), \qquad \eta(X,Y) \doteq \frac{\partial x}{\partial \beta}(t(X,Y), \beta(t, x(X,Y))) . \tag{3.21}$$

Finally, we define some variables to track the propagation of R_i and S_i, instead of $\mathbf{R}$ and $\mathbf{S}$, in order to get a system with unique solution. We define $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3), \boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$ by setting

$$\xi_i(X,Y) \doteq \frac{2c(n_1(X,Y))}{p(X,Y)} n_{i,X}(X,Y) , \qquad \zeta_i(X,Y) \doteq \frac{2c(n_1(X,Y))}{q(X,Y)} n_{i,Y}(X,Y) , \tag{3.22}$$

for $i = 1, 2, 3$, where we use the fact that the functions c, p, q are strictly positive. Based on Rademacher's theorem and Lemmas 3.1 and 3.3, we see that the above derivatives are a.e. well defined. Moreover,

$$p(X,Y) = q(X,Y) = 1 \quad \text{if} \quad t(X,Y) = 0.$$

The main goal of this subsection is to prove that these variables satisfy the following semilinear system with smooth coefficients

$$\begin{cases} n_{i,X} = \dfrac{1}{2c}\xi_i p, \qquad n_{i,Y} = \dfrac{1}{2c}\zeta_i q, \\ x_X = \dfrac{1}{2}\nu p, \qquad x_Y = -\dfrac{1}{2}\eta q, \\ t_X = \dfrac{1}{2c}\nu p, \qquad t_Y = \dfrac{1}{2c}\eta q, \\ p_Y = \dfrac{c'}{4c^2}(\zeta_1 - \xi_1) pq, \\ q_X = \dfrac{c'}{4c^2}(\xi_1 - \zeta_1) pq, \\ \nu_Y = \dfrac{c'}{4c^2}(\nu - \eta)\xi_1 q, \\ \eta_X = -\dfrac{c'}{4c^2}(\nu - \eta)\zeta_1 p, \\ \xi_{i,Y} = \dfrac{n_i}{8c^3}\big[(c^2 - \lambda_i)(\eta + \nu - 2\eta\nu) - 2(3c^2 - \lambda_i)\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\big] q - \dfrac{c'}{2c}(\zeta_1 - \zeta_i)\xi_i q - \dfrac{c'}{4c^2}(\zeta_i - \xi_i)\xi_1 q, \\ \zeta_{i,X} = \dfrac{n_i}{8c^3}\big[(c^2 - \lambda_i)(\eta + \nu - 2\eta\nu) - 2(3c^2 - \lambda_i)\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\big] p + \dfrac{c'}{2c}(\xi_i - \xi_1)\zeta_i p + \dfrac{c'}{4c^2}(\zeta_i - \xi_i)\zeta_1 p, \end{cases} \tag{3.23}$$

with $i = 1, 2, 3$, $\lambda_1 = \gamma$ and $\lambda_2 = \lambda_3 = \alpha$. Below, we state the main theorem of this subsection.

Theorem 3.1. *By possibly changing the functions $p, q, \nu, \eta, \boldsymbol{\xi}, \boldsymbol{\zeta}$ on a set of measure zero in the X-Y plane, the following holds.*

(i) *For a.e. $X_0 \in \mathbb{R}$, the functions $t, x, \mathbf{n}, p, \nu, \boldsymbol{\xi}$ are absolutely continuous on every vertical segment of the form $S_0 \doteq \{(X_0, Y) ;\ a < Y < b\}$. Their partial derivatives w.r.t. Y satisfy a.e. the corresponding equations in (3.23).*

(ii) *For a.e. $Y_0 \in \mathbb{R}$, the functions $t, x, \mathbf{n}, q, \eta, \boldsymbol{\zeta}$ are absolutely continuous on every horizontal segment of the form $S_0 \doteq \{(X, Y_0) ;\ a < X < b\}$. Their partial derivatives w.r.t. X satisfy a.e. the corresponding equations in (3.23).*

For this purpose, we recall a technical result in [8].

Lemma 3.4 ([8])**.** *Let $\Gamma = (a,b)\times(c,d)$ be a rectangle in the X-Y plane. Assume that $u \in \mathbf{L}^\infty(\Gamma)$ and $f \in \mathbf{L}^1(\Gamma)$. Moreover assume that there exists null sets $\mathcal{N}_X \subset (a,b)$ and $\mathcal{N}_Y \subset (c,d)$ such that the following holds.*

For every $\overline{X}_1, \overline{X}_2 \notin \mathcal{N}_X$ and $\overline{Y}_1, \overline{Y}_2 \notin \mathcal{N}_Y$ with $\overline{X}_1 < \overline{X}_2$ and $\overline{Y}_1 < \overline{Y}_2$, one has

$$\int_{\overline{Y}_1}^{\overline{Y}_2} \Big[u(\overline{X}_2, Y) - u(\overline{X}_1, Y)\Big]\, dY \;=\; \int_{\overline{Y}_1}^{\overline{Y}_2}\int_{\overline{X}_1}^{\overline{X}_2} f(X,Y)\, dX dY\,.$$

Then, by possibly modifying u on a set of measure zero, the following holds. For a.e. $Y_0 \in]c,d[$, the map $X \mapsto u(X,Y_0)$ is absolutely continuous and

$$\frac{\partial}{\partial X} u(X,Y_0) \;=\; f(X,Y_0) \qquad\qquad \textit{for a.e. } X \in (a,b)\,.$$

Based on above results and definitions, we begin to study the representation for the variables $\nu, \eta, \boldsymbol{\xi}, \boldsymbol{\zeta}$ in terms of $\mathbf{R}$ and $\mathbf{S}$. Then, we have the following results, which can be proved in a similar procedure as [8]. We omit the proof here for simplicity.

Lemma 3.5. (i) *If $(X,Y) \in \mathcal{G}$ then*

$$\frac{p(X,Y)}{x_X(X,Y)} \;=\; 2(1+\mathbf{R}^2)\,, \qquad\qquad \frac{q(X,Y)}{x_Y(X,Y)} \;=\; 2(1+\mathbf{S}^2)\,,$$

$$\begin{cases} \nu(X,Y) &=\; \frac{1}{1+\mathbf{R}^2}\,, \\ \\ \eta(X,Y) &=\; \frac{1}{1+\mathbf{S}^2}\,, \end{cases} \qquad\qquad \begin{cases} \boldsymbol{\xi}(X,Y) &=\; \frac{\mathbf{R}}{1+\mathbf{R}^2}\,, \\ \\ \boldsymbol{\zeta}(X,Y) &=\; \frac{\mathbf{S}}{1+\mathbf{S}^2}\,, \end{cases}$$

where the right hand sides are evaluated at the point $(t(X,Y), x(X,Y))$.

(ii) *For a.e. $(X,Y) \in \Omega$, one has*

$$\nu(X,Y) \;=\; \eta(X,Y) \;=\; 0$$

and

$$\boldsymbol{\xi}(X,Y) \;=\; \boldsymbol{\zeta}(X,Y) \;=\; \mathbf{0}.$$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. In what follows, we will show the variables $t, x, \mathbf{n}, p, q, \eta, \nu, \boldsymbol{\xi}, \boldsymbol{\zeta}$ in (3.23) indeed satisfy the assumptions of Lemma 3.4. Hence, consider any rectangle

$$\mathcal{Q} \doteq [X_1, X_2] \times [Y_1, Y_2]$$

in the X-Y plane.

1 - Equations for n. In view of Lemma 3.3, we see that the function $\mathbf{n}$ is Lipschitz continuous w.r.t. X, Y. Thus, by the definitions (3.22), it is straightforward to verify that

$$\mathbf{n}_X \;=\; \frac{1}{2c}\,\boldsymbol{\xi} p\,, \qquad\qquad \mathbf{n}_Y \;=\; \frac{1}{2c}\,\boldsymbol{\zeta} q.$$

2 - Equations for x and t. From Lemma 3.3 and the definitions (3.21)-(3.22), we have

$$\begin{aligned} \frac{\partial}{\partial X} x(X,Y) &= \frac{1}{2}\frac{\partial}{\partial \bar\alpha} x(t(X,Y), \alpha(t(X,Y), \bar\alpha)) \\ &= \frac{1}{2}\frac{\partial x}{\partial \alpha}(t(X,Y), \alpha(t(X,Y), x(X,Y)))\frac{\partial \alpha}{\partial \bar\alpha}(t(X,Y), \bar\alpha)\Big|_{\bar\alpha = X} \\ &= \frac{1}{2}\nu p. \end{aligned}$$

Similarly, we can obtain the equation for x_Y. To that end, it follows from the equations for x that

$$t_X = \frac{x_X}{c(n_1)} = \frac{1}{2c}\nu p, \qquad t_Y = -\frac{x_Y}{c(n_1)} = \frac{1}{2c}\eta q.$$

3 - Equations for p and q. Consider the domain

$$\mathcal{D} \doteq \Big\{(X,Y); \quad X \in [X_1,\, X_2], \quad Y \in [Y_1, Y_2], \qquad \det\, D\Lambda(X,Y) > 0\Big\}.$$

By (3.17) and (3.19), it holds that

$$\begin{aligned}
&\int_{X_1}^{X_2} p(X,Y_2) - p(X,Y_1)\,dX \\
&= \int_{X_1}^{X_2} \left[\frac{\partial\alpha(\tau,X)}{\partial X}\bigg|_{\tau=\tau(X,Y_2)} - \frac{\partial\alpha(\tau,X)}{\partial X}\bigg|_{\tau=\tau(X,Y_1)} \right] dX \\
&= \int_{X_1}^{X_2} \left[\frac{\partial}{\partial X} \int_{x(X,Y_1)}^{x(X,Y_2)} \int_{\tau(\widetilde{X},Y_1)}^{\tau(\widetilde{X},Y_2)} \frac{c'}{2c}(S_1 - R_1 + \mathbf{R}^2 S_1 - \mathbf{S}^2 R_1)\,dt\,dx \right] d\widetilde{X} \\
&= \iint_{\Lambda(\mathcal{D})} \frac{c'}{2c}(S_1 - R_1 + \mathbf{R}^2 S_1 - \mathbf{S}^2 R_1)\,dx\,dt \\
&= \iint_{\mathcal{D}} \frac{c'}{2c}(S_1 - R_1 + \mathbf{R}^2 S_1 - \mathbf{S}^2 R_1)\cdot \det D\Lambda(X,Y)\,dX\,dY \\
&= \iint_{\mathcal{D}} \frac{c'}{4c^2}(S_1 - R_1 + \mathbf{R}^2 S_1 - \mathbf{S}^2 R_1)\frac{1}{1+\mathbf{R}^2}\frac{1}{1+\mathbf{S}^2} pq\,dX\,dY \\
&= \iint_{\mathcal{Q}} \frac{c'}{4c^2}(\zeta_1 - \xi_1)pq\,dX\,dY.
\end{aligned}$$

The last equality follows from Lemma 3.5, part (i) for the integral over $\mathcal{D}$ and part (ii) for the integral over $\mathcal{Q}\setminus\mathcal{D}$. Thus the above estimate implies

$$p_Y = \frac{c'}{4c^2}(\zeta_1 - \xi_1)pq. \tag{3.24}$$

In a similar way, we have

$$q_X = -\frac{c'}{4c^2}(\zeta_1 - \xi_1)pq.$$

4 - Equations for η and ν. From (3.20), (3.21) and Remark 3.1, we obtain

$$\begin{aligned}
&\int_{X_1}^{X_2} \Big[p\nu(X,Y_2) - p\nu(X,Y_1)\Big]\,dX = \int_{X_1}^{X_2} \left[\frac{\partial x(\tau,X)}{\partial X}\bigg|_{\tau=t(X,Y_2)} - \frac{\partial x(\tau,X)}{\partial X}\bigg|_{\tau=t(X,Y_1)} \right] dX \\
&= \int_{X_1}^{X_2} \left[\frac{\partial}{\partial X} \int_{x(X,Y_1)}^{x(X,Y_2)} \int_{t(\widetilde{X},Y_1)}^{t(\widetilde{X},Y_2)} \frac{c'}{2c}(S_1 - R_1)\,dt\,dx \right] d\widetilde{X} = \iint_{\Lambda(\mathcal{D})} \frac{c'}{2c}(S_1 - R_1)\,dx\,dt \\
&= \int_{\mathcal{D}} \frac{c'}{2c}(S_1 - R_1)\cdot \det D\Lambda(X,Y)\,dX\,dY \\
&= \int_{\mathcal{D}} \frac{c'}{4c^2}(S_1 - R_1)\frac{1}{1+\mathbf{R}^2}\frac{1}{1+\mathbf{S}^2} pq\,dX\,dY \\
&= \int_{\mathcal{Q}} \frac{c'}{4c^2}(\nu\zeta_1 - \xi_1\eta)pq\,dX\,dY.
\end{aligned}$$

Thus, from Lemma 3.4 one has

$$(p\nu)_Y \;=\; \frac{c'}{4c^2}(\nu\zeta_1 - \xi_1\eta)pq,$$

which from the equation (3.24) for p further implies

$$\nu_Y = \frac{c'}{4c^2}(\nu - \eta)\xi_1 q.$$

Similarly, we have

$$\eta_X = -\frac{c'}{4c^2}(\nu - \eta)\zeta_1 p.$$

5 - Equations for $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$. Now, it remains to study the equations for $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$. In fact, we want to characterize the distributional derivative $\mathbf{n}_{XY}$. More precisely, for any values $X_1 < X_2$ and $Y_1 > Y_2$, we wish to find a function $f \in \mathbf{L}^1_{loc}(\mathbb{R}^2)$ such that

$$[n_i(X_2, Y_1) - n_i(X_1, Y_1)] - [n_i(X_2, Y_2) - n_i(X_1, Y_2)] \;=\; \int_{X_1}^{X_2}\int_{Y_2}^{Y_1} f(X,Y)\, dX\, dY$$

for $i = 1, 2, 3$.

However, to obtain this equation, we need more subtle estimate on the weak solutions.

(i). As shown in the Fig. 1, we first denote

$$P_1 \doteq (t_1, x_1) = \Lambda(X_1, Y_1), \qquad P_2 \doteq (t_2, x_2) = \Lambda(X_2, Y_1),$$

$$P_3 \doteq (t_3, x_3) = \Lambda(X_1, Y_2), \qquad P_4 \doteq (t_4, x_4) = \Lambda(X_2, Y_2).$$

Then we denote

- Backward characteristic $t \mapsto x_1^-(t)$ passing through P_1, P_3. (Corresponding to $X = X_1$).
- Backward characteristic $t \mapsto x_2^-(t)$ passing through P_2, P_4. (Corresponding to $X = X_2$).
- Forward characteristic $t \mapsto x_1^+(t)$ passing through P_1, P_2. (Corresponding to $Y = Y_1$).
- Forward characteristic $t \mapsto x_2^+(t)$ passing through P_3, P_4. (Corresponding to $Y = Y_2$).

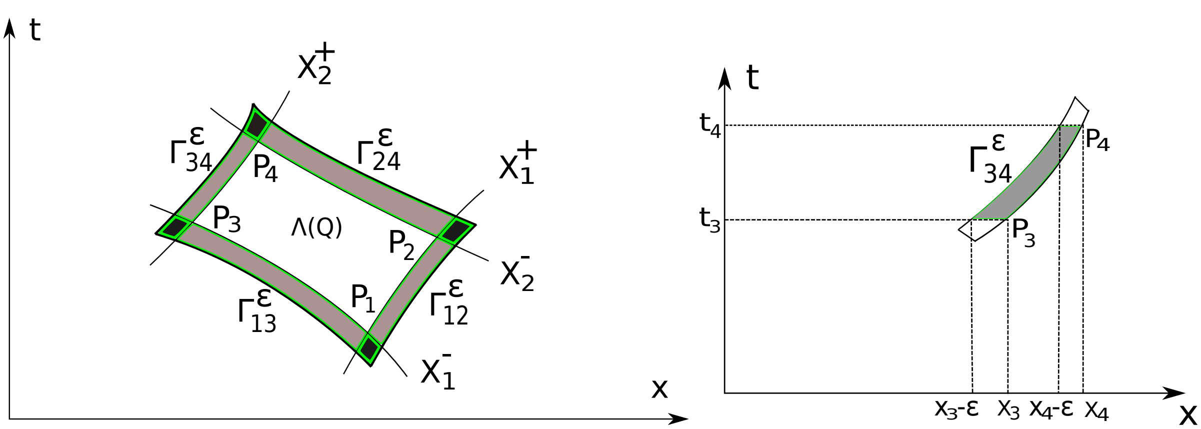

FIGURE 1. Left: The support of the test function ϕ^ϵ in (3.25). Right: An enlarged picture of Γ^ε_{34}, which is used in (3.35) where we only do the calculation in the shaded region, becuase the unshaded region can be omitted as $\varepsilon \to 0$.

We now construct a family of test functions ϕ^ϵ approaching the characteristic function of the set $\Lambda(\mathcal{Q})$, where $\mathcal{Q} \doteq [X_1, X_2] \times [Y_1, Y_2]$. More precisely, set

$$\phi^\epsilon(s,y) \;\doteq\; \min\{\varrho^\epsilon(s,y),\, \varsigma^\epsilon(s,y)\}, \tag{3.25}$$

where

$$\varrho^\epsilon(s,y) \;\doteq\; \begin{cases} 0 & \text{if } \; y \;\le\; x_1^-(s)-\epsilon, \\ 1+\epsilon^{-1}(y-x_1^-(s)) & \text{if } \; x_1^-(s)-\epsilon \;\le\; y \le x_1^-(s), \\ 1 & \text{if } \; x_1^-(s) \;\le\; y \;\le\; x_2^-(s), \\ 1-\epsilon^{-1}(y-x_2^-(s)) & \text{if } \; x_2^-(s) \;\le\; y \;\le\; x_2^-(s)+\epsilon, \\ 0 & \text{if } \; y \;\ge\; x_2^-(s)+\epsilon\,, \end{cases} \tag{3.26}$$

$$\varsigma^\epsilon(s,y) \;\doteq\; \begin{cases} 0 & \text{if } \; y \;\le\; x_1^+(s)-\epsilon, \\ 1+\epsilon^{-1}(y-x_1^+(s)) & \text{if } \; x_1^+(s)-\epsilon \;\le\; y \;\le\; x_1^+(s), \\ 1 & \text{if } \; x_1^+(s) \;\le\; y \;\le\; x_2^+(s), \\ 1-\epsilon^{-1}(y-x_2^+(s)) & \text{if } \; x_2^+(s) \;\le\; y \;\le\; x_2^+(s)+\epsilon, \\ 0 & \text{if } \; y \;\ge\; x_2^+(s)+\epsilon\,. \end{cases} \tag{3.27}$$

For every test function $\varphi \in \mathcal{C}_c^1(\mathbb{R}^2)$, one has from $(2.2)_1$ that

$$\begin{aligned} \iint R_1\Big[\varphi_t-(c\varphi)_x\Big]\,dx\,dt \;&=\; -\iint \frac{1}{4c^2}\big\{(c^2-\gamma)(|\mathbf{R}|^2+|\mathbf{S}|^2)-2(3c^2-\gamma)\mathbf{R}\cdot\mathbf{S}\big\}n_1\varphi\,dx\,dt \\ &\quad -\iint \frac{c'}{2c}(R_1-S_1)R_1\varphi\,dx\,dt\,. \end{aligned}$$

Noticing that for $i=1,2,3$ $R_i \in \mathbf{L}^2_{loc}(\mathbb{R}^2), c(n_1) \in H^1_{loc}(\mathbb{R}^2), \frac{1}{c(n_1)} \in H^1_{loc}(\mathbb{R}^2)$, we choose a sequence of test functions φ_n such that, as $n \to \infty$,

$$\varphi_n \;\to\; \frac{\phi^\epsilon}{c(n_1)} \qquad \text{in } H^1(\mathbb{R}^2).$$

Taking the limit, we have

$$\begin{aligned} \iint R_1\big[(\frac{\phi^\epsilon}{c})_t-\phi^\epsilon_x\big]\,dx\,dt \;&=\; -\iint \frac{1}{4c^3}\big\{(c^2-\gamma)(|\mathbf{R}|^2+|\mathbf{S}|^2)-2(3c^2-\gamma)\mathbf{R}\cdot\mathbf{S}\big\}n_1\phi^\epsilon\,dx\,dt \\ &\quad -\iint \frac{c'}{2c^2}(R_1-S_1)R_1\phi^\epsilon\,dx\,dt\,. \end{aligned}$$

From (2.1), it follows that

$$\begin{aligned} \iint \frac{R_1}{c}(\phi^\epsilon_t-c\phi^\epsilon_x)\,dx\,dt \;&=\; -\iint \frac{1}{4c^3}\big\{(c^2-\gamma)(|\mathbf{R}|^2+|\mathbf{S}|^2)-2(3c^2-\gamma)\mathbf{R}\cdot\mathbf{S}\big\}n_1\phi^\epsilon\,dx\,dt \\ &\quad +\iint \frac{c'}{c^2}R_1S_1\phi^\epsilon\,dx\,dt\,. \end{aligned} \tag{3.28}$$

(ii). On the other hand, consider the four boundary strips Γ^ϵ_{12}, Γ^ϵ_{13}, Γ^ϵ_{24} and Γ^ϵ_{34} of the support of ϕ^ϵ in Fig. 1. For example, Γ^ϵ_{34} is the strip enclosed by $x_2^+(t)-\epsilon$, $x_2^+(t)$, $x_1^-(t)-\varepsilon$ and $x_2^-(t)+\epsilon$. These sets overlap near the points $P_i=(t_i,x_i)$, $i=1,2,3,4$. However, each of these intersections is contained in a ball of radius $\mathcal{O}(\varepsilon)$. For example, $\Gamma^\epsilon_{12}\cap\Gamma^\epsilon_{13} \;\subset\; B(P_1,K\epsilon)$, for some constant K and all $\epsilon>0$. We first prove some estimates on $\Gamma^\epsilon_{12}\cap\Gamma^\epsilon_{13}$, which basically says that these regions can be omitted.

In this way, we have

$$\left|\iint_{\Gamma_{12}^\epsilon\cap\Gamma_{13}^\epsilon} \frac{R_1}{c(n_1)}(\phi_t^\epsilon - c\phi_x^\epsilon)\,dx\,dt\right| \;\le\; \frac{C_0}{\epsilon}\iint_{B(P_1,K\epsilon)} |R_1|\,dx\,dt$$
$$\le\; \frac{C_0}{\epsilon}\int_{t_1-K\epsilon}^{t_1+K\epsilon}\left(\int_{x_1-2K\epsilon}^{x_1+2K\epsilon} R_1^2(t,x)dx\right)^{1/2}(2K\epsilon)^{1/2}\,dt \;\le\; \frac{C_0}{\epsilon}E_0^{1/2}(2K\epsilon)^{3/2},$$

for suitable constants C_0, K, and all $\epsilon > 0$. The same arguments used for the other three intersections yield

$$\begin{aligned}\lim_{\epsilon\to 0}\iint \frac{R_1}{c(n_1)}(\phi_t^\epsilon - c\phi_x^\epsilon)\,dx\,dt \;&=\; \lim_{\epsilon\to 0}\iint_{\Gamma_{12}^\epsilon\cup\Gamma_{13}^\epsilon\cup\Gamma_{24}^\epsilon\cup\Gamma_{34}^\epsilon} \frac{R_1}{c(n_1)}(\phi_t^\epsilon - c\phi_x^\epsilon)\,dx\,dt\\ &=\; \lim_{\epsilon\to 0}\left(\iint_{\Gamma_{12}^\epsilon} + \iint_{\Gamma_{13}^\epsilon} + \iint_{\Gamma_{24}^\epsilon} + \iint_{\Gamma_{34}^\epsilon}\right)\frac{R_1}{c(n_1)}(\phi_t^\epsilon - c\phi_x^\epsilon)\,dx\,dt\,.\end{aligned}\tag{3.29}$$

(iii). For the integral over Γ_{13}^ϵ, assume $t_1, t_2, t_3, t_4 \in [0, T]$, by the Cauchy's inequality and (3.27), we obtain

$$\begin{aligned}&\lim_{\epsilon\to 0}\iint_{\Gamma_{13}^\epsilon} \frac{R_1(t,x)}{c(n_1(t,x))}\cdot\frac{c(n_1(t,x_1^-(t))) - c(n_1(t,x))}{\epsilon}\,dx\,dt\\ &\le\; \mathcal{O}(1)\cdot\lim_{\epsilon\to 0}\iint_{\Gamma_{34}^\epsilon}\frac{|c(n_1(t,x_1^-(t))) - c(n_1(t,x))|}{\epsilon}|R_1(t,x)|dx\,dt\\ &=\; \frac{\mathcal{O}(1)}{\epsilon}\cdot\lim_{\epsilon\to 0}\int_0^T\int_{0\le x_1^-(t)-x\le\varepsilon}|x_1^-(t)-x|^{1/2}\,|R_1(t,x)|\,dx\,dt\\ &=\; \mathcal{O}(1)\cdot\lim_{\epsilon\to 0}\int_0^T\left(\int_{0\le x_1^-(t)-x\le\varepsilon}|R_1(t,x)|^2\,dx\right)^{1/2} dt = 0\,.\end{aligned}\tag{3.30}$$

In a same way as (3.30), it suffices to estimate the integral over Γ_{24}^ϵ by

$$\lim_{\epsilon\to 0}\iint_{\Gamma_{24}^\epsilon}\frac{R_1}{c(n_1)}(\phi_t^\epsilon - c\phi_x^\epsilon)\,dx\,dt \;=\; 0\,.\tag{3.31}$$

As for the integral over Γ_{34}^ϵ, it holds that

$$\begin{aligned}&\lim_{\epsilon\to 0}\iint_{\Gamma_{34}^\epsilon}\frac{R_1(t,x)}{c(n_1(t,x))}\cdot\frac{c(n_1(t,x_2^+(t))) + c(n_1(t,x))}{\epsilon}\,dx\,dt\\ &=\; \lim_{\epsilon\to 0}\iint_{\Gamma_{34}^\epsilon}\frac{R_1(t,x)}{c(n_1(t,x))}\frac{2c(n_1(t,x))}{\epsilon}dx\,dt\\ &\qquad + \lim_{\epsilon\to 0}\iint_{\Gamma_{34}^\epsilon}\frac{R_1(t,x)}{c(n_1(t,x))}\frac{c(n_1(t,x_2^+(t))) - c(n_1(t,x))}{\epsilon}\,dx\,dt\\ &=\; \lim_{\epsilon\to 0}\frac{2}{\epsilon}\iint_{\Gamma_{34}^\epsilon} R_1(t,x)\,dx\,dt\,.\end{aligned}\tag{3.32}$$

In a similar fashion, we can obtain the integral over Γ_{12}^ϵ as

$$\lim_{\epsilon\to 0}\iint_{\Gamma_{12}^\epsilon}\frac{R_1}{c(n_1)}(\phi_t^\epsilon - c\phi_x^\epsilon)\,dx\,dt \;=\; -\lim_{\epsilon\to 0}\frac{2}{\epsilon}\iint_{\Gamma_{12}^\epsilon} R_1(t,x)\,dx\,dt\,.\tag{3.33}$$

Since $\big\{(c^2-\gamma)(|\mathbf{R}|^2+|\mathbf{S}|^2)-2(3c^2-\gamma)\mathbf{R}\cdot\mathbf{S}\big\}n_1 \in \mathbf{L}^1_{loc}(\mathbb{R}^2)$ and $R_1S_1 \in \mathbf{L}^1_{loc}(\mathbb{R}^2)$, we derive

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\iint \Big(-\frac{1}{4c^3}\big\{(c^2-\gamma)(|\mathbf{R}|^2+|\mathbf{S}|^2)-2(3c^2-\gamma)\mathbf{R}\cdot\mathbf{S}\big\}n_1+\frac{c'}{c^2}R_1S_1\Big)\phi^\epsilon\,dx\,dt\\
&= \iint_{\Lambda(\mathcal{Q})}\Big(-\frac{1}{4c^3}\big\{(c^2-\gamma)(|\mathbf{R}|^2+|\mathbf{S}|^2)-2(3c^2-\gamma)\mathbf{R}\cdot\mathbf{S}\big\}n_1+\frac{c'}{c^2}R_1S_1\Big)\,dx\,dt\,.
\end{aligned}
\tag{3.34}
$$

Thus, it follows from (3.28)–(3.34) that

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\iint_{\Gamma_{34}^\epsilon}R_1\,dx\,dt-\lim_{\epsilon\to 0}\frac{1}{\epsilon}\iint_{\Gamma_{12}^\epsilon}R_1\,dx\,dt\\
&= \iint_{\Lambda(\mathcal{Q})}\Big(-\frac{1}{8c^3}\big\{(c^2-\gamma)(|\mathbf{R}|^2+|\mathbf{S}|^2)-2(3c^2-\gamma)\mathbf{R}\cdot\mathbf{S}\big\}n_1+\frac{c'}{2c^2}R_1S_1\Big)\,dx\,dt\,.
\end{aligned}
\tag{3.35}
$$

(iv). In addition, using $n_1 \in H^1_{loc}$, we further have that (Fig. 1, right)

$$
\begin{aligned}
n_1(P_4)-n_1(P_3) &= \lim_{\varepsilon\to 0}\frac{1}{\epsilon}\Big(\int_{x_4-\epsilon}^{x_4}n_1(t_4,y)\,dy-\int_{x_3-\epsilon}^{x_3}n_1(t_3,y)\,dy\Big)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int\!\!\int_{\Gamma_{34}}\Big[n_{1,t}+c(n_1(t,x_2^+(t)))n_{1,x}\Big]\,dx\,dt\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int\!\!\int_{\Gamma_{34}}\Big[n_{1,t}+c(n_1(t,x))n_{1,x}\Big]\,dx\,dt\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int\!\!\int_{\Gamma_{34}}R_1\,dx\,dt\,.
\end{aligned}
$$

This together with a similar estimate for $n_1(P_2)-n_1(P_1)$ implies

$$
\begin{aligned}
&[n_1(P_4)-n_1(P_3)]-[n_1(P_2)-n_1(P_1)]\\
&= \iint_{\Lambda(\mathcal{Q})}\Big(-\frac{1}{8c^3}\big\{(c^2-\gamma)(|\mathbf{R}|^2+|\mathbf{S}|^2)-2(3c^2-\gamma)\mathbf{R}\cdot\mathbf{S}\big\}n_1+\frac{c'}{2c^2}R_1S_1\Big)\,dx\,dt\,.
\end{aligned}
\tag{3.36}
$$

Here, in view of Remark 3.1, we can write the right hand side of (3.36) as an integral w.r.t. the variables X, Y

$$
\begin{aligned}
&[n_1(X_2,Y_2)-n_1(X_1,Y_2)]-[n_1(X_2,Y_1)-n_1(X_1,Y_1)]\\
&= -\iint_{\mathcal{Q}\cap\mathcal{G}}\frac{n_1}{8c^3}\big\{(c^2-\gamma)(|\mathbf{R}|^2+|\mathbf{S}|^2)-2(3c^2-\gamma)\mathbf{R}\cdot\mathbf{S}\big\}\cdot\frac{p}{(1+\mathbf{R}^2)}\frac{q}{2c(1+\mathbf{S}^2)}\,dXdY\\
&\quad+\iint_{\mathcal{Q}\cap\mathcal{G}}\frac{c'}{2c^2}R_1S_1\cdot\frac{p}{(1+\mathbf{R}^2)}\frac{q}{2c(1+\mathbf{S}^2)}\,dXdY\,.
\end{aligned}
$$

Therefore, it follows from Lemma 3.4 that the weak derivative $n_{1,XY}$ exists and if $\det D\Lambda(X,Y)>0$, then

$$
\begin{aligned}
n_{1,XY}(X,Y) &= \frac{n_1}{16c^4}\big\{(c^2-\gamma)(|\mathbf{R}|^2+|\mathbf{S}|^2)-2(3c^2-\gamma)\mathbf{R}\cdot\mathbf{S}\big\}\cdot\frac{pq}{(1+\mathbf{R}^2)(1+\mathbf{S}^2)}\\
&\quad-\frac{c'}{4c^3}R_1S_1\cdot\frac{pq}{(1+\mathbf{R}^2)(1+\mathbf{S}^2)},
\end{aligned}
$$

if $\det D\Lambda(X,Y)=0$, then

$$
n_{1,XY}(X,Y) = 0.
$$

Finally, the definition (3.22) shows that

$$n_{1,XY} = \frac{n_1}{16c^4}\big[(c^2-\gamma)(\eta+\nu-2\eta\nu)-2(3c^2-\gamma)\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\big]pq-\frac{c'}{4c^3}\xi_1\zeta_1 pq\,. \tag{3.37}$$

(v). Hence, from Lemma 3.4, we see that, for a.e. X, the map $Y\mapsto n_{1,X}(X,Y) = (\frac{1}{2c}\xi_1 p)(X,Y)$ is absolutely continuous and its derivative is given by (3.37). Recalling the equations for p and n_1, and the fact that p remains uniformly positive on bounded sets, we get

$$\xi_{1,Y} = \frac{n_1}{8c^3}\big[(c^2-\gamma)(\eta+\nu-2\eta\nu)-2(3c^2-\gamma)\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\big]q-\frac{c'}{4c^2}(\zeta_1-\xi_1)\xi_1 q\,.$$

Similarly, we can obtain

$$\zeta_{1,X} = \frac{n_1}{8c^3}\big[(c^2-\gamma)(\eta+\nu-2\eta\nu)-2(3c^2-\gamma)\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\big]p+\frac{c'}{4c^2}(\zeta_1-\xi_1)\zeta_1 p\,.$$

Then $\xi_{j,Y},\zeta_{j,X}, j=2,3$ can be proved in a similar argument. This completes the proof of Theorem 3.1. □

3.3. **Proof of Theorem 2.2.** Now, we are in a position to prove Theorem 2.2 on the uniqueness of conservative solutions to equation (1.1). Let the initial data $n_{i0}\in H^1(\mathbb{R})$, $n_{i1}\in \mathbf{L}^2(\mathbb{R})$, $i=1,2,3$ be given. These data uniquely determine a curve γ in the X-Y plane, parameterized by

$$X(x) \doteq x+\int_{-\infty}^{x}\mathbf{R}^2(0,y)\,dy\,, \qquad Y(x) \doteq x+\int_{-\infty}^{x}\mathbf{S}^2(0,y)\,dy.$$

Notice that the values of $(\mathbf{n},x,t,p,q,\nu,\eta,\boldsymbol{\xi},\boldsymbol{\zeta})$ are all determined by the data n_{i0},n_{i1} along γ. Thus, at the point $(X(x),Y(x))\in\gamma$, we have

$$\begin{cases} t &= 0,\\ x &= x\,, \end{cases} \qquad \begin{cases} n &= n_{i0}(x),\\ p &= q = 1\,, \end{cases}$$

$$\begin{cases} \nu &= \dfrac{1}{1+\mathbf{R}^2(0,x)}\,,\\[2ex] \eta &= \dfrac{1}{1+\mathbf{S}^2(0,x)}\,, \end{cases} \qquad \begin{cases} \boldsymbol{\xi} &= \dfrac{\mathbf{R}(0,x)}{1+\mathbf{R}^2(0,x)}\,,\\[2ex] \boldsymbol{\zeta} &= \dfrac{\mathbf{S}(0,x)}{1+\mathbf{S}^2(0,x)}\,, \end{cases}$$

for $i=1,2,3$. Also, by the definition (2.1), one has

$$\mathbf{R}(0,x) = \mathbf{n}_1(x)+c(n_{10}(x))\,\mathbf{R}_{0,x}\,, \qquad \mathbf{S}(0,x) = \mathbf{n}_1(x)-c(n_{10}(x))\,\mathbf{n}_{0,x}\,.$$

Using an argument analog to the one in [13] for a semi-linear system, we can prove that the right hand side of (3.23) is Lipschitz. Then by a standard contraction mapping theory, we can prove that the system (3.23) has a unique solution in the X-Y plane. We refer the reader to [13]. Moreover, the functions $(X,Y)\mapsto (x,t,\mathbf{n})(X,Y)$ are uniquely determined, up to a set of zero measure in the X-Y plane. Since the map $(x,t)\mapsto \mathbf{n}(x,t)$ is continuous, we conclude that $\mathbf{n}$ is uniquely determined, pointwise in the x-t plane. This completes the proof of Theorem 2.2. □

4. Generic regularity of conservative solutions

Because of the large amount of variables used in this paper, it is hard to find a entirely different set of variables to use in this section. So we **note** that the symbols in this section have no relation to those used in section 3, except the solution $\mathbf{n}(\mathbf{t},\mathbf{x})$.

From now on, we are devoted to concerning the generic regularity of conservative solutions to (1.1)–(1.4) of Theorem 2.3. Roughly speaking, we prove that, for generic smooth initial data $(\mathbf{n}_0,\mathbf{n}_1)$, the corresponding solution is piecewise smooth in the t–x plane, with singularities occurring along a finite set of smooth curves. In this section we always assume that

$$\alpha \neq \gamma.$$

Define the forward and backward characteristics as follows

$$\begin{cases} \frac{d}{ds}x^{\pm}(s,t,x) = \pm c(n_1(s,x^{\pm}(s,t,x))), \\ x^{\pm}|_{s=t} = x. \end{cases}$$

Then define the coordinate transformation as

$$X \doteq \int_0^{x^-(0,t,x)} [1+\mathbf{R}^2(0,y)]\,dy, \quad \text{and} \quad Y \doteq \int_{x^+(0,t,x)}^0 [1+\mathbf{S}^2(0,y)]\,dy.$$

This implies

$$X_t - c(n_1)X_x = 0, \quad Y_t + c(n_1)Y_x = 0.$$

Thus, for any smooth function f, we have

$$\begin{cases} f_t + c(n_1)f_x = (X_t + c(n_1)X_x)f_X = 2c(n_1)X_x f_X, \\ f_t - c(n_1)f_x = (Y_t - c(n_1)Y_x)f_Y = -2c(n_1)Y_x f_Y. \end{cases} \tag{4.1}$$

Now, we introduce

$$\begin{aligned} & p = \frac{1+|\mathbf{R}|^2}{X_x}, \qquad q = \frac{1+|\mathbf{S}|^2}{-Y_x}, \\ & \mathbf{L} = (l_1,l_2,l_3) = \frac{\mathbf{R}}{1+|\mathbf{R}|^2}, \quad \mathbf{m} = (m_1,m_2,m_3) = \frac{\mathbf{S}}{1+|\mathbf{S}|^2}, \\ & h_1 = \frac{1}{1+|\mathbf{R}|^2}, \quad h_2 = \frac{1}{1+|\mathbf{S}|^2}. \end{aligned} \tag{4.2}$$

We have the semi-linear system, c.f. [13]

$$\begin{cases} \partial_Y l_i = & \frac{q}{8c^3(n_1)}[(c^2(n_1)-\lambda_i)(h_1+h_2-2h_1h_2) - 2(3c^2(n_1)-\lambda_i)\mathbf{L}\cdot\mathbf{m}]n_i \\ & +\frac{c'(n_1)}{4c^2(n_1)}l_1 q(l_i - m_i), \\ \partial_X m_i = & \frac{p}{8c^3(n_1)}[(c^2(n_1)-\lambda_i)(h_1+h_2-2h_1h_2) - 2(3c^2(n_1)-\lambda_i)\mathbf{L}\cdot\mathbf{m}]n_i \\ & -\frac{c'(n_1)}{4c^2(n_1)}m_1 p(l_i - m_i), \\ \partial_Y \mathbf{n} = & \frac{q}{2c(n_1)}\mathbf{m}, \qquad (\text{or} \quad \partial_X \mathbf{n} = \frac{p}{2c(n_1)}\mathbf{L}), \\ \partial_Y h_1 = & \frac{c'(n_1)}{4c^2(n_1)}ql_1(h_1-h_2), \qquad \partial_X h_2 = \frac{c'(n_1)}{4c^2(n_1)}pm_1(h_2-h_1), \\ p_Y = & -\frac{c'(n_1)}{4c^2(n_1)}pq(l_1-m_1), \qquad q_X = \frac{c'(n_1)}{4c^2(n_1)}pq(l_1-m_1). \end{cases} \tag{4.3}$$

with $i = 1, 2, 3$, $\lambda_1 = \gamma$ and $\lambda_2 = \lambda_3 = \alpha$. Using (4.1), by letting $f = t$ or x, we obtain the equations

$$t_X = \frac{ph_1}{2c(n_1)}, \quad t_Y = \frac{qh_2}{2c(n_1)}, \quad x_X = \frac{ph_1}{2}, \quad x_Y = -\frac{qh_2}{2}. \tag{4.4}$$

Given the initial data (1.4), the corresponding boundary data for (4.3) can be determined as follows. In the X–Y plane, consider the line

$$\gamma_0 = \{(X, Y);\ X + Y = 0\} \subset \mathbb{R}^2$$

parameterized as $x \mapsto (\bar{X}(x), \bar{Y}(x)) \doteq (x, -x)$. Along the curve γ_0 we assign the boundary data $(\bar{\mathbf{n}}, \bar{\mathbf{L}}, \bar{\mathbf{m}}, \bar{h_1}, \bar{h_2}, \bar{p}, \bar{q})$ as

$$\begin{aligned} &\bar{\mathbf{n}} = \mathbf{n}_0(x), \quad \bar{\mathbf{L}} = \mathbf{R}(0, x)\bar{h}_1, \quad \bar{\mathbf{m}} = \mathbf{S}(0, x)\bar{h}_2, \\ &\bar{h}_1 = \frac{1}{1 + |\mathbf{R}(0, x)|^2}, \quad \bar{h}_2 = \frac{1}{1 + |\mathbf{S}(0, x)|^2}, \\ &\bar{p} = 1 + |\mathbf{R}(0, x)|^2, \quad \bar{q} = 1 + |\mathbf{S}(0, x)|^2, \end{aligned} \tag{4.5}$$

where

$$\mathbf{R}(0, x) = \mathbf{n}_1 + c(n_{10}(x))\mathbf{n}_0'(x), \quad \mathbf{S}(0, x) = \mathbf{n}_1 - c(n_{10}(x))\mathbf{n}_0'(x).$$

It suffices to express the solution $\mathbf{n}(X, Y)$ in terms of the original variables (t, x), then according to the results in [13], we have

Lemma 4.1. *Let $(\mathbf{n}, \mathbf{L}, \mathbf{m}, h_1, h_2, p, q, x, t)$ be a smooth solution to the system (4.3)–(4.4) with $p, q > 0$. Then the set of points*

$$\big\{(t(X, Y), x(X, Y), \mathbf{n}(X, Y));\ (X, Y) \in \mathbb{R}^2\big\} \tag{4.6}$$

is the graph of a conservative solution to (1.1).

4.1. **Compatible boundary data.** More generally, Instead of (4.5), along a line $\gamma = \{(X, Y);\ X + Y = \kappa\}$, we can assign the boundary data for (4.3). That is

$$\begin{aligned} &\mathbf{n}(s, \kappa - s) = \bar{\mathbf{n}}(s), \quad \mathbf{L}(s, \kappa - s) = \bar{\mathbf{L}}(s), \quad \mathbf{m}(s, \kappa - s) = \bar{\mathbf{m}}(s), \\ &h_1(s, \kappa - s) = \bar{h}_1(s), \quad h_2(s, \kappa - s) = \bar{h}_2(s), \\ &p(s, \kappa - s) = \bar{p}_1(s), \quad q(s, \kappa - s) = \bar{q}_1(s). \end{aligned} \tag{4.7}$$

If both equations in $(4.3)_3$ hold, then the boundary data should satisfy the compatibility condition

$$\frac{d}{ds}\bar{\mathbf{n}}(s) = \frac{d}{ds}\mathbf{n}(s, \kappa - s) = (\partial_X \mathbf{n} - \partial_Y \mathbf{n})(s, \kappa - s) = \frac{\bar{p}(s)}{2c(\bar{n}_1(s))}\bar{\mathbf{L}}(s) - \frac{\bar{q}(s)}{2c(\bar{n}_1(s))}\bar{\mathbf{m}}(s). \tag{4.8}$$

Now, it suffices to see that, if the compatibility condition (4.8) hold, then any smooth solution satisfying one of the equations in $(4.3)_3$ satisfies the other as well. More specifically, we have

Lemma 4.2. *Assume the compatibility condition (4.8) is satisfied. Let $(\mathbf{n}, \mathbf{L}, \mathbf{m}, h_1, h_2, p, q)(X, Y)$ be smooth solutions of the system(4.3) with the boundary conditions (4.7) along the line $\gamma = \{(X, Y);\ X + Y = \kappa\}$. Then, for any $(X, Y) \in \mathbb{R}^2$, it holds that*

$$\partial_Y \mathbf{n} = \frac{q}{2c(n_1)}\mathbf{m} \tag{4.9}$$

if and only if

$$\partial_X \mathbf{n} = \frac{p}{2c(n_1)}\mathbf{L}. \tag{4.10}$$

Proof. Assume that (4.9) holds, namely, $\partial_Y n_i = \frac{q}{2c(n_1)} m_i, i = 1,2,3$. To begin with, we define a smooth function

$$\varphi(n_1) = 2\int_0^{n_1} c(s)\,ds,$$

then (4.9) for $i = 1$ can be expressed as

$$\varphi(n_1)_Y = 2c(n_1)\partial_Y n_1 = q m_1. \tag{4.11}$$

Integrating (4.11) in Y to obtain

$$\varphi(n_1(X,Y)) = \varphi(n_1(X,\kappa - X)) + \int_{\kappa - X}^{Y} (q m_1)(X,y)\,dy.$$

Differentiating the above identity w.r.t. X, thus, it follows from (4.3) and (4.8) that

$$\begin{aligned}
\varphi(n_1(X,Y))_X =& \varphi'(n_1(X,\kappa - X)) \cdot (\partial_X n_1 - \partial_Y n_1)(X,\kappa - X) + (q m_1)(X,\kappa - X) \\
&+ \int_{\kappa - X}^{Y} [q_X m_1 + q\partial_X m_1](X,y)\,dy \\
=& p l_1(X,\kappa - X) + \int_{\kappa - X}^{Y} \Big(\frac{pq}{8c^3(n_1)} [(c^2(n_1) - \gamma)(h_1 + h_2 - 2h_1 h_2) \\
&\qquad - 2(3c^2(n_1) - \gamma)\mathbf{L}\cdot\mathbf{m}] n_1 \Big)(X,y)\,dy \\
=& p l_1(X,\kappa - X) + \int_{\kappa - X}^{Y} \frac{\partial}{\partial Y}(p l_1)(X,y)\,dy \\
=& p l_1(X,Y),
\end{aligned}$$

which yields

$$\partial_X n_1 = \frac{p}{2c(n_1)} l_1. \tag{4.12}$$

On the other hand, we directly integrate (4.9) for $n_i, i = 2,3$ to deduce

$$n_i(X,Y) = n_i(X,\kappa - X) + \int_{\kappa - X}^{Y} \frac{q m_i}{2c(n_1)}(X,y)\,dy.$$

Differentiating it w.r.t. X, and then utilizing the estimates (4.3), (4.8), (4.12) to get

$$\begin{aligned}\partial_X n_i(X,Y) =&(\partial_X n_i - \partial_Y n_i)(X,\kappa - X) + \frac{q m_i}{2c(n_1)}(X,\kappa-X)\\ &+\int_{\kappa-X}^{Y}[\frac{\partial_X q}{2c(n_1)}m_i + \frac{q\partial_X m_i}{2c(n_1)} - \frac{q m_i}{2c^2(n_1)}c'(n_1)\partial_X n_1](X,y)\,dy\\ =&\frac{p l_i}{2c(n_1)}(X,\kappa-X) - \int_{\kappa-X}^{Y}\frac{c'(n_1)pq}{8c^3(n_1)}(l_1 m_i + l_i m_1)(X,y)\,dy\\ &+\int_{\kappa-X}^{Y}\Big(\frac{pq}{16c^4(n_1)}[(c^2(n_1)-\alpha)(h_1+h_2-2h_1h_2)\\ &\qquad - 2(3c^2(n_1)-\alpha)\mathbf{L}\cdot\mathbf{m}]n_i\Big)(X,y)\,dy\\ =&\frac{p l_i}{2c(n_1)}(X,\kappa-X) + \int_{\kappa-X}^{Y}\frac{\partial}{\partial Y}(\frac{p l_i}{2c(n_1)})(X,y)\,dy\\ =&\frac{p l_i}{2c(n_1)}(X,Y).\end{aligned}$$

This together with (4.12) implies that (4.10) holds. In a same fashion, we can prove the converse implication. This completes the proof of Lemma 4.2. □

Now, we setting the boundary data for t, x along the line $\gamma = \{(X,Y);\ X+Y=\kappa\}$ by

$$x(s,\kappa-s) = \bar{x}(s), \quad t(s,\kappa-s) = \bar{t}(s). \tag{4.13}$$

In accordance with (4.4), we get the compatibility conditions

$$\frac{d}{ds}\bar{x}(s) = \frac{\bar{p}(s)\bar{h}_1(s) + \bar{q}(s)\bar{h}_2(s)}{2}, \tag{4.14}$$

$$\frac{d}{ds}\bar{t}(s) = \frac{\bar{p}(s)\bar{h}_1(s) - \bar{q}(s)\bar{h}_2(s)}{2c(\bar{n}_1(s))}. \tag{4.15}$$

Then, we have the following results, which can be proved in a similar procedure as [5], we omit it here for brevity.

Lemma 4.3. *Let $(\mathbf{n},\mathbf{L},\mathbf{m},h_1,h_2,p,q)(X,Y)$ be smooth solutions of the system(4.3). Then there exists a solution $(t,x)(X,Y)$ of (4.4) with the boundary data (4.13) if and only if the compatibility conditions (4.14)–(4.15) are satisfied.*

4.2. **Families of perturbed solutions.** Given a point (X_0,Y_0), and consider the line

$$\gamma = \{(X,Y);\ X+Y=\kappa\}, \quad \kappa \doteq X_0+Y_0. \tag{4.16}$$

Now, for a fixed solution of (4.3), we are going to construct several families of perturbed solutions.

Lemma 4.4. *Assume generic condition holds. Let $(\mathbf{n},\mathbf{L},\mathbf{m},h_1,h_2,p,q)$ be a smooth solution to the system (4.3) and let a point $(X_0,Y_0)\in\mathbb{R}^2$ be given.*
(1) *If $(h_1,\mathbf{L}_X,\mathbf{L}_{XX})(X_0,Y_0) = (0,\mathbf{0},\mathbf{0})$, then there exists a 7-parameter family of smooth solutions $(\mathbf{n}^\theta,\mathbf{L}^\theta,\mathbf{m}^\theta,h_1^\theta,h_2^\theta,p^\theta,q^\theta)$, depending smoothly on $\theta\in\mathbb{R}^7$, such that the following holds.*

(i) When $\theta = \mathbf{0}\in\mathbb{R}^7$ one recovers the original solution, namely $(\mathbf{n}^0,\mathbf{L}^0,\mathbf{m}^0,h_1^0,h_2^0,p^0,q^0) = (\mathbf{n},\mathbf{L},\mathbf{m},h_1,h_2,p,q)$.

(ii) At a point (X_0,Y_0), when $\theta=\mathbf{0}$ one has

$$rank\ D_\theta(h_1,\mathbf{L}_X,\mathbf{L}_{XX}) \ = \ 7. \tag{4.17}$$

(2) *If* $(h_1, h_2, \mathbf{L}_X)(X_0, Y_0) = (0, 0, \mathbf{0})$, *then there exists a 5-parameter family of smooth solutions* $(\mathbf{n}^\theta, \mathbf{L}^\theta, \mathbf{m}^\theta, h_1^\theta, h_2^\theta, p^\theta, q^\theta)$, *satisfying (i)-(ii) above, with* (4.17) *replaced by*

$$\text{rank } D_\theta(h_1, h_2, \mathbf{L}_X) \;=\; 5. \tag{4.18}$$

(3) *If* $(h_1, (\alpha-\gamma)(1-n_1^2)n_1, \mathbf{L}_X)(X_0, Y_0) = (0, 0, \mathbf{0})$, *then there exists a 5-parameter family of smooth solutions* $(\mathbf{n}^\theta, \mathbf{L}^\theta, \mathbf{m}^\theta, h_1^\theta, h_2^\theta, p^\theta, q^\theta)$ *satisfying (i)(ii) as above, with* (4.17) *replaced by*

$$\text{rank } D_\theta(h_1, (\alpha-\gamma)(1-n^2)n_1, \mathbf{L}_X) \;=\; 5. \tag{4.19}$$

Proof. Let $(\mathbf{n}^\theta, \mathbf{L}^\theta, \mathbf{m}^\theta, h_1^\theta, h_2^\theta, p^\theta, q^\theta)$ be a smooth solution of the semilinear system (4.3). Given the point (X_0, Y_0), let the line γ and the values $(\bar{\mathbf{n}}, \bar{\mathbf{L}}, \bar{\mathbf{m}}, \bar h_1, \bar h_2, \bar p, \bar q)$ as in (4.16) and (4.7), respectively. To begin with, we calculate the values of $\partial_X l_i$ and $\partial_{XX} l_i$ $(i = 1, 2, 3)$ at the point (X_0, Y_0) for later use. Indeed, it is easy to see that, at any point $(s, \kappa - s) \in \gamma$,

$$\begin{aligned}
&\bar l_i'(s) = (\partial_X l_i - \partial_Y l_i)(s, \kappa - s), \quad \bar m_i'(s) = (\partial_X m_i - \partial_Y m_i)(s, \kappa - s), \quad i = 1, 2, 3,\\
&\bar h_j'(s) = (\partial_X h_j - \partial_Y h_j)(s, \kappa - s), \quad \bar q'(s) = (\partial_X q - \partial_Y q)(s, \kappa - s), \quad j = 1, 2.
\end{aligned}$$

Here and in the sequel, a prime denotes derivative w.r.t. the parameter s along the line γ. Thus, in view of (4.3), we have

$$\begin{aligned}
&\partial_X l_i(X_0, Y_0) = \bar l_i' + \bar q f_i + \frac{c'(\bar n_1)}{4c^2(\bar n_1)} \bar l_1 \bar q (\bar l_i - \bar m_i),\\
&\partial_Y m_i(X_0, Y_0) = -\bar m_i' + \bar p f_i - \frac{c'(\bar n_1)}{4c^2(\bar n_1)} \bar m_1 \bar p (\bar l_i - \bar m_i),\\
&\partial_X h_1(X_0, Y_0) = \bar h_1' + \frac{c'(\bar n_1)}{4c^2(\bar n_1)} \bar l_1 \bar q (\bar h_1 - \bar h_2),\\
&\partial_Y h_2(X_0, Y_0) = -\bar h_2' + \frac{c'(\bar n_1)}{4c^2(\bar n_1)} \bar m_1 \bar p (\bar h_2 - \bar h_1),\\
&\partial_Y q(X_0, Y_0) = -\bar q' + \frac{c'(\bar n_1)}{4c^2(\bar n_1)} \bar p \bar q (\bar l_1 - \bar m_1),
\end{aligned} \tag{4.20}$$

where all terms on the right hand sides are evaluated at $s = X_0$ and we have denoted

$$f_i \;\doteq\; \frac{1}{8c^3(\bar n_1)}[(c^2(\bar n_1) - \lambda_i)(\bar h_1 + \bar h_2 - 2\bar h_1 \bar h_2) - 2(3c^2(\bar n_1) - \lambda_i)\bar{\mathbf{L}} \cdot \bar{\mathbf{m}}]\bar n_i,$$

with $i = 1, 2, 3$. A further differentiation yields

$$\frac{d^2}{ds^2}\bar l_i(s) = \frac{d}{ds}[\partial_X l_i(s, \kappa - s) - \partial_Y l_i(s, \kappa - s)] = (\partial_{XX} l_i + \partial_{YY} l_i - 2\partial_{XY} l_i)(s, \kappa - s), \tag{4.21}$$

for $i = 1, 2, 3$, where, by virtue of (4.3), the relations for $\partial_{YY} l_i$ and $\partial_{XY} l_i$ can be bounded as

$$\begin{aligned}
\partial_{YY} l_i(X_0, Y_0) =& \bar q \partial_Y f_i + f_i \partial_Y \bar q + \big(\frac{c'(\bar n_1)}{4c^2(\bar n_1)}\big)' \bar l_1 \bar q (\bar l_i - \bar m_i) \partial_Y \bar n_1\\
&+ \frac{c'(\bar n_1)}{4c^2(\bar n_1)}[(\bar q \partial_Y \bar l_1 + \bar l_1 \partial_Y \bar q)(\bar l_i - \bar m_i) + \bar l_1 \bar q (\partial_Y \bar l_i - \partial_Y \bar m_i)]\\
=& g_{i1}, \qquad \text{for} \quad i = 1, 2, 3,
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\partial_{XY} l_i(X_0,Y_0) =&\bar{q}\partial_X f_i + f_i\partial_X\bar{q} + \big(\frac{c'(\bar{n}_1)}{4c^2(\bar{n}_1)}\big)'\bar{l}_1\bar{q}(\bar{l}_i-\bar{m}_i)\partial_X\bar{n}_1\\ &+\frac{c'(\bar{n}_1)}{4c^2(\bar{n}_1)}[(\bar{q}\partial_X\bar{l}_1+\bar{l}_1\partial_X\bar{q})(\bar{l}_i-\bar{m}_i)+\bar{l}_1\bar{q}(\partial_X\bar{l}_i-\partial_X\bar{m}_i)]\\ =&g_{i2}, \qquad \text{for} \quad i=1,2,3.\end{aligned} \tag{4.23}$$

Here, for $r = X$ or Y, a straightforward computation gives rise to

$$\begin{aligned}\partial_r f_i =&\frac{3c'(\bar{n}_1)}{c(\bar{n}_1)} f_i\partial_r\bar{n}_1 + \frac{1}{8c^3(\bar{n}_1)}\Big[2c(\bar{n}_1)c'(\bar{n}_1)\partial_r\bar{n}_1(\bar{h}_1+\bar{h}_2-2\bar{h}_1\bar{h}_2)+(c^2(\bar{n}_1)-\lambda_i)\\ &\cdot(\partial_r\bar{h}_1+\partial_r\bar{h}_2-2\bar{h}_2\partial_r\bar{h}_1-2\bar{h}_1\partial_r\bar{h}_2)-12c(\bar{n}_1)c'(\bar{n}_1)\partial_r\bar{n}_1\bar{\mathbf{L}}\cdot\bar{\mathbf{m}}\\ &-2(3c^2(\bar{n}_1)-\lambda_i)(\partial_r\bar{\mathbf{L}}\cdot\bar{\mathbf{m}}+\bar{\mathbf{L}}\cdot\partial_r\bar{\mathbf{m}})\Big]\bar{n}_i+\frac{1}{8c^3(\bar{n}_1)}[(c^2(\bar{n}_1)-\lambda_i)\\ &\cdot(\bar{h}_1+\bar{h}_2-2\bar{h}_1\bar{h}_2)-2(3c^2(\bar{n}_1)-\lambda_i)\bar{\mathbf{L}}\cdot\bar{\mathbf{m}}]\partial_r\bar{n}_i, \qquad \text{for} \quad i=1,2,3.\end{aligned}$$

Hence, in light of (4.21)–(4.23), we obtain

$$\partial_{XX} l_i = \bar{l}_i'' - g_{i1} + 2g_{i2}, \qquad \text{for } i=1,2,3. \tag{4.24}$$

Now, we construct families solution $(\bar{\mathbf{n}}^\theta, \bar{\mathbf{L}}^\theta, \bar{\mathbf{m}}^\theta, \bar{h}_1^\theta, \bar{h}_2^\theta, \bar{p}^\theta, \bar{q}^\theta)$ of perturbation of the data (4.7) along the curve γ, so that the matrices in (4.17)–(4.19) have full rank at the point (X_0,Y_0). These perturbations will have the form

$$\begin{cases} \bar{l}_i^\theta(s) = \bar{l}_i(s) + \sum_{j=1}^N \theta_j L_{ij}(s),\\ \bar{m}_i^\theta(s) = \bar{m}_i(s) + \sum_{j=1}^N \theta_j M_{ij}(s),\\ \bar{h}_1^\theta(s) = \bar{h}_1(s) + \sum_{j=1}^N \theta_j H_{1j}(s), \end{cases} \qquad \begin{cases} \bar{h}_2^\theta(s) = \bar{h}_2(s) + \sum_{j=1}^N \theta_j H_{2j}(s),\\ \bar{p}^\theta(s) = \bar{p}(s) + \sum_{j=1}^N \theta_j P_j(s),\\ \bar{q}^\theta(s) = \bar{q}(s) + \sum_{j=1}^N \theta_j Q_j(s), \end{cases}$$

for some suitable functions $L_{ij}, M_{ij}, H_{1j}, H_{2j}, P_j, Q_j \in \mathcal{C}_0^\infty(\mathbb{R})$, $i=1,2,3$, $j=1,\cdots,N$. Note that the construction in (1) needs $N=7$ and in (2)–(3), we take $N=5$. Moreover, at point $s=X_0$, we set

$$\bar{n}_i^\theta(X_0) = \bar{n}_i(X_0) + \sum_{j=1}^N \theta_j \mathcal{N}_{ij}(X_0), \qquad \text{for } i=1,2,3,$$

which, together with the compatibility condition (4.8) determine the values of $\bar{n}_i^\theta(s)$ for all $s\in\mathbb{R}$.

Thus, by choosing suitable perturbations $(\bar{\mathbf{n}}^\theta, \bar{\mathbf{L}}^\theta, \bar{\mathbf{m}}^\theta, \bar{h}_1^\theta, \bar{h}_2^\theta, \bar{p}^\theta, \bar{q}^\theta)$, so that at the point $s = X_0$ and $\theta = \mathbf{0}$, the Jacobian matrix of first order derivatives w.r.t. θ is given by

$$D_\theta \begin{pmatrix} \bar{n}_1 \\ \bar{n}_2 \\ \bar{n}_3 \\ \bar{l}_1 \\ \bar{l}_2 \\ \bar{l}_3 \\ \bar{m}_1 \\ \bar{m}_2 \\ \bar{m}_3 \\ \bar{l}_1' \\ \bar{l}_2' \\ \bar{l}_3' \\ \bar{h}_1 \\ \bar{h}_2 \\ \bar{p} \\ \bar{q} \\ \bar{l}_1'' \\ \bar{l}_2'' \\ \bar{l}_3'' \\ \bar{h}_1' \\ \bar{h}_2' \\ \bar{m}_1' \\ \bar{m}_2' \\ \bar{m}_3' \\ \bar{q}' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

At the point (X_0, Y_0), this relation, (4.20) and (4.24) imply that

$$D_\theta \begin{pmatrix} h_1 \\ \partial_X l_1 \\ \partial_X l_2 \\ \partial_X l_3 \\ \partial_{XX} l_1 \\ \partial_{XX} l_2 \\ \partial_{XX} l_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 1 & 0 & 0 & 0 \\ * & * & * & * & 1 & 0 & 0 \\ * & * & * & * & 0 & 1 & 0 \\ * & * & * & * & 0 & 0 & 1 \end{pmatrix}.$$

That means (4.17) holds.

On the other hand, we choose suitable perturbations $(\bar{\mathbf{n}}^\theta, \bar{\mathbf{L}}^\theta, \bar{\mathbf{m}}^\theta, \bar{h}_1^\theta, \bar{h}_2^\theta, \bar{p}^\theta, \bar{q}^\theta)$, such that, at the point $s = X_0$ and $\theta = \mathbf{0}$, we have

$$D_\theta \begin{pmatrix} \bar{n}_1 \\ \bar{n}_2 \\ \bar{n}_3 \\ \bar{l}_1 \\ \bar{l}_2 \\ \bar{l}_3 \\ \bar{m}_1 \\ \bar{m}_2 \\ \bar{m}_3 \\ \bar{h}_1 \\ \bar{h}_2 \\ \bar{p} \\ \bar{q} \\ \bar{l}_1' \\ \bar{l}_2' \\ \bar{l}_3' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

At the point (X_0, Y_0), the above construction and (4.20) yields

$$D_\theta \begin{pmatrix} h_1 \\ h_2 \\ \partial_X l_1 \\ \partial_X l_2 \\ \partial_X l_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 \\ * & * & 0 & 1 & 0 \\ * & * & 0 & 0 & 1 \end{pmatrix}.$$

This achieves (4.18).

Finally, if at the point (X_0, Y_0), $(h_1, (\alpha-\gamma)(1-n_1^2)n_1, \mathbf{L}_X) = (0, 0, \mathbf{0})$, then the generic condition $\alpha \neq \gamma$ implies that

$$n_1(X_0, Y_0) = 0 \quad \text{or} \quad \pm 1. \tag{4.25}$$

Hence, we choose suitable perturbations $(\bar{\mathbf{n}}^\theta, \bar{\mathbf{L}}^\theta, \bar{\mathbf{m}}^\theta, \bar{h}_1^\theta, \bar{h}_2^\theta, \bar{p}^\theta, \bar{q}^\theta)$, such that, at the point $s = X_0$ and $\theta = \mathbf{0}$, we have

$$D_\theta \begin{pmatrix} \bar{n}_1 \\ \bar{n}_2 \\ \bar{n}_3 \\ \bar{l}_1 \\ \bar{l}_2 \\ \bar{l}_3 \\ \bar{m}_1 \\ \bar{m}_2 \\ \bar{m}_3 \\ \bar{h}_1 \\ \bar{h}_2 \\ \bar{p} \\ \bar{q} \\ \bar{l}_1' \\ \bar{l}_2' \\ \bar{l}_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

At the point (X_0, Y_0), it follows from (4.20) that

$$D_\theta \begin{pmatrix} h_1 \\ (\alpha-\gamma)(1-n_1^2)n_1 \\ \partial_X l_1 \\ \partial_X l_2 \\ \partial_X l_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & (\alpha-\gamma)(1-3n_1^2) & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 \\ * & * & 0 & 1 & 0 \\ * & * & 0 & 0 & 1 \end{pmatrix}.$$

This together with (4.25) implies (4.19). This completes the proof of Lemma 4.4. □

4.3. **Proof of the main Theorem 2.3.** We start with the following lemma, which shows for almost all of the solutions the level sets $\{(X,Y);\ h_1(X,Y) = 0\}$ and $\{(X,Y);\ h_2(X,Y) = 0\}$ satisfy a number of generic properties. The proof relies on an application of Thom's transversality theorem [3, 5, 19]. The proof is similar to [5], we omit it here for brevity.

Lemma 4.5. *Let a compact domain of the form*

$$\Omega \doteq \{(X,Y);\ |X| + |Y| \le M\},$$

and define $\mathcal{U}$ be the family of all $\mathcal{C}^2$ solutions $(\mathbf{n}, \mathbf{L}, \mathbf{m}, h_1, h_2, p, q)$ to the semilinear system (4.3), with $p, q > 0$ for all $(X,Y) \in \mathbb{R}^2$. Moreover, define $\mathcal{U}' \subset \mathcal{U}$ be the subfamily of all solutions $(\mathbf{n}, \mathbf{L}, \mathbf{m}, h_1, h_2, p, q)$, such that for $(X,Y) \in \Omega$, none of the following values is attained:

$$\begin{cases} (h_1, \mathbf{L}_X, \mathbf{L}_{XX}) = (0, \mathbf{0}, \mathbf{0}), \\ (h_2, \mathbf{m}_Y, \mathbf{m}_{YY}) = (0, \mathbf{0}, \mathbf{0}), \end{cases}$$
$$\begin{cases} (h_1, h_2, \mathbf{L}_X) = (0, 0, \mathbf{0}), \\ (h_1, h_2, \mathbf{m}_Y) = (0, 0, \mathbf{0}), \end{cases} \tag{4.26}$$
$$\begin{cases} (h_1, (\alpha-\gamma)(1-n_1^2)n_1, \mathbf{L}_X) = (0, 0, \mathbf{0}), \\ (h_2, (\alpha-\gamma)(1-n_1^2)n_1, \mathbf{m}_Y) = (0, 0, \mathbf{0}). \end{cases}$$

Then $\mathcal{U}'$ is a relatively open and dense subset of $\mathcal{U}$, in the topology induced by $\mathcal{C}^2(\Omega)$.

Remark 4.1. *There are many forward and backward singular curves on which $h_1 = 0$ and $h_2 = 0$, respectively. The conditions in (4.26) are corresponding to: starting or ending points of the singular curves; intersection points of two singular curves in different directions; inner points of the singular curves, respectively.*

Moreover, we have that, if the initial $(n_{i0}, n_{i1}) \in \mathcal{N}$, the space $\mathcal{N}$ is defined in the (4.27), then the solution remains smooth for all $|x|$ sufficiently large. That is,

Lemma 4.6. *Assume $(n_{i0}, n_{i1}) \in \mathcal{N}$ and let $T > 0$ be given. Then there exists $r > 0$ sufficiently large so that the solution $\mathbf{n} = \mathbf{n}(t, x)$ of (1.1)–(1.4) remains $\mathcal{C}^2$ on the domain $\{(t, x);\ t \in [0, T],\ |x| \geq r\}$.*

Now, we are ready to prove the Theorem 2.3.

Proof of Theorem 2.3. First, we denote

$$\mathcal{N} \doteq \Big(\mathcal{C}^3(\mathbb{R}) \cap H^1(\mathbb{R})\Big) \times \Big(\mathcal{C}^2(\mathbb{R}) \cap L^2(\mathbb{R})\Big), \tag{4.27}$$

with norm

$$\|(n_{i0}, n_{i1})\|_{\mathcal{N}} \doteq \|n_{i0}\|_{\mathcal{C}^3} + \|n_{i0}\|_{H^1} + \|n_{i1}\|_{\mathcal{C}^2} + \|n_{i1}\|_{L^2}.$$

Here and in the rest of this manuscript, $i = 1, 2, 3$. Given initial data $(\hat{n}_{i0}, \hat{n}_{i1}) \in \mathcal{N}$ and denote the open ball

$$B_\delta \doteq \{(n_{i0}, n_{i1}) \in \mathcal{N};\ \|(n_{i0}, n_{i1}) - (\hat{n}_{i0}, \hat{n}_{i1})\|_{\mathcal{N}} < \delta\}.$$

1. Let $(\hat{n}_{i0}, \hat{n}_{i1}) \in \mathcal{N}$, by the definition of the space $\mathcal{N}$, we have

$$\hat{n}_{i0}(x) \to 0, \quad \partial_x \hat{n}_{i0}(x) \to 0, \quad \text{and} \quad \hat{n}_{i1}(x) \to 0, \quad \text{as } |x| \to \infty.$$

Thus, the corresponding functions $\mathbf{R}, \mathbf{S}$ in (2.1) satisfy, as $|x| \to \infty$,

$$R_i(0, x) \to 0, \quad \text{and} \quad S_i(0, x) \to 0.$$

So the functions $\hat{R}_i, \hat{S}_i$ are uniformly bounded on a domain of the form $\{(t, x); t \in [0, T], |x| \geq \rho\}$, for $\rho > 0$ large enough. More specifically, we can choose $\delta > 0$, such that for every initial data $(n_{i0}, n_{i1}) \in B_\delta$, the corresponding solution $\mathbf{n}(t, x)$ remains twice continuously differentiable on the outer domain $\{(t, x);\ t \in [0, T], |x| \geq \varrho\}$, for some $\varrho > 0$ sufficiently large. This implies that the singularities of $\mathbf{n}(t, x)$ in the set $[0, T] \times \mathbb{R}$ only occur on the compact set $\mathcal{M} \doteq [0, T] \times [-\varrho, \varrho]$.

Next, for any $(n_{i0}, n_{i1}) \in B_\delta$, let Λ be the map of $(X, Y) \mapsto \Lambda(X, Y) \doteq (t(X, Y), x(X, Y))$, and let Ω be a domain as in Lemma 4.5. Choosing M large enough and by possibly shrinking the radius δ, we can obtain the inclusion $\mathcal{M} \subset \Lambda(\Omega)$.

Now, the subset $\tilde{\mathcal{D}} \subset B_\delta$ is defined as follows. $(n_{i0}, n_{i1}) \in \tilde{\mathcal{D}}$ if $(n_{i0}, n_{i1}) \in B_\delta$ and for the corresponding solution $(\mathbf{n}, \mathbf{L}, \mathbf{m}, h_1, h_2, p, q)$ of (4.3) with boundary data (4.5), the values (4.26) are never attained, for any (X, Y) such that $(t(X, Y), x(X, Y)) \in \mathcal{M}$.

2. At this step, we claim the set $\tilde{\mathcal{D}}$ is open, in the topology of $\mathcal{C}^3 \times \mathcal{C}^2$. Indeed, consider a sequence of initial data $(n_{i0}^\nu, n_{i1}^\nu)_{\nu \geq 1}$ such that the sequence converges to (n_{i0}, n_{i1}), with $(n_{i0^\nu}, n_{i1}^\nu) \notin \tilde{\mathcal{D}}$. By the definition of $\tilde{\mathcal{D}}$, there exist points (X^ν, Y^ν) at which the corresponding solutions $(\mathbf{n}^\nu, \mathbf{L}^\nu, \mathbf{m}^\nu, h_1^\nu, h_2^\nu, p^\nu, q^\nu)$ satisfy

$$(h_1^\nu, \mathbf{L}_X^\nu, \mathbf{L}_{XX}^\nu)(X^\nu, Y^\nu) = (0, \mathbf{0}, \mathbf{0}), \quad (t^\nu(X^\nu, Y^\nu), x^\nu(X^\nu, Y^\nu)) \in \mathcal{M},$$

for all $\nu \geq 1$. Observe that the domain Ω is compact, we can choose a subsequence, denote still by (X^ν, Y^ν), such that $(X^\nu, Y^\nu) \to (\bar{X}, \bar{Y})$ for some point $(\bar{X}, \bar{Y})$. By continuity,

$$(h_1, \mathbf{L}_X, \mathbf{L}_{XX})(\bar{T}, \bar{Y}) = (0, \mathbf{0}, \mathbf{0}), \quad (t(\bar{X}, \bar{Y}), x(\bar{X}, \bar{Y})) \in \mathcal{M},$$

which implies $(n_{i0}, n_{i1}) \notin \tilde{\mathcal{D}}$. Repeating the same procedure on the other cases in (4.26), we can obtain $\tilde{\mathcal{D}}$ is open.

3. Now, we will prove the set $\tilde{\mathcal{D}}$ is dense in B_δ. Given $(n_{i0}, n_{i1}) \in B_\delta$, by a small perturbation, we can assume that $n_{i0}, n_{i1} \in \mathcal{C}^\infty$.

From Lemma 4.5, we can construct a sequence of solutions $(\mathbf{n}^\nu, \mathbf{L}^\nu, \mathbf{m}^\nu, h_1^\nu, h_2^\nu, p^\nu, q^\nu, x^\nu, t^\nu)$ of (4.3), such that, for every $\nu \geq 1, (X,Y) \in \Omega$, the values in (4.26) are never attained, and the $\mathcal{C}^k, k \geq 1$ norm satisfies

$$\lim_{\nu\to\infty} \|(n_i^\nu - n_i, l_i^\nu - l_i, m_i^\nu - m_i, h_1^\nu - h_1, h_2^\nu - h_2, p^\nu - p, q^\nu - q, x^\nu - x, t^\nu - t)\|_{\mathcal{C}^k(\Gamma)} = 0,$$

for every bounded set $\Gamma \subset \mathbb{R}^2$. Thus, consider the corresponding solution $\mathbf{n}^\nu(t,x)$ of (1.1), with graph $\{(t^\nu(X,Y), x^\nu(X,Y), \mathbf{n}^\nu(X,Y);\ (X,Y) \subset \mathbb{R}^2)\}$. And for $t = 0$, the corresponding sequence of initial values satisfies

$$\lim_{\nu\to\infty} \|n_{i0}^\nu - n_{i0}\|_{\mathcal{C}^k(I)} = 0, \quad \lim_{\nu\to\infty} \|n_{i1}^\nu - n_{i1}\|_{\mathcal{C}^k(I)} = 0, \tag{4.28}$$

for every bounded set $I \subset \mathbb{R}$.

Consider a cutoff function $\psi(x) \in \mathcal{C}_0^\infty$, such that

$$\psi(x) = 1, \qquad \text{if } |x| \leq \eta,$$
$$\psi(x) = 0, \qquad \text{if } |x| \geq \eta + 1,$$

where $\eta \gg \varrho$ is large enough. Then for every $\nu \geq 1$, define the following initial data

$$\tilde{n}_{i0}^\nu \doteq \psi n_{i0}^\nu + (1-\psi)n_{i0}, \quad \tilde{n}_{i1}^\nu \doteq \psi n_{i1}^\nu + (1-\psi)n_{i1},$$

which together with (4.28) implies

$$\lim_{\nu\to\infty} \|(\tilde{n}_{i0}^\nu - n_{i0}, \tilde{n}_{i1}^\nu - n_{i1}\|_{\mathcal{N}} = 0.$$

Choosing $\eta > 0$ sufficiently large, such that for every $(t,x) \in \mathcal{M}$,

$$\tilde{n}_i^\nu(t,x) = n_i^\nu(t,x).$$

While $\tilde{n}_i^\nu(t,x)$ remains $\mathcal{C}^2$ on the outer domain $\{(t,x);\ t \in [0,\mathcal{T}], |x| \geq \varrho\}$. Thus, we have proved for every $\nu \geq 1$ sufficiently large, $(\tilde{n}_{i0}^\nu, \tilde{n}_{i1}^\nu) \in \tilde{D}$, which means that $\tilde{D}$ is dense in B_δ.

4. Finally, we shall show that for every initial data $(n_{i0}, n_{i1}) \in \tilde{D}$, the corresponding solution $n_i(t,x)$ of (1.1) is piecewise $\mathcal{C}^2$ on the domain $[0,T] \times \mathbb{R}$. Indeed, we know $n_i(t,x)$ is $\mathcal{C}^2$ on the outer domain $\{(t,x);\ t \in [0,T], |x| \geq \varrho\}$ by the previous argument. So we need to study the singularity of solutions on the inner domain $\mathcal{M}$.

Recall from step **1**, every point in $\mathcal{M}$ is contained in the image of the domain Ω. Thus, for every point $(X_0, Y_0) \in \Omega$, we have two cases.

Case I. $h_1(X_0,Y_0) \neq 0$ and $h_2(X_0,Y_0) \neq 0$. From the coordinate change (4.4), the determinant of the Jacobian matrix is computed by

$$\det \begin{pmatrix} x_X & x_Y \\ t_X & t_Y \end{pmatrix} = \frac{pqh_1h_2}{2c} > 0.$$

From this, we know that the map $(X,Y) \mapsto (t,x)$ is locally invertible in a neighborhood of (X_0,Y_0). Therefore, the function n_i is $\mathcal{C}^2$ in a neighbouhood of $(t_0(X_0,Y_0), x(X_0,Y_0))$.

Case II. $h_1(X_0,Y_0) = 0$. Since $h_1 = 0$ implies $\mathbf{L} = \mathbf{0}$, in this case, we have either $\mathbf{L}_X \neq \mathbf{0}$ or $\mathbf{L}_Y \neq \mathbf{0}$. Indeed, at the point (X_0,Y_0), it follows from (4.3) that,

$$\partial_Y l_i(X_0,Y_0) = \frac{q}{8c^3(n_1)}[(c^2(n_1) - \lambda_i)h_2]n_i,$$

with $i = 1, 2, 3$. By the definition of $\tilde{D}$, we know the values $(h_1, h_2, \mathbf{L}_X) = (0, 0, \mathbf{0})$ and $(h_1, (\alpha - \gamma)(1 - n_1^2)n_1, \mathbf{L}_X) = (0, 0, \mathbf{0})$ are never attained in Ω. Thus, we get $\mathbf{L}_X \neq \mathbf{0}$ or $\mathbf{L}_Y \neq \mathbf{0}$.

5. By continuity, there exists $\eta > 0$, such that the values in (4.26) are never attained in the open neighborhood

$$\Omega' \ \doteq \ \{(X, Y);\ |X| < M + \eta, |Y| < M + \eta\}.$$

Thanks to the implicit function theorem, we derive that the sets

$$\mathcal{U}^l \ \doteq \ \{(X, Y) \in \Omega';\ \mathbf{L}(X, Y) = \mathbf{0},\ h_1(X, Y) = 0\},$$

and

$$\mathcal{U}^m \ \doteq \ \{(X, Y) \in \Omega';\ \mathbf{m}(X, Y) = \mathbf{0},\ h_2(X, Y) = 0\}$$

are 1–dimensional embedded manifold of class $\mathcal{C}^2$.

Now, we claim that the number of connected components of $\mathcal{U}^l$ that intersect the compact set Ω is finite. Assume, by contradiction, that $P_1, P_2, \cdots$ is a sequence of points in $\mathcal{U}^l \cap \Omega$ belonging to distinct components. Thus, we can choose a subsequence P_i, such that $P_i \to \bar{P}$ for some $\bar{P} \in \mathcal{U}^l \cap \Omega$. By assumption, $(\mathbf{L}_X, \mathbf{L}_Y)(\bar{P}) \neq (\mathbf{0}, \mathbf{0})$. Hence, by the implicit function theorem, there is a neighborhood Γ of $\bar{P}$ such that $\gamma := \mathcal{U}^l \cap \Gamma$ is a connected $\mathcal{C}^2$ curve. Thus, $P_i \in \gamma$ on all i large enough, providing a contradiction.

6. To complete the proof, we need to study in more detail the image of the singular set $\mathcal{U}^l$ and $\mathcal{U}^m$, since the set of points (t, x) where $\mathbf{n}$ is singular coincides with the image of the two sets $\mathcal{U}^l, \mathcal{U}^m$ under the C^2 map $(X, Y) \mapsto \Lambda(X, Y) = (t(X, Y), x(X, Y))$.

By the argument in step 5, inside the compact set Ω', there are only finite many points where $h_1 = 0, \mathbf{L} = \mathbf{0}$, and $\mathbf{L}_X = \mathbf{0}$, say $P_i = (X_i, Y_i), i = 1, \cdots, m$, and by (4.26), at a point $(X_0, Y_0) \in \mathcal{U}^l \cap \mathcal{U}^m$, we have $\mathbf{L}_X \neq \mathbf{0}, \mathbf{L}_Y = \mathbf{0}, \mathbf{m}_X = \mathbf{0}, \mathbf{m}_Y \neq \mathbf{0}$. Thus, the two curves $h_1 = 0$ and $h_2 = 0$ intersect perpendicular. Therefore, there are only finitely many such intersection points inside Ω, say $Q_\jmath = (t'_\jmath, Y'_\jmath), \jmath = 1, \cdots, n$. Moreover, the set $\mathcal{U}^l \backslash \{P_1, \cdots, P_m, Q_1, \cdots, Q_n\}$ has finitely many connected components which intersect Ω. Consider any one of these components. This is a connected curve, say γ_j, such that $h_1 = 0, \mathbf{L} = \mathbf{0}$ and $\mathbf{L}_X \neq \mathbf{0}$ for any $(X, Y) \in \gamma_j$. Thus, this curve can be expressed in the form

$$\gamma_j = \{(X, Y) :\ X = \phi_j(Y), a_j < Y < b_j\},$$

for a suitable function ϕ_j.

At this stage, we claim that the image $\Lambda(\gamma_j)$ is a $\mathcal{C}^2$ curve in the t–x plane. Indeed, it suffices to show that, on the open interval (a_j, b_j), the differential of the map $Y \mapsto (t(\phi_j(Y), Y), x(\phi_j(Y), Y))$ does not vanish. This is true, because by (4.4), we have

$$\frac{d}{dY} t(\phi_j(Y), Y)) = t_X \phi'_j + t_Y = \frac{q h_2}{2c(n_1)} > 0,$$

since $h_2, c(n_1), q > 0$. Hence, the singular set $\Lambda(\mathcal{U}^l)$ is thus the union of the finitely points $p_i = \Lambda(P_i), i = 1, \cdots, m$, $q_\jmath = \Lambda(Q_\jmath), \jmath = 1, \cdots, n$, together with finitely many $\mathcal{C}^2$–curve $\Lambda(\gamma_j)$. Obviously, the same representation is valid for the image $\Lambda(\mathcal{U}^m)$. This completes the proof of Theorem 2.3. □

Appendix

In the Appendix, we give the proof of Lemma 3.2. The proof is very similar to the corresponding one in [8]. We add it here for the readers' convenience.

Proof. We claim that there exists a unique function $t \mapsto \alpha(t)$ such that

$$x^-(t) \;=\; x(t,\alpha(t)) \tag{A.1}$$

satisfies the equations in (3.1)-(3.2) and (3.11). We begin to prove the claim on the interval $t \in [0,1]$, then iterate the argument by induction. We prove this lemma by several steps.

1. We first derive an equation for $\alpha(t)$. Summing the first equation in (3.1) with (3.11) and integrating w.r.t. time we obtain

$$\begin{aligned} & x^-(t) + \mu_-^t\Big(\big]-\infty\,,\ x^-(t)\big[\Big) + \theta\cdot\mu_-^t\Big(\{x^-(t)\}\Big) \\ &= \bar y + \int_{-\infty}^{\bar y} \mathbf{R}^2(0,x)\,dx + \int_0^t \left(-c(n_1(s,x^-(s))) + \int_{-\infty}^{x^-(s)} \frac{c'(\mathbf{R}^2 S_1 - R_1\mathbf{S}^2)}{2c}\,dx\right) ds\,, \end{aligned} \tag{A.2}$$

for some $\theta \in [0,1]$. In view of the definition (2.1), one has

$$c(n_1(t,x)) \;=\; c(0) + \int_{-\infty}^{x} c'(n_1(t,\xi))\,n_{1,x}(t,\xi)\,d\xi \;=\; c(0) + \int_{-\infty}^{x} \frac{c'(R_1 - S_1)}{2c}\,d\xi\,. \tag{A.3}$$

This together with (A.1)–(A.2) gives an integral equation for α,

$$\alpha(t) \;=\; \bar\alpha + \int_0^t \left(-c(0) + \int_{-\infty}^{x(s,\alpha(s))} \frac{c'(S_1 - R_1 + \mathbf{R}^2 S_1 - R_1\mathbf{S}^2)}{2c}\,dx\right) ds\,. \tag{A.4}$$

Here

$$\bar\alpha \;=\; \alpha(0) \;=\; \bar y + \int_{-\infty}^{\bar y} \mathbf{R}^2(0,x)\,dx\,. \tag{A.5}$$

We further observe that the equation (A.4) is in accordance with

$$\dot\alpha(t) \;=\; G(t,\alpha(t)) \;\doteq\; -c(0) + \int_{-\infty}^{x(t,\alpha(t))} \frac{c'(S_1 - R_1 + \mathbf{R}^2 S_1 - R_1\mathbf{S}^2)}{2c}\,dx\,. \tag{A.6}$$

2. Now, we will get the existence of a solution to (A.4). More precisely, define the Picard map $\mathcal{P} : \mathcal{C}^0([0,1]) \mapsto \mathcal{C}^0([0,1])$ by setting

$$\mathcal{P}\alpha(t) \;\doteq\; \bar\alpha + \int_0^t \left[-c(0) + \int_{-\infty}^{x(s,\alpha(s))} \frac{c'(S_1 - R_1 + \mathbf{R}^2 S_1 - R_1\mathbf{S}^2)}{2c}\,dx\right] ds\,.$$

We can prove $\mathcal{P}$ is a continuous transformation of a compact convex set $\mathcal{K} \subset \mathcal{C}^0([0,1])$ into itself. Here the set $\mathcal{K}$ is a set of Hölder continuous functions, defined by

$$\mathcal{K} \;\doteq\; \{f \in \mathcal{C}^{1/2}([0,1])\,;\ \|f\|_{\mathcal{C}^{1/2}} \;\le\; C_K,\ f(0) = \bar\alpha\},$$

for a suitable constant C_K. The detail can be found in [8]. By Schauder's fixed point theorem, we conclude that the integral equation (A.4) has at least one solution. Iterating the argument, this solution can be extended to any time interval $t \in [0,T]$.

3. Using a generalized characteristic idea, in this and the next step we prove that $x^-(\tau) \doteq x(\tau,\alpha(\tau))$ satisfies the first equation in (3.1) at a.e. time τ. Since $\frac{c'}{2c}[\mathbf{R}^2 S_1 - R_1\mathbf{S}^2] \in \mathbf{L}^1([0,T]\times\mathbb{R})$, a classical theorem of Lebesgue implies that

$$\lim_{r\to 0+} \frac{1}{\pi r^2} \iint_{(\tau-t)^2+(y-x)^2\le r^2} \frac{c'}{2c}[\mathbf{R}^2 S_1 - R_1\mathbf{S}^2](\tau,y)\,dy\,d\tau \;=\; \frac{c'}{2c}[\mathbf{R}^2 S_1 - R_1\mathbf{S}^2](t,x),$$

for all $(t,x) \in\,]0,T[\,\times\mathbb{R}$ outside a null set $\mathcal{N}_2$ whose 2-dimensional measure is zero.

If one divides by r instead of r^2, by Corollary 3.2.3 in [31] there is a set $\mathcal{N}_1 \subset \mathcal{N}_2$ whose 1-dimensional Hausdorff measure is zero and such that

$$\limsup_{r\to 0^+} \frac{1}{r}\iint_{(\tau-t)^2+(y-x)^2\le r^2} \frac{c'}{2c}[\mathbf{R}^2 S_1 - R_1\mathbf{S}^2](\tau,y)dyd\tau \;=\; 0$$

for every $(t,x)\notin\mathcal{N}_1$.

Moreover, by the definition of absolutely continuity and the fact that the map $\alpha\mapsto x(t,\alpha)$ is contractive, it is easy to prove the map $t\mapsto x^-(t)\doteq x(t,\alpha(t))$ is absolutely continuous. Use [8] as a reference.

Thanks to the above, there exists a null 1-dimensional set $\mathcal{N}\subset[0,T]$ with the properties

(i) For every $\tau\notin\mathcal{N}$ and $x\in\mathbb{R}$ one has $(\tau,x)\notin\mathcal{N}_1$;
(ii) If $\tau\notin\mathcal{N}$ then the map $t\mapsto\zeta(t)$ in (A.7) is differentiable at $t=\tau$. Moreover, τ is a Lebesgue point of the derivative ζ';
(iii) The functions $t\mapsto x^-(t)$ and $t\mapsto\alpha(t)$ are differentiable at each point $\tau\in[0,T]\setminus\mathcal{N}$. Moreover, each point $\tau\notin\mathcal{N}$ is a Lebesgue point of the derivatives $\dot{x}^-$ and $\dot{\alpha}$.

Here the function

$$\zeta(\tau) \;\doteq\; \int_0^\tau\int_{-\infty}^{+\infty}\left|\frac{c'}{2c}(\mathbf{R}^2 S_1 - R_1\mathbf{S}^2)\right| dx\,dt \tag{A.7}$$

is locally Hölder continuous, nondecreasing, with sub-linear growth [8].

4. Let $\tau\notin\mathcal{N}$. In this step, we will show that the map $t\mapsto x^-(t)\doteq x(t,\alpha(t))$ satisfies the first equation in (3.1) at time $t=\tau$. Assume, on the contrary, that $\dot{x}^-(\tau)\neq -c(n_1(\tau,x^-(\tau)))$. Without loss of generality, let

$$\dot{x}^-(\tau) \;=\; -c(n_1(\tau,x^-(\tau)))+2\varepsilon_0 \tag{A.8}$$

for some $\varepsilon_0>0$. (The case $\varepsilon_0<0$ can be handled similarly). To begin with, we choose $\delta>0$ small enough so that,

$$X(t) \;\doteq\; x^-(\tau)+(t-\tau)[-c(n_1(\tau,x(\tau)))+\varepsilon_0] \;<\; x^-(t) \tag{A.9}$$

for all $t\in]\tau,\tau+\delta]$. Notice that the identity in (1.1) holds in distributional sense for test function $\varphi\in C_c^1(\mathbb{R}^+\times\mathbb{R})$, then it remains valid for any Lipschitz continuous function φ with compact support. Given $\tau<t<\tau+\delta$, for $\epsilon>0$ small we shall construct a Lipschitz approximation φ^ϵ to the characteristic function of the set

$$\Omega \;\doteq\; \big\{(s,y)\,;\; s\in[\tau,t]\,,\; y\in[\epsilon^{-1},\; X(s)]\big\}.$$

Define the Lipschitz function with compact support

$$\varphi^\epsilon(s,y) \;\doteq\; \min\{\varrho^\epsilon(s,y),\,\chi^\epsilon(s)\}\,.$$

where

$$\rho^\epsilon(s,y) \;\doteq\; \begin{cases} 0 & \text{if } \; y\le-\epsilon^{-1},\\ \epsilon^{-1}(y+\epsilon^{-1}) & \text{if } \; -\epsilon^{-1}\le y\le\epsilon-\epsilon^{-1},\\ 1 & \text{if } \; \epsilon-\epsilon^{-1}\le y\le X(s),\\ 1-\epsilon^{-1}(y-X(s)) & \text{if } \; X(s)\le y\le X(s)+\epsilon,\\ 0 & \text{if } \; y\ge X(s)+\epsilon, \end{cases}$$

$$\chi^\epsilon(s) \doteq \begin{cases} 0 & \text{if} \quad s \le \tau - \epsilon, \\ \epsilon^{-1}(s - \tau + \epsilon) & \text{if} \quad \tau - \epsilon \le s \le \tau, \\ 1 & \text{if} \quad \tau \le s \le t, \\ 1 - \epsilon^{-1}(s - t) & \text{if} \quad t \le s < t + \epsilon, \\ 0 & \text{if} \quad s \ge t + \epsilon. \end{cases} \tag{A.10}$$

In light of the first equation in (2.4), we get

$$\int \Big[\int (\varphi_t^\epsilon - c\varphi_x^\epsilon)\, d\mu_-^t + \int \frac{c'}{2c} (\mathbf{R}^2 S_1 - R_1 \mathbf{S}^2)\, \varphi^\epsilon\, dx \Big]\, dt \;=\; 0. \tag{A.11}$$

Moreover, if t is sufficiently close to τ, then for $s \in [\tau, t]$ and x close to $x^-(\tau)$, one has

$$0 \;=\; \varphi_t^\epsilon + [-c(n_1(\tau, x(\tau))) + \varepsilon_0]\varphi_x^\epsilon \;\le\; \varphi_t^\epsilon - c(n_1(s,x))\varphi_x^\epsilon\,,$$

since $-c(n_1(s,x)) < -c(n_1(\tau, x(\tau))) + \varepsilon_0$ and $\varphi_x^\epsilon \le 0$. Since the family of measures μ_-^t depends continuously on t in the topology of weak convergence, taking the limit of (A.11) as $\epsilon \to 0$, for $\tau, t \notin \mathcal{N}$ we obtain

$$0 \;\ge\; \mu_-^\tau\Big((-\infty,\; x^-(\tau)]\Big) - \mu_-^t\Big((-\infty,\; X(t)]\Big) + \int_\tau^t \int_{-\infty}^{X(s)} \frac{c'}{2c}[\mathbf{R}^2 S^1 - R_1 \mathbf{S}^2]\, dxds. \tag{A.12}$$

This implies

$$\begin{aligned} &\mu_-^t\Big((-\infty,\; x^-(t)]\Big) \;\ge\; \mu_-^t\Big((-\infty,\; X(t)]\Big) \\ &\ge\; \mu_-^\tau\Big((-\infty,\; x^-(\tau)]\Big) + \int_\tau^t \int_{-\infty}^{x^-(s)} \frac{c'}{2c}[\mathbf{R}^2 S_1 - R_1 \mathbf{S}^2]\, dxds + o_1(t-\tau). \end{aligned} \tag{A.13}$$

Here the last term

$$o_1(t-\tau) \;\doteq\; -\int_\tau^t \int_{X(s)}^{x^-(s)} \frac{c'}{2c}[\mathbf{R}^2 S_1 - R_1 \mathbf{S}^2]\, dyds$$

satisfies

$$\lim_{t\to\tau} \frac{o_1(t-\tau)}{t-\tau} \;=\; 0$$

because $\tau \notin \mathcal{N}$. Using (A.13) one obtains

$$\begin{aligned} \alpha(t) - \alpha(\tau) \;&\ge\; \Big[x^-(t) + \mu_-^t\Big((-\infty,\, x^-(t)[\Big)\Big] - \Big[x^-(\tau) + \mu_-^\tau\Big((-\infty,\, x^-(\tau)]\Big)\Big] \\ &\ge\; \Big[-c\big(n_1(\tau, x^-(\tau))\big) + 2\varepsilon_0\Big](t-\tau) + \int_\tau^t \int_{-\infty}^{x^-(s)} \frac{c'}{2c}[\mathbf{R}^2 S_1 - R_1 \mathbf{S}^2]\, dyds + o_1(t-\tau). \end{aligned} \tag{A.14}$$

Differentiating (A.14) w.r.t. t at $t = \tau$, we obtain

$$\dot\alpha(\tau) \;\ge\; \Big[-c\big(n_1(\tau, x^-(\tau))\big) + 2\varepsilon_0\Big] + \int_{-\infty}^{x^-(\tau)} \frac{c'}{2c}[\mathbf{R}^2 S_1 - R_1 \mathbf{S}^2]\, dyds$$

in contradiction with (A.6). Therefore, the first equation in (3.1) must hold.

5. Now the main issue is how to prove the uniqueness of solution to (A.5)-(A.6) by controlling the highest order terms in (4.3). Essentially, the appearance of these terms shows that the forward or backward energy might increase due to wave interaction. We consider the weight

$$W(t, \alpha) \;\doteq\; e^{\kappa A^+(t,\alpha)} \tag{A.15}$$

with

$$A^+(t,\alpha) \doteq \mu_+^t\Big((-\infty,\ x(t,\alpha)]\Big) + [\zeta(T)-\zeta(t)]. \tag{A.16}$$

Here ζ is the function defined at (A.7), while

$$\kappa \doteq \frac{M}{2c_0^2}\,. \tag{A.17}$$

We recall that $\zeta(T)-\zeta(t)$ provides an upper bound on the energy transferred from backward to forward moving waves and conversely, during the time interval $[t,T]$. In turn, $A^+(t,\alpha)$ yields an upper bound on the total energy of forward moving waves that can cross the backward characteristic $x(\cdot,\alpha)$ during the time interval $[t,T]$. For any $\alpha_1<\alpha_2$ and $t\geq 0$, we define a weighted distance by setting

$$d^{(t)}(\alpha_1,\alpha_2) \doteq \int_{\alpha_1}^{\alpha_2} W(t,\alpha)\,d\alpha. \tag{A.18}$$

Thanks to the measures $c'(n_1)\cdot\mu_-^t$ and $c'(n_))\cdot\mu_+^t$ are absolutely continuous w.r.t. Lebesgue measure for a.e. time t and the Gronwall's lemma, we have

$$d^{(t)}\big(\alpha_1(t),\alpha_2(t)\big) \leq e^{C_0 t}\,d^{(0)}\big(\alpha_1(0),\alpha_2(0)\big), \tag{A.19}$$

with $C_0 \doteq \|c'/2c\|_{\mathbf{L}^\infty}$. The detail can be found in [8], we omit it here for brevity. Thus, for every initial value $\bar\alpha$, the solution of (A.5)-(A.6) is unique.

6. Finally, the uniqueness result of α proved in step **5** implies the uniqueness of solution to the first equation in (3.1). □

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Hong Cai
School of Mathematical Sciences, Xiamen University, Fujian, Xiamen, 361005, China
E-mail address: caihong19890418@163.com

Geng Chen
Department of Mathematics, University of Kansas, Lawrence, KS 66045
E-mail address: gengchen@ku.edu

Yi Du
Mathematics Department, JINAN University, GuangZhou, China
E-mail address: duyidy@jnu.edu.cn