Unique Conservative Solutions to a Variational Wave Equation

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November 7, 2014

Abstract

Relying on the analysis of characteristics, we prove the uniqueness of conservative solutions to the variational wave equation $u_{tt} - c(u)(c(u)u_x)_x = 0$. Given a solution $u(t,x)$, even if the wave speed $c(u)$ is only Hölder continuous in the t-x plane, one can still define forward and backward characteristics in a unique way. Using a new set of independent variables X, Y, constant along characteristics, we prove that t, x, u, together with other variables, satisfy a semilinear system with smooth coefficients. From the uniqueness of the solution to this semilinear system, one obtains the uniqueness of conservative solutions to the Cauchy problem for the wave equation with general initial data $u(0,\cdot) \in H^1(I\!R)$, $u_t(0,\cdot) \in L^2(I\!R)$.

1 Introduction

Consider the Cauchy problem for the quasilinear second order wave equation

$$u_{tt} - c(u)\big(c(u)u_x\big)_x = 0\,, \tag{1.1}$$

with initial data

$$u(0,x) = u_0(x)\,, \qquad u_t(0,x) = u_1(x)\,. \tag{1.2}$$

Here $u_0 \in H^1(I\!R)$ while $u_1 \in \mathbf{L}^2(I\!R)$. We assume that the wave speed $c : I\!R \mapsto I\!R_+$ is a smooth, bounded, uniformly positive function, satisfying

$$0 \ < \ c_0 \ \leq \ c(u) \ < \ M, \qquad\qquad |c'(u)| \ < \ M \qquad\quad \text{for all} \ \ u. \tag{1.3}$$

The above Cauchy problem has been studied in several papers [5, 8, 10, 11, 12]. In particular, the analysis in [8] shows that the problem (1.1)-(1.2) has a weak solution which conserves the total energy. Indeed, a global flow of such solutions can be constructed, both forward and backward in time, exhibiting some kind of continuous dependence on the initial data. The approach developed in [8] relied on the introduction of a set of auxiliary variables. Using these variables, one obtains a semilinear system of equations having unique solutions. In terms of the original variables, this yields a solution $u = u(t,x)$ of the Cauchy problem (1.1)-(1.2), for which the total energy is a.e. conserved. The main results in [8] can be summarized as follows.

Theorem 1. *Let $c : I\!R \mapsto I\!R$ be a smooth function satisfying (1.3). Assume that the initial data u_0 in (1.2) is absolutely continuous, and that $(u_0)_x \in \mathbf{L}^2$, $u_1 \in \mathbf{L}^2$. Then the Cauchy problem (1.1)-(1.2) admits a weak solution $u = u(t,x)$, defined for all $(t,x) \in I\!R \times I\!R$. In the t-x plane, the function u is locally Hölder continuous with exponent $1/2$. This solution $t \mapsto u(t,\cdot)$ is continuously differentiable as a map with values in $\mathbf{L}^p_{\rm loc}$, for all $1 \le p < 2$. Moreover, it is Lipschitz continuous w.r.t. the $\mathbf{L}^2$ distance, i.e.*

$$\big\|u(t,\cdot) - u(s,\cdot)\big\|_{\mathbf{L}^2} \le L\,|t-s| \tag{1.4}$$

for all $t, s \in I\!R$. The equation (1.1) is satisfied in integral sense, i.e.

$$\iint \Big[\phi_t\, u_t - \big(c(u)\phi\big)_x c(u)\, u_x\Big]\, dxdt = 0 \tag{1.5}$$

for all test functions $\phi \in \mathcal{C}^1_c$. Moreover, the maps $t \mapsto u_t(t,\cdot)$ and $t \mapsto u_x(t,\cdot)$ are continuous with values in $\mathbf{L}^p_{loc}(I\!R)$, for every $p \in [1,2[$.

In general, the solution constructed in Theorem 1 is not unique. To select a unique solution, additional properties must be imposed. In particular, one can require that the total energy be conserved.

It is convenient to introduce the variables

$$\begin{cases} R \doteq u_t + c(u)u_x\,, \\ S \doteq u_t - c(u)u_x\,, \end{cases} \tag{1.6}$$

so that

$$u_t = \frac{R+S}{2}\,, \qquad u_x = \frac{R-S}{2c}\,. \tag{1.7}$$

By (1.1), the variables R, S satisfy

$$\begin{cases} R_t - cR_x = \frac{c'}{4c}(R^2 - S^2), \\ S_t + cS_x = \frac{c'}{4c}(S^2 - R^2). \end{cases} \tag{1.8}$$

Multiplying the first equation in (1.8) by R and the second one by S, one obtains balance laws for R^2 and S^2, namely

$$\begin{cases} (R^2)_t - (cR^2)_x = \frac{c'}{2c}(R^2S - RS^2)\,, \\ (S^2)_t + (cS^2)_x = -\frac{c'}{2c}(R^2S - RS^2)\,. \end{cases} \tag{1.9}$$

As a consequence, the following quantities are conserved:

$$E \doteq 2\big(u_t^2 + c^2u_x^2\big) = R^2 + S^2\,, \qquad \mathcal{M} \doteq -\,u_t u_x = \frac{S^2 - R^2}{4c}\,. \tag{1.10}$$

One can think of R^2 and S^2 as the energy of backward and forward moving waves, respectively. Notice that these are not separately conserved. Indeed, by (1.9) energy is transferred from forward to backward waves, and vice versa.

Theorem 2. *Under the previous assumptions, a solution $u = u(t,x)$ can be constructed which is* conservative *in the following sense.*

There exists two families of positive Radon measures on the real line: $\{\mu_-^t\}$ and $\{\mu_+^t\}$, depending continuously on t in the weak topology of measures, with the following properties.

(i) At every time t one has

$$\mu_-^t(I\!R) + \mu_+^t(I\!R) \;=\; E_0 \;\doteq\; 2\int_{-\infty}^{\infty} \Big[u_1^2(x) + \big(c(u_0(x))u_{0,x}(x)\big)^2\Big]\,dx\,. \tag{1.11}$$

(ii) For each t, the absolutely continuous parts of μ_-^t and μ_+^t w.r.t. the Lebesgue measure have densities respectively given by

$$R^2 \;=\; \big(u_t + c(u)u_x\big)^2, \qquad\qquad S^2 \;=\; \big(u_t - c(u)u_x\big)^2. \tag{1.12}$$

(iii) For almost every $t \in I\!R$, the singular parts of μ_-^t and μ_+^t are concentrated on the set where $c'(u) = 0$.

(iv) The measures μ_-^t and μ_+^t provide measure-valued solutions respectively to the balance laws

$$\left\{\begin{array}{rcl} w_t - (cw)_x &=& \frac{c'}{2c}(R^2S - RS^2)\,,\\ z_t + (cz)_x &=& -\frac{c'}{2c}(R^2S - RS^2)\,. \end{array}\right. \tag{1.13}$$

Remark 1. In principle, the equations (1.13) should be written as

$$\left\{\begin{array}{rcl} w_t - (cw)_x &=& \frac{c'}{2c}(Sw - Rz)\,,\\ z_t + (cz)_x &=& -\frac{c'}{2c}(Sw - Rz)\,. \end{array}\right. \tag{1.14}$$

This reflects the fact that, if $w = w^a + w^s$ is a measure with an absolutely continuous and a singular part, then both of these multiply S. However, we are here making the assumption that the solution is *conservative*, so that by (iii) the product $c'(u)w^s = 0$ for a.e. time t. For this reason, on the right hand side of (1.14) we can replace w with the measure w^a having density R^2 w.r.t. Lebesgue measure. Similarly, we can replace z with the measure z^a having density S^2 w.r.t. Lebesgue measure.

Observe that the total energy represented by the sum $\mu_-^t + \mu_+^t$ is conserved in time. Occasionally, some of this energy is concentrated on a set of measure zero. At the times τ when this happens, $\mu_-^t + \mu_+^t$ has a non-trivial singular part and

$$E(\tau) \;\doteq\; \int_{-\infty}^{\infty} \Big[u_t^2(\tau,x) + c^2(u(\tau,x))u_x^2(\tau,x)\Big]\,dx \;<\; E_0\,.$$

The condition (iii) puts some restrictions on the set of such times τ. In particular, if $c'(u) \neq 0$ for all u, then this set has measure zero.

Our present goal is to understand whether these conservative solutions are unique. In a way, our approach is the inverse of [8]. Given a conservative solution $u = u(t,x)$, we define a set of independent variables X, Y and dependent variables u, w, z, p, q, and show that these satisfy a suitable semilinear system of equations. By proving that this semilinear system has unique solutions, we eventually obtain the uniqueness of solutions to the original equation (1.1).

In essence, this semilinear system describes the evolution of u and its derivatives along characteristic curves, i.e. curves $t \mapsto x^{\pm}(t)$ which satisfy the ODEs

$$\dot{x}^-(t) \;=\; -\,c(u(t,x^-(t))), \qquad \dot{x}^+(t) \;=\; c(u(t,x^+(t))). \tag{1.15}$$

At this naive level, the approach runs into a fundamental difficulty. Namely, since the solution u is only Hölder continuous, for a given $\bar{y} \in I\!R$ the Cauchy problems for the ODEs in (1.15) with initial data

$$x^{\pm}(0) \;=\; \bar{y}, \tag{1.16}$$

may well have multiple solutions. To overcome this difficulty, our analysis relies on two ideas. To simplify the exposition, we here assume that the measures μ_-^t, μ_+^t are absolutely continuous for a.e. t.

- The two balance laws (1.9) imply

$$\frac{d}{dt}\int_{-\infty}^{x^-(t)} R^2(t,x)\,dx \;=\; \int_{-\infty}^{x^-(t)} \frac{c'}{2c}(R^2S - RS^2)\,dx\,, \tag{1.17}$$

$$\frac{d}{dt}\int_{-\infty}^{x^+(t)} S^2(t,x)\,dx \;=\; -\int_{-\infty}^{x^+(t)} \frac{c'}{2c}(R^2S - RS^2)\,dx\,. \tag{1.18}$$

  While the Cauchy problem (1.15)-(1.16) can have multiple solutions, the characteristic curves $t \mapsto x^{\pm}(t)$ can be uniquely determined by combining all the equations in (1.15)–(1.18).

- Instead of the variables (t,x), it is convenient to work with an adapted set of variables $x(t,\alpha)$, $y(t,\beta)$, where

$$x(t,\alpha) + \int_{-\infty}^{x(t,\alpha)} R^2(t,\xi)\,d\xi \;=\; \alpha\,, \tag{1.19}$$

$$y(t,\beta) + \int_{-\infty}^{y(t,\beta)} S^2(t,\xi)\,d\xi \;=\; \beta\,. \tag{1.20}$$

  Here the parameter α singles out a backward characteristic, while β singles out a forward characteristic.

Our main result is the following.

Theorem 3. *Let $c : I\!R \mapsto I\!R$ be a smooth function satisfying (1.3). For any initial data $u_0 \in H^1(I\!R)$, $u_1 \in \mathbf{L}^2(I\!R)$, the conservative solution to Cauchy problem (1.1)-(1.2), which satisfies all conditions (i)–(iv) in Theorem 2, is unique.*

The main technique used in the proof is similar to the paper [3] by the same authors, dealing with the Camassa-Holm equation, and was inspired by the uniqueness result in [9]. However, in the present case of a second order wave equation, the analysis is harder. Indeed, for the Camassa-Holm equation one has a single family of characteristics. After a change of variables, each characteristic is obtained by solving an ODE with Lipschitz continuous right hand side.

On the other hand, for the wave equation (1.1) one has two families of characteristics moving forward and backward, respectively. After a change of variables, the ODEs which determine these characteristics are still no better than Hölder continuous. However, the singularities are transversal. Uniqueness and Lipschitz continuous dependence of solutions on the initial data can thus be established using ideas from [1, 4, 6].

The paper is organized as follows. In Section 2 we review the basic equations and prove an a priori estimate on the total amount of wave interactions. In Section 3 we show that, for a given conservative solution $u = u(t,x)$, one can uniquely determine a forward and a backward characteristic through each initial point. In Section 4 we introduce the characteristic coordinates (X,Y) and prove Lipschitz continuity of map $(X,Y) \mapsto (t,x,u)$. In Section 5 we introduce some additional variables and show that they satisfy a semilinear system with smooth coefficients. The uniqueness of solutions to this semilinear system yields the uniqueness of conservative solutions to the original wave equation (1.1).

2 Adapted variables and wave interaction estimate

Recalling (1.3), for notation convenience, we introduce the constant

$$C_0 \doteq \left\| \frac{c'}{2c} \right\|_{\mathbf{L}^\infty} \leq \frac{M}{2c_0}. \tag{2.1}$$

Let $u = u(t,x)$ be a conservative solution of (1.1), having all the properties listed in Theorems 1 and 2. For any time t and any $\alpha, \beta \in I\!R$, we define the points $x(t,\alpha)$ and $y(t,\beta)$ by setting

$$x(t,\alpha) \doteq \sup\Big\{x\,;\;\; x + \mu_-^t(\,]-\infty,x]\,) \;<\; \alpha\Big\}, \tag{2.2}$$

$$y(t,\beta) \doteq \sup\Big\{x\,;\;\; x + \mu_+^t(\,]-\infty,x]\,) \;<\; \beta\Big\}. \tag{2.3}$$

Notice that the above holds if and only if, for some $\theta, \theta' \in [0,1]$, one has

$$x(t,\alpha) + \mu_-^t\Big(]-\infty\,,\; x(t,\alpha)[\Big) + \theta\cdot\mu_-^t\Big(\{x(t,\alpha)\}\Big) \;=\; \alpha\,, \tag{2.4}$$

$$y(t,\beta) + \mu_+^t\Big(]-\infty\,,\; y(t,\beta)[\Big) + \theta'\cdot\mu_-^t\Big(\{y(t,\beta)\}\Big) \;=\; \beta\,. \tag{2.5}$$

Since the measures μ_-^t, μ_+^t are both positive and bounded, it is clear that these points are well defined. In the absolutely continuous case, the equations (2.4)-(2.5) are equivalent to (1.19)-(1.20).

Lemma 1. *For every fixed t, the maps $\alpha \mapsto x(t,\alpha)$ and $\beta \mapsto y(t,\beta)$ are both Lipschitz continuous with constant 1. Moreover, for fixed α, β, the maps $t \mapsto x(t,\alpha)$ and $t \mapsto y(t,\beta)$ are absolutely continuous and locally Hölder continuous with exponent* $1/2$.

Proof. 1. The first part is straightforward. Indeed, if

$$x_1 \doteq x(t,\alpha_1) < x(t,\alpha_2) \doteq x_2\,,$$

then

$$x_2 - x_1 \le x_2 - x_1 + \mu^t_-\big(\,]x_1,x_2[\,\big) \le \alpha_2-\alpha_1\,.$$

The same argument applies to the map $\beta \mapsto y(t,\beta)$.

2. To prove the second statement, denote by $\mu^t_-\otimes\mu^t_+$ the product measure on $I\!R^2$ and consider the wave interaction potential

$$Q(t) \doteq (\mu^t_-\otimes\mu^t_+)\Big(\big\{(x,y)\,;\quad x>y\big\}\Big). \tag{2.6}$$

We recall that $\mu^t_-(I\!R)+\mu^t_+(I\!R)=E_0$ is the total energy, constant in time. Since $R^2(t,\cdot)$ and $S^2(t,\cdot)$ provide the absolutely continuous parts of μ^t_- and μ^t_+, respectively, recalling (1.3) and using the balance laws (1.13) we obtain

$$\begin{aligned}
\frac{d}{dt}Q(t) \;&\le\; -\,c_0\int_{-\infty}^{+\infty} S^2(t,x)R^2(t,x)\,dx + \frac{M}{2c_0}\int\left(\int_x^{+\infty}|R^2S-RS^2|(t,y)\,dy\right)d\mu^t_-(x)\\
&\qquad +\frac{M}{2c_0}\int\left(\int_{-\infty}^{y}|R^2S-RS^2|(t,x)\,dx\right)d\mu^t_+(y)\\
&\le\; -\,c_0\int_{-\infty}^{+\infty} S^2R^2\,dx + \frac{M}{2c_0}\big(\mu^t_-(I\!R)+\mu^t_+(I\!R)\big)\int_{-\infty}^{+\infty}|R^2S-RS^2|\,dx\\
&\le\; -\,c_0\int_{-\infty}^{+\infty} S^2R^2\,dx + \frac{M}{2c_0}E_0\int_{-\infty}^{+\infty}\big(|R^2S|+|RS^2|\big)\,dx\\
&\le\; -\,\frac{c_0}{2}\int_{-\infty}^{+\infty} S^2R^2\,dx + \int_{\{2ME_0>c_0^2|S|\}}|R^2S|\,dx + \int_{\{2ME_0>c_0^2|R|\}}|RS^2|\,dx\\
&\le\; -\,\frac{c_0}{2}\int_{-\infty}^{+\infty} S^2R^2\,dx + \frac{2ME_0}{c_0^2}\int_{-\infty}^{+\infty}(R^2+S^2)\,dx\\
&\le\; -\,\frac{c_0}{2}\int_{-\infty}^{+\infty} S^2R^2\,dx + \frac{2ME_0^2}{c_0^2}\,.
\end{aligned} \tag{2.7}$$

Since $Q(t)\le E_0^2$ for every time t, from (2.7) it follows

$$\int_0^T\int_{-\infty}^{+\infty} R^2(t,x)S^2(t,x)\,dx\,dt \;\le\; \frac{2}{c_0}\cdot\left[\big(Q(0)-Q(T)\big)+\frac{2ME_0^2}{c_0^2}\,T\right] \;\le\; \frac{2E_0^2}{c_0}+\frac{4ME_0^2}{c_0^3}\,T\,. \tag{2.8}$$

3. For a given τ and any $\varepsilon \in]0,1]$, we now estimate

$$\begin{aligned}
&\int_\tau^{\tau+\varepsilon}\int_{-\infty}^{+\infty} |R^2S - RS^2|\,dx\,dt \\
&\leq \int_\tau^{\tau+\varepsilon}\int_{S\leq\varepsilon^{-1/2}} |R^2S|\,dx\,dt + \int_\tau^{\tau+\varepsilon}\int_{R\leq\varepsilon^{-1/2}} |RS^2|\,dx\,dt \\
&\quad + \int_\tau^{\tau+\varepsilon}\int_{S>\varepsilon^{-1/2}} |R^2S|\,dx\,dt + \int_\tau^{\tau+\varepsilon}\int_{R>\varepsilon^{-1/2}} |RS^2|\,dx\,dt \\
&\leq \varepsilon^{-1/2}\int_\tau^{\tau+\varepsilon}\int_{-\infty}^{+\infty} (R^2+S^2)\,dx\,dt \qquad (2.9)\\
&\quad + \int_\tau^{\tau+\varepsilon}\int_{S\geq\varepsilon^{-1/2}} (R^2S^2)\frac{|R^2S|}{R^2S^2}\,dx\,dt + \int_\tau^{\tau+\varepsilon}\int_{R\geq\varepsilon^{-1/2}} (R^2S^2)\frac{|RS^2|}{R^2S^2}\,dx\,dt \\
&\leq E_0\varepsilon^{1/2} + 2\varepsilon^{1/2}\Big(\frac{2E_0^2}{c_0} + \frac{4ME_0^2}{c_0^3}\varepsilon\Big) \\
&\leq \left[E_0 + \frac{4E_0^2}{c_0} + \frac{8ME_0^2}{c_0^3}\right]\varepsilon^{1/2}.
\end{aligned}$$

As a consequence, the function ζ defined by

$$\zeta(\tau) \doteq \int_0^\tau\int_{-\infty}^{+\infty} \left|\frac{c'}{2c}(R^2S - RS^2)\right|\,dx\,dt \qquad (2.10)$$

is locally Hölder continuous, nondecreasing, with sub-linear growth. Since $R^2S - RS^2 \in \mathbf{L}^1([0,T]\times I\!R)$, by Fubini's theorem the map $t \mapsto \int \left|\frac{c'}{2c}(R^2S-RS^2)\right|\,dx$ is in $\mathbf{L}^1([0,T])$. By its definition at (2.10), the function ζ is absolutely continuous. Recalling (1.3), for $0 < t_2 - t_1 \leq 1$ we have

$$\zeta(t_2) - \zeta(t_1) \leq C_1(t_1 - t_2)^{1/2},$$

where the constant C_1 is defined as

$$C_1 \doteq \frac{M}{c_0}\left[E_0 + \frac{4E_0^2}{c_0} + \frac{8ME_0^2}{c_0^3}\right]. \qquad (2.11)$$

4. We recall that the family of measures μ_-^t satisfies the balance law in (1.13) with velocity $-c(u) \in [-M, 0]$. For any $t_1 < t_2$ and any α, this yields the inequalities

$$\mu_-^{t_2}\Big(\big]-\infty,\ x(t_1,\alpha)\big[\Big) \geq \mu_-^{t_1}\Big(\big]-\infty,\ x(t_1,\alpha)\big[\Big) - \big[\zeta(t_2)-\zeta(t_1)\big], \qquad (2.12)$$

$$\mu_-^{t_2}\Big(\big]-\infty,\ x(t_1,\alpha) - M(t_2-t_1)\big[\Big) \leq \mu_-^{t_1}\Big(\big]-\infty,\ x(t_1,\alpha)\big[\Big) + \big[\zeta(t_2)-\zeta(t_1)\big]. \qquad (2.13)$$

From the definition (2.4) it thus follows

$$x(t_1,\alpha) - M(t_2-t_1) - \big[\zeta(t_2)-\zeta(t_1)\big] \leq x(t_2,\alpha) \leq x(t_1,\alpha) + \big[\zeta(t_2)-\zeta(t_1)\big]. \qquad (2.14)$$

By the properties of the function ζ, proved in step **3**, this achieves the proof. Of course, the same argument can be applied to the map $t \mapsto y(t,\beta)$. □

Remark 2. For each fixed t, the map $\alpha \mapsto x(t,\alpha)$ is Lipschitz continuous, hence a.e. differentiable. We can define the set of singular points Ω^t and the set of singular values V^t according to

$$\Omega^t \doteq \left\{\alpha \in I\!R\,;\ \ \frac{\partial}{\partial\alpha}x(t,\alpha) \ = \ 0 \ \text{ or else this partial derivative does not exist}\right\}, \tag{2.15}$$

$$V^t \doteq \left\{x(t,\alpha)\,;\ \ \alpha\in\Omega^t\right\}. \tag{2.16}$$

Observe that V^t has zero Lebesgue measure.

In general, the map $\alpha \mapsto x(t,\alpha)$ is onto but not one-to-one. However, for every regular value $x_0 \in I\!R \setminus V^t$ there exists a unique α_0 such that $x_0 = x(t,\alpha_0)$.

If now $f \in \mathbf{L}^1(I\!R)$, the composition $\tilde f(\alpha) \doteq f(x(t,\alpha))$ is well defined for a.e. $\alpha \in I\!R\setminus\Omega^t$. The integral of f can be computed by a change of variables:

$$\int f(x)\,dx \ = \ \int_{I\!R\setminus\Omega^t} \tilde f(\alpha)\cdot\frac{\partial}{\partial\alpha}x(t,\alpha)\,d\alpha \ = \ \int_{I\!R\setminus\Omega^t}\tilde f(\alpha)\cdot\frac{1}{1+R^2(t,x(t,\alpha))}\,d\alpha\,. \tag{2.17}$$

3 Recovering the characteristic curves

The next lemma, which plays a crucial role in our analysis, shows that for a conservative solution the characteristic curves can be uniquely determined. Observe that, in the general case where the measures μ_-^t, μ_+^t need not be absolutely continuous, the identities (1.17)-(1.18) can be written in the equivalent integrated form

$$\int_{-\infty}^{\bar y} R^2(0,x)\,dx + \int_0^t\int_{-\infty}^{x^-(s)} \frac{c'(R^2S-RS^2)}{2c}\,dx\,ds \ = \ \mu_-^t\Big(\,]-\infty, x^-(t)[\,\Big) + \theta(t,\bar y)\cdot\mu_-^t\Big(\{x^-(t)\}\Big), \tag{3.1}$$

$$\int_{-\infty}^{\bar y} S^2(0,x)\,dx - \int_0^t\int_{-\infty}^{x^+(s)} \frac{c'(R^2S-RS^2)}{2c}\,dx\,ds \ = \ \mu_+^t\Big(\,]-\infty, x^-(t)[\,\Big) + \theta'(t,\bar y)\cdot\mu_+^t\Big(\{x^+(t)\}\Big), \tag{3.2}$$

for some functions $\theta, \theta' \in [0,1]$.

Lemma 2. *Let u be a conservative solution of (1.1), satisfying the properties stated in Theorems 1 and 2. Then, for any $\bar y \in I\!R$, there exists unique Lipschitz continuous maps $t \mapsto x^\pm(t)$ which satisfy (1.15)-(1.16) together with (3.1)-(3.2).*

Proof. We claim that there exists a unique function $t \mapsto \alpha(t)$ such that

$$x^-(t) \ = \ x(t,\alpha(t)) \tag{3.3}$$

satisfies the equations in (1.15)-(1.16) and (1.17). It suffices to prove the claim on the time interval $t\in[0,1]$, then iterate the argument by induction. The proof will be given in several steps.

1. Integrating the first equation in (1.15) w.r.t. time and summing it with (3.1) we obtain

$$x^-(t) + \mu_-^t\Big(\big]-\infty\,,\; x^-(t)\big[\Big) + \theta(t)\cdot\mu_-^t\Big(\{x^-(t)\}\Big)$$

$$= \bar y + \int_{-\infty}^{\bar y} R^2(0,x)\,dx + \int_0^t \left(-c(u(s,x^-(s))) + \int_{-\infty}^{x^-(s)} \frac{c'(R^2S - RS^2)}{2c}\,dx\right) ds\,, \tag{3.4}$$

for some $\theta(t)\in[0,1]$.

From (3.3) and (3.4) we obtain an integral equation for α, namely

$$\alpha(t) \;=\; \bar\alpha + \int_0^t \left(-c(u(s,x^-(s))) + \int_{-\infty}^{x(s,\alpha(s))} \frac{c'(R^2S - RS^2)}{2c}\,dx\right) ds\,. \tag{3.5}$$

Here

$$\bar\alpha \;=\; \alpha(0) \;=\; \bar y + \int_{-\infty}^{\bar y} R^2(0,x)\,dx\,. \tag{3.6}$$

Notice that the equation (3.5) is equivalent to

$$\dot\alpha(t) \;=\; G(t,\alpha(t)) \;\doteq\; -\,c\big(u(t,x(t,\alpha(t)))\big) + \int_{-\infty}^{x(t,\alpha(t))} \frac{c'(R^2S - RS^2)}{2c}\,dx\,, \tag{3.7}$$

with initial data (3.6). We take (3.5) as the starting point for our analysis. In the following steps we will show that this integral equation has a unique solution $t\mapsto\alpha(t)$. Moreover, the function $t\mapsto x^-(t) = x(t,\alpha(t))$ satisfies the first equation in (1.15), as well as (1.17).

2. We first prove the existence of a solution to (3.5) on the time interval $[0,1]$. Consider the Picard map $\mathcal{P}:\mathcal{C}^0([0,1])\mapsto\mathcal{C}^0([0,1])$, defined as

$$\mathcal{P}\alpha(t) \;\doteq\; \bar\alpha + \int_0^t \left[-c\big(u(t,x(t,\alpha(t)))\big) + \int_{-\infty}^{x(t,\alpha(t))} \frac{c'(R^2S - RS^2)}{2c}\,dx\right] ds\,. \tag{3.8}$$

We claim that $\mathcal{P}$ is a continuous transformation of a compact convex set $\mathcal{K}\subset\mathcal{C}^0([0,1])$ into itself, with the usual norm

$$\|f\|_{\mathcal{C}^0} \;\doteq\; \max_{t\in[0,1]} |f(t)|.$$

Here the set $\mathcal{K}$ is a set of Hölder continuous functions, defined by

$$\mathcal{K} \;\doteq\; \{f\in\mathcal{C}^{1/2}([0,1])\,;\quad \|f\|_{\mathcal{C}^{1/2}} \;\le\; C_K,\quad f(0)=\bar\alpha\}, \tag{3.9}$$

for a suitable constant C_K, to be determined later. Indeed, for any $t \in [0,1]$ one has

$$
\begin{aligned}
& |\mathcal{P}\alpha_1(t) - \mathcal{P}\alpha_2(t)| \\
= \;& \left| \int_0^t \Big(c(u(s,x(s,\alpha_1(s)))) - c(u(s,x(s,\alpha_2(s)))) \Big) + \int_{x(s,\alpha_1(s))}^{x(s,\alpha_2(s))} \frac{c'(R^2S - RS^2)}{2c}\, dxds \right| \\
\le \;& \int_0^t M\|u\|_{C^{1/2}} |\alpha_2 - \alpha_1|^{1/2} ds + \int_0^t \int_{x(s,\alpha_1(s))}^{x(s,\alpha_2(s))} \frac{|c'|}{2c} \Big(|RS| \big(|R| + |S| \big) \Big)\, dxds \\
\le \;& C_0 \int_0^1 E_0^{1/2} |\alpha_2 - \alpha_1|^{1/2} ds \\
& + C_0 \left(\int_0^1 \int_{x(s,\alpha_1(s))}^{x(s,\alpha_2(s))} R^2S^2 dxds \right)^{1/2} \left(\int_0^1 \int_{x(s,\alpha_1(s))}^{x(s,\alpha_2(s))} (R^2 + S^2) dxds \right)^{1/2}.
\end{aligned}
\tag{3.10}
$$

Here we have used Hölder's inequality and the fact that $\alpha \mapsto x(t,\alpha)$ is Lipschitz continuous with constant 1.

Consider any function $\alpha_1 \in \mathcal{C}^0([0,1])$. As $\|\alpha_2 - \alpha_1\|_{\mathcal{C}^0} \to 0$ we have $\|x(\cdot,\alpha_2) - x(\cdot,\alpha_1)\|_{\mathcal{C}^0} \to 0$ as well. Since the functions R^2S^2, R^2, and S^2 are all in $\mathbf{L}^1([0,1] \times I\!R)$, the right hand side of (3.10) approaches zero. This proves the continuity of the map $\mathcal{P}$, in the $\mathcal{C}^0$ norm.

Next, we need to show that, for a suitable choice of C_K, the transformation $\mathcal{P}$ maps the compact convex set $\mathcal{K}$ in (3.9) into itself. For $0 < t_2 - t_1 \le 1$, recalling (2.10) we obtain

$$
\begin{aligned}
\big|\mathcal{P}\alpha(t_2) - \mathcal{P}\alpha(t_1)\big| \;&=\; \left| \int_{t_1}^{t_2} \left(-c(u(s,x(s,\alpha(s)))) + \int_{-\infty}^{x(s,\alpha(s))} \frac{c'(R^2S - RS^2)}{2c}\, dx \right) ds \right| \\
&\le\; \int_{t_1}^{t_2} M\, dt + \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \left| \frac{c'}{2c} (R^2S - RS^2) \right| dxdt \\
&\le\; M \cdot (t_2 - t_1) + [\zeta(t_2) - \zeta(t_1)] \\
&\le\; (M + C_1)(t_2 - t_1)^{1/2}.
\end{aligned}
\tag{3.11}
$$

The above computation also yields

$$
\max_{t \in [0,1]} |\mathcal{P}\alpha(t)| \;\le\; \bar{\alpha} + M + C_1\,. \tag{3.12}
$$

Together, (3.11) and (3.12) yield an a priori bound on the Hölder norm

$$
\|\mathcal{P}\alpha\|_{\mathcal{C}^{1/2}} \;\le\; \bar{\alpha} + 2(M + C_1),
$$

where C_1 is the constant in (2.11).

By (3.8), $\mathcal{P}\alpha(0) = \bar{\alpha}$. Choosing the constant $C_K \doteq \bar{\alpha} + 2(M + C_1)$ in (3.9), we obtain that $\mathcal{P}$ maps $\mathcal{K}$ into $\mathcal{K}$.

By Schauder's fixed point theorem, the integral equation (3.5) has at least one solution. Iterating the argument, this solution can be extended to any time interval $t \in [0, T]$.

3. In this step and the next one we prove that $x^-(\tau) \doteq x(\tau, \alpha(\tau))$ satisfies the first equation in (1.15) at a.e. time τ.

Since $\frac{c'}{2c}[R^2S - RS^2] \in \mathbf{L}^1([0,T] \times I\!R)$, a classical theorem of Lebesgue implies that

$$\lim_{r\to 0+} \frac{1}{\pi r^2} \iint_{\{(\tau-t)^2+(y-x)^2 \le r^2\}} \frac{c'}{2c}[R^2S - RS^2](\tau, y)\,dy\,d\tau \;=\; \frac{c'}{2c}[R^2S - RS^2](t, x),$$

for all $(t,x) \in \;]0,T[\,\times I\!R$ outside a null set $\mathcal{N}_2$ whose 2-dimensional measure is zero.

If one divides by r instead of r^2, by Corollary 3.2.3 in [13] there is a set $\mathcal{N}_1 \subset \mathcal{N}_2$ whose 1-dimensional Hausdorff measure is zero and such that

$$\limsup_{r\to 0+} \frac{1}{r} \iint_{\{(\tau-t)^2+(y-x)^2 \le r^2\}} \frac{c'}{2c}[R^2S - RS^2](\tau, y)\,dy\,d\tau \;=\; 0$$

for every $(t,x) \notin \mathcal{N}_1$. Therefore, there exists a null 1-dimensional set $\mathcal{N} \subset [0,T]$ with the properties

(i) For every $\tau \notin \mathcal{N}$ and $x \in I\!R$ one has $(\tau, x) \notin \mathcal{N}_1$,

(ii) If $\tau \notin \mathcal{N}$ then the map $t \mapsto \zeta(t)$ in (2.10) is differentiable at $t = \tau$. Moreover, τ is a Lebesgue point of the derivative ζ'.

We conclude this step by observing that the map $t \mapsto x^-(t) \doteq x(t, \alpha(t))$ is absolutely continuous. Indeed, consider a finite sequence of times such that

$$0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_\nu < t_\nu \,.$$

Using (2.14) and the fact that the map $\alpha \mapsto x(t, \alpha)$ is contractive, we obtain

$$\begin{aligned} S \;&\doteq\; \sum_{k=1}^{\nu} |x^-(t_k) - x^-(s_k)| \;=\; \sum_{k=1}^{\nu} \big|x^-(t_k, \alpha(t_k)) - x^-(s_k, \alpha(s_k))\big| \\ &\le\; \sum_{k=1}^{\nu} \big|x^-(t_k, \alpha(t_k)) - x^-(t_k, \alpha(s_k))\big| + \sum_{k=1}^{\nu} \big|x^-(t_k, \alpha(s_k)) - x^-(s_k, \alpha(s_k))\big| \\ &\le\; \sum_{k=1}^{\nu} \big|\alpha(t_k) - \alpha(s_k))\big| + \sum_{k=1}^{\nu} \Big(M(t_k - s_k) + \big|\zeta(t_k) - \zeta(s_k)\big|\Big)\,. \end{aligned} \tag{3.13}$$

Since the two maps $\alpha(\cdot)$ and $\zeta(\cdot)$ are both absolutely continuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$\sum_{k=1}^{\nu} |t_k - s_k| \;\le\; \delta$$

implies that both summations on the right hand side of (3.13) are $< \varepsilon/2$. This proves the absolute continuity of the map $t \mapsto x^-(t)$.

By possibly enlarging the null set $\mathcal{N} \subset [0,T]$ we can assume that, in addition to (i)-(ii) above, one has

(iii) The functions $t \mapsto x^-(t)$ and $t \mapsto \alpha(t)$ are differentiable at each point $\tau \in [0,T] \setminus \mathcal{N}$. Moreover, each point $\tau \notin \mathcal{N}$ is a Lebesgue point of the derivatives $\dot{x}^-$ and $\dot{\alpha}$.

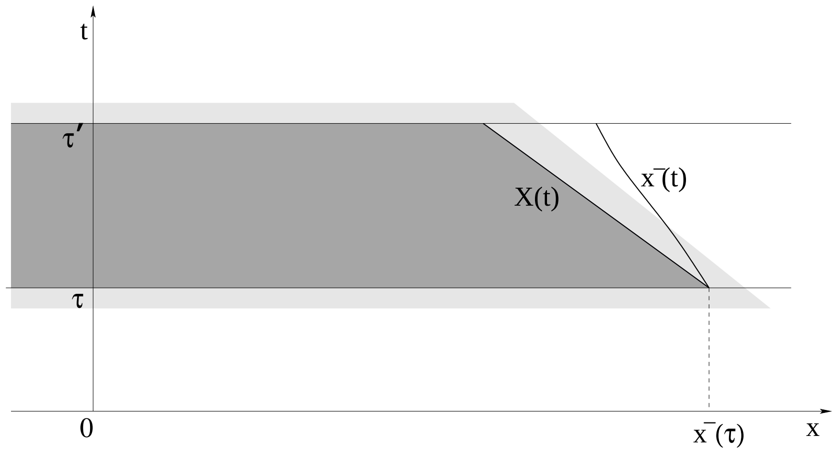

Figure 1: The construction used to prove that $\dot{x}^-(\tau) = -c(u(\tau, x^-(\tau))$. The shaded area is the support of the test function φ^ϵ.

4. Let now $\tau \notin \mathcal{N}$. We claim that the map $t \mapsto x^-(t) \doteq x(t, \alpha(t))$ satisfies the first equation in (1.15) at time $t = \tau$.

Assume, on the contrary, that $\dot{x}^-(\tau) \neq -c(u(\tau, x^-(\tau)))$. To fix the ideas, let

$$\dot{x}^-(\tau) \;=\; -\,c(u(\tau, x^-(\tau))) + 2\varepsilon_0 \tag{3.14}$$

for some $\varepsilon_0 > 0$. The case $\varepsilon_0 < 0$ is entirely similar. To derive a contradiction we choose $\delta > 0$ small enough so that, as shown in Fig. 1,

$$X(t) \;\doteq\; x^-(\tau) + (t-\tau)[-c(u(\tau, x^-(\tau))) + \varepsilon_0] \;<\; x^-(t) \tag{3.15}$$

for all $t \in]\tau, \tau+\delta]$. Since u is continuous while $u_t, u_x \in \mathbf{L}^2$, by an approximation argument the identity in (1.5) remains valid for any Lipschitz continuous function φ with compact support. Given $\tau < \tau' < \tau + \delta$, for $\epsilon > 0$ small we shall construct a Lipschitz approximation φ^ϵ to the characteristic function of the set

$$\Omega \;\doteq\; \big\{(t,y)\,;\;\; t \in [\tau, \tau'], \quad y \in [\epsilon^{-1},\; X(t)]\big\}.$$

For this purpose, consider the functions

$$\rho^\epsilon(t,y) \;\doteq\; \begin{cases} 0 & \text{if } \; y \leq -\epsilon^{-1}, \\ \epsilon^{-1}(y + \epsilon^{-1}) & \text{if } \; -\epsilon^{-1} \leq y \leq \epsilon - \epsilon^{-1}, \\ 1 & \text{if } \; \epsilon - \epsilon^{-1} \leq y \leq X(t), \\ 1 - \epsilon^{-1}(y - X(t)) & \text{if } \; X(t) \leq y \leq X(t) + \epsilon, \\ 0 & \text{if } \; y \geq X(t) + \epsilon, \end{cases}$$

$$\chi^\epsilon(t) \;\doteq\; \begin{cases} 0 & \text{if } \; t \leq \tau - \epsilon, \\ \epsilon^{-1}(s - \tau + \epsilon) & \text{if } \; \tau - \epsilon \leq t \leq \tau, \\ 1 & \text{if } \; \tau \leq t \leq \tau', \\ 1 - \epsilon^{-1}(t - \tau') & \text{if } \; \tau' \leq t < \tau' + \epsilon, \\ 0 & \text{if } \; t \geq \tau' + \epsilon. \end{cases} \tag{3.16}$$

Define the Lipschitz function with compact support

$$\varphi^\epsilon(t,y) \doteq \min\{\rho^\epsilon(t,y),\, \chi^\epsilon(t)\}. \tag{3.17}$$

Using φ^ϵ as test function, since the family of measures μ_-^t satisfy the first equation in (1.13), we obtain

$$\int\left[\int (\varphi_t^\epsilon - c\varphi_x^\epsilon)\, d\mu_-^t + \int \frac{c'}{2c}\,(R^2S - RS^2)\,\varphi^\epsilon\, dx\right]dt \;=\; 0. \tag{3.18}$$

We now observe that, if τ' is sufficiently close to τ, then for any $t \in [\tau, \tau']$ and x close to $x^-(\tau)$, one has

$$0 \;=\; \varphi_t^\epsilon + [-c(u(\tau, x(\tau))) + \varepsilon_0]\varphi_x^\epsilon \;\leq\; \varphi_t^\epsilon - c(u(t,x))\varphi_x^\epsilon\,,$$

because $-c(u(t,x)) < -c(u(\tau, x(\tau))) + \varepsilon_0$ and $\varphi_x^\epsilon \leq 0$.

Since the measures μ_-^t depend continuously on t in the topology of weak convergence, taking the limit of (3.18) as $\epsilon \to 0$, for $\tau, \tau' \notin \mathcal{N}$ we obtain

$$\mu_-^{\tau'}\Big(\,]-\infty,\, X(\tau')]\Big) \;\geq\; \mu_-^{\tau}\Big(\,]-\infty,\, X(\tau)]\Big) + \int_\tau^{\tau'}\int_{-\infty}^{X(t)} \frac{c'}{2c}[R^2S - RS^2]\, dxdt\,. \tag{3.19}$$

By (3.15), for $t \in]\tau,\, \tau+\delta]$ one has

$$\mu_-^{t}\Big(\,]-\infty,\, X(t)]\Big) \;\leq\; \mu_-^{t}\Big(\,]-\infty,\, x^-(t)[\Big). \tag{3.20}$$

Using (3.19)-(3.20) one obtains

$$\begin{aligned}\alpha(t) - \alpha(\tau) \;&\geq\; \Big[x^-(t) + \mu_-^t\Big(\,]-\infty,\, x^-(t)[\Big)\Big] - \Big[x^-(\tau) + \mu_-^\tau\Big(\,]-\infty,\, x^-(\tau)]\Big)\Big]\\ &\geq\; \Big[-c\big(u(\tau, x^-(\tau))\big) + 2\varepsilon_0\Big](t-\tau) + \int_\tau^t\int_{-\infty}^{x^-(s)} \frac{c'}{2c}[R^2S - RS^2]\, dyds + o(t-\tau).\end{aligned} \tag{3.21}$$

Since $\tau \notin \mathcal{N}$, the term

$$o(t-\tau) \;=\; \int_\tau^t\int_{X(s)}^{x^-(s)} \frac{c'}{2c}[R^2S - RS^2]\, dyds$$

is a higher order infinitesimal, namely $\frac{o(t-\tau)}{t-\tau} \to 0$ as $t \to \tau+$. Differentiating (3.21) w.r.t. t at $t = \tau$, we obtain

$$\dot\alpha(\tau) \;\geq\; \Big[-c\big(u(\tau, x^-(\tau))\big) + 2\varepsilon_0\Big] + \int_{-\infty}^{x^-(\tau)} \frac{c'}{2c}[R^2S - RS^2]\, dyds$$

in contradiction with (3.7).

5. In this step we prove the uniqueness of the solution to (3.6)-(3.7). Consider the weight

$$W(t,\alpha) \;\doteq\; e^{\kappa A^+(t,\alpha)} \tag{3.22}$$

with

$$A^+(t,\alpha) \doteq \mu_+^t\Big(\,]-\infty,\ x(t,\alpha)]\Big) + [\zeta(T)-\zeta(t)]. \tag{3.23}$$

Here ζ is the function defined at (2.10), while

$$\kappa \doteq \frac{M}{2c_0^2}\,. \tag{3.24}$$

We recall that $\zeta(T)-\zeta(t)$ provides an upper bound on the energy transferred from backward to forward moving waves and conversely, during the time interval $[t,T]$. In turn, $A^+(t,\alpha)$ yields an upper bound on the total energy of forward moving waves that can cross the backward characteristic $x(\cdot,\alpha)$ during the time interval $[t,T]$.

For any $\alpha_1<\alpha_2$ and $t\geq 0$, we define a weighted distance by setting

$$d^{(t)}(\alpha_1,\alpha_2) \doteq \int_{\alpha_1}^{\alpha_2} W(t,\alpha)\,d\alpha. \tag{3.25}$$

Consider two solutions of (3.7), say $\alpha_1(t)\leq\alpha_2(t)$. For convenience, we use the shorter notation

$$x_i(t) \doteq x^-(t,\alpha_i(t)),\qquad\qquad i=1,2.$$

We recall that, by the definition of conservative solution, the measures $c'(u)\cdot\mu_-^t$, and $c'(u)\cdot\mu_+^t$ are absolutely continuous w.r.t. Lebesgue measure for a.e. time t. As before, R^2 and S^2 denote the density of the absolutely continuous part of μ_-^t, and μ_+^t, respectively. One has the estimate

$$
\begin{aligned}
&\int_{\alpha_1(t)}^{\alpha_2(t)} W(t,\alpha)\,d\alpha \;-\int_{\alpha_1(\tau)}^{\alpha_2(\tau)} W(\tau,\alpha)\,d\alpha \\
=\;& \int_\tau^t \dot\alpha_2 W(s,\alpha_2)-\dot\alpha_1 W(s,\alpha_1)+\int_{\alpha_1(t)}^{\alpha_2(t)}\frac{\partial}{\partial t}W(s,\alpha)\,d\alpha ds \\
=\;& \int_\tau^t\int_{\alpha_1(s)}^{\alpha_2(s)}\left\{\frac{\partial}{\partial\alpha}\left[\left(-c(u(s,x(s,\alpha)))+\int_{-\infty}^{x(s,\alpha)}\frac{c'(R^2S-RS^2)}{2c}\,dx\right)W(s,\alpha)\right]+\frac{\partial}{\partial t}W(s,\alpha)\right\}d\alpha ds \\
=\;& \int_\tau^t\int_{\alpha_1(s)}^{\alpha_2(s)}\left[\frac{c'(S-R)}{2c(1+R^2)}+\frac{c'\,(R^2S-RS^2)}{2c(1+R^2)}\right]W\,d\alpha \\
&\quad+\int_{\alpha_1(s)}^{\alpha_2(s)}\left(-c+\int_{-\infty}^{x(s,\alpha)}\frac{c'(R^2S-RS^2)}{2c}\,dx\right)\frac{S^2}{1+R^2}\kappa W\,d\alpha ds \\
&\quad-\int_\tau^t\int_{\alpha_1(s)}^{\alpha_2(s)}\left[\frac{1+2R^2}{1+R^2}cS^2+\frac{1+R^2+S^2}{1+R^2}\int_{-\infty}^{x(s,\alpha)}\frac{c'(R^2S-RS^2)}{2c}\,dx+\dot\zeta(s)\right]\cdot\kappa W\,d\alpha ds \\
=\;& \int_\tau^t\int_{\alpha_1(s)}^{\alpha_2(s)}\left[\frac{c'(S-R)}{2c(1+R^2)}+\frac{c'\,(R^2S-RS^2)}{2c(1+R^2)}\right]W\,d\alpha ds-\int_\tau^t\int_{\alpha_1(s)}^{\alpha_2(s)}2cS^2\kappa W\,d\alpha ds \\
&\quad-\int_\tau^t\int_{\alpha_1(s)}^{\alpha_2(s)}\left(\int_{-\infty}^{x(s,\alpha)}\frac{c'(R^2S-RS^2)}{2c}\,dx+\dot\zeta(s)\right)\kappa W\,d\alpha ds \\
\le\;& \int_\tau^t\int_{\alpha_1(s)}^{\alpha_2(s)}\left\{\left\|\frac{c'}{2c}\right\|_{\mathbf{L}^\infty}\left(1+|S|+|S|+|S^2|\right)-2cS^2\kappa\right\}W\,d\alpha ds \\
\le\;& \int_\tau^t\int_{\alpha_1(s)}^{\alpha_2(s)}\left\{\left\|\frac{c'}{2c}\right\|_{\mathbf{L}^\infty}2(1+S^2)-2cS^2\kappa\right\}W\,d\alpha ds\;\le\;\left\|\frac{c'}{2c}\right\|_{\mathbf{L}^\infty}\cdot\int_\tau^t\int_{\alpha_1(s)}^{\alpha_2(s)}W\,d\alpha ds\,.
\end{aligned}
\tag{3.26}
$$

By Gronwall's lemma this implies

$$
d^{(t)}\big(\alpha_1(t),\alpha_2(t)\big)\;\le\;e^{C_0t}\,d^{(0)}\big(\alpha_1(0),\alpha_2(0)\big),
\tag{3.27}
$$

with $C_0\doteq\|c'/2c\|_{\mathbf{L}^\infty}$. For every initial value $\bar\alpha$ this yields the uniqueness of the solution of (3.6)-(3.7).

6. Finally, we claim that, for any initial data (1.16), there exists a unique function $x^-(t)$ which satisfies the first equation in (1.15) together with (3.1).

Indeed, let $x_1^-(t)$ and $x_2^-(t)$ be two solutions with $x_1^-(0)=x_2^-(0)=\bar y$. For $i=1,2$, consider the functions

$$
\alpha_i(t)\;\doteq\;x_i^-(t)+\int_{-\infty}^{\bar y}R^2(0,x)\,dx+\int_0^t\int_{-\infty}^{x_i^-(t)}\frac{c'\,(R^2S-RS^2)}{2c}\,dx\,ds\,.
$$

Then $x_i^-(t) = x(t, \alpha_i(t))$, and both α_1, α_2 are solutions to the Cauchy problem (3.5), with initial data

$$\alpha_1(0) \;=\; \alpha_2(0) \;=\; \bar{y} + \int_{-\infty}^{\bar{y}} R^2(0,x)\,dx\,.$$

The uniqueness result proved in step **5** now implies $x_1^-(t) = x(t, \alpha_1(t)) = x(t, \alpha_2(t)) = x_2^-(t)$.
□

4 Lipschitz continuity in characteristic coordinates

Let $u = u(t,x)$ be a conservative solution to the wave equation (1.1) with initial data (1.2). Given $(X,Y) \in I\!R^2$, there exists unique initial points $\bar{x} = x_0(X)$ and $\bar{y} = y_0(Y)$ such such that

$$X \;=\; \bar{x} + \int_{-\infty}^{\bar{x}} R^2(0,x)dx, \qquad\qquad Y \;=\; \bar{y} + \int_{-\infty}^{\bar{y}} S^2(0,x)dx. \tag{4.1}$$

By Lemma 2, there exists a unique backward characteristic $t \mapsto x^-(t,\bar{x})$ starting at $\bar{x}$, and a unique forward characteristic $t \mapsto x^+(t,\bar{y})$ starting at $\bar{y}$. Indeed, recalling (2.4)-(2.5), we can write

$$x^-(t,\bar{x}) \;=\; x(t,\alpha(t)), \qquad\qquad x^+(t,\bar{y}) \;=\; y(t,\beta(t)), \tag{4.2}$$

where $\alpha(\cdot)$ provides a solution to (3.7) with initial data $\alpha(0) = X$, and similarly for $\beta(\cdot)$.

Assuming that $\bar{x} \geq \bar{y}$, we define

$$P(X,Y) \;=\; (t(X,Y), x(X,Y))$$

to be the unique point where these two characteristics cross. That means

$$x^-(t(X,Y),\,\bar{x}) \;=\; x^+(t(X,Y),\,\bar{y}) \;=\; x(X,Y). \tag{4.3}$$

We then define

$$u(X,Y) \;\doteq\; u\big(t(X,Y), x(X,Y)\big). \tag{4.4}$$

Lemma 3. *The map $(X,Y) \mapsto (t,x,u)(X,Y)$ is locally Lipschitz continuous.*

Proof. 1. For a fixed Y, we show that the map $X \mapsto (t,x,u)(X,Y)$ is locally Lipschitz continuous. Fix an interval $[0,T]$ and let $X_1 < X_2$. Recalling the notation in (2.4)-(2.5), consider the backward characteristics

$$t \mapsto x_1(t) \;\doteq\; (t,\alpha_1(t)), \qquad\qquad t \mapsto x_2(t) \;\doteq\; x(t,\alpha_2(t)), \tag{4.5}$$

where α_1, α_2 are the solutions of (3.7) with initial data $\alpha_1(0) = X_1$ and $\alpha_2(0) = X_2$, respectively. Similarly, let $t \mapsto y(t,\beta(t))$ be the forward characteristic, with $\beta(0) = Y$.

As shown in Fig. 2, assuming that $y(0,Y) \leq x(0,X_1)$, let t_1, t_2 be the times when these characteristics cross, so that $x(t_i, \alpha_i(t_i)) = y(t_i, \beta(t_i))$, $i = 1,2$.

2. According to (3.27), we have

$$\alpha_2(t_1) - \alpha_1(t_1) \;\leq\; C\,(\alpha_2(0) - \alpha_1(0)) \;=\; C\,(X_2 - X_1), \tag{4.6}$$

for some constant C uniformly bounded as t_1 ranges over a bounded interval. In turn this implies

$$x(X_2,Y)-x(X_1,Y) \;\leq\; x(t_1,\alpha_2(t_1))-x(t_1,\alpha_1(t_1)) \;\leq\; \alpha_2(t_1)-\alpha_1(t_1) \;\leq\; C\,(X_2-X_1), \tag{4.7}$$

proving the Lipschitz continuity of the map $X \mapsto x(X,Y)$.

In turn, we have

$$t(X_2,Y)-t(X_1,Y) \;\leq\; \|c(u)\|_{\mathbf{L}^\infty}\cdot\big(x(X_2,Y)-x(X_1,Y)\big) \leq\; \|c(u)\|_{\mathbf{L}^\infty}\cdot C\,(X_2-X_1), \tag{4.8}$$

showing that the map $X \mapsto t(X,Y)$ is Lipschitz continuous as well.

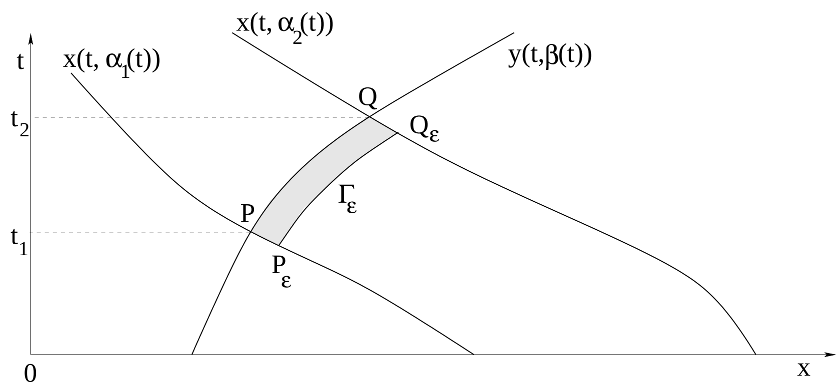

Figure 2: Proving the Lipschitz continuity of t,x,u as functions of the variables X,Y.

3. It remains to show that the map $X \mapsto u(X,Y)$ is Lipschitz continuous. Let $X_1 < X_2$ and Y be given. As in step **1**, consider the backward characteristics $t \mapsto x_i(t) \doteq x(t,\alpha_i(t))$, $i=1,2$, and the forward characteristic $t \mapsto y(t) \doteq y(t,\beta(t))$, with $\alpha_i(0)=X_i$ and $\beta(0)=Y$.

As shown in Fig. 2, consider the intersection points

$$P \;=\; (t_1, x(t_1,\alpha_1(t_1))), \qquad Q=(t_2,x(t_2,\alpha_2(t_2))).$$

Moreover, for $\eta>0$ small, consider the curve

$$t \;\mapsto\; \gamma_\eta(t) \;\doteq\; y(t,\beta(t))+\eta\,,$$

and call P_η, Q_η the points where this curve intersects the two backward characteristics (4.5).

Since $u=u(t,x)$ is continuous, one has

$$u(X_2,Y)-u(X_1,Y) \;=\; u(Q)-u(P) \;=\; \lim_{\varepsilon\to 0+}\;\frac{1}{\varepsilon}\int_0^\varepsilon \Big(u(Q_\eta)-u(P_\eta)\Big)\,d\eta\,. \tag{4.9}$$

Call Γ_ε the rectangular region bounded by the two backward characteristics in (4.5) and by the curves $\gamma_0,\gamma_\varepsilon$ (the shaded region in Fig. 2). We now compute

$$\begin{aligned}\int_0^\varepsilon u(Q_\eta)-u(P_\eta)\,d\eta \;&=\; \iint_{\Gamma_\varepsilon}\Big[u_t+c(u(t,y(t)))u_x-R(t,x)\Big]\,dxdt+\iint_{\Gamma_\varepsilon} R\,dxdt\\ &\doteq\; I_1+I_2\,.\end{aligned} \tag{4.10}$$

We estimate the two integrals on the right hand side of (4.10). Assuming $0 \leq t_1 < t_2 \leq T$, by Cauchy's inequality we obtain

$$\begin{aligned}
|I_1| &\leq \iint_{\Gamma_\varepsilon} \Big| c(u(t,y(t)) - c(u(t,x)) \Big| \, |u_x(t,x)| \, dxdt \\
&= \mathcal{O}(1) \cdot \int_0^T \left(\int_{0<x-y(t)<\varepsilon} |x-y(t)|^{1/2} \, |u_x(t,x)| \, dx \right) dt \\
&= \mathcal{O}(1) \cdot \int_0^T \varepsilon \left(\int_{0<x-y(t)<\varepsilon} |u_x(t,x)|^2 \, dx \right)^{1/2} dt \, .
\end{aligned}$$

This implies

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \, I_1 \;=\; 0. \tag{4.11}$$

Next,

$$|I_2| \;\leq\; \int_{\Gamma_\varepsilon} (1+R^2) \, dxdt \tag{4.12}$$

To estimate the above integral, consider the weight function

$$V^\varepsilon(t,x) \;\doteq\; \begin{cases} 0 & \text{if} \quad x \leq y(t), \\[2mm] \dfrac{x-y(t)}{\varepsilon} & \text{if} \quad x \in [y(t), \; y(t)+\varepsilon], \\[2mm] 1 & \text{if} \quad x \geq y(t)+\varepsilon \, . \end{cases}$$

Since the family of measures μ_-^t satisfies the balance equation in (1.13) with speed $< -c_0$, for a.e. time t such that $x_1(t) \;\leq\; y(t) \;\leq\; y(t)+\varepsilon \;\leq\; x_2(t)$ we have

$$\begin{aligned}
\frac{1}{\varepsilon} \int_{y(t)}^{y(t)+\varepsilon} R^2(t,x) \, dx \quad &\leq \; \frac{1}{\varepsilon} \, \mu_-^t \big( ]x_1(t), \; x_2(t)[\big) \\
&\leq \; -\frac{1}{2c_0} \frac{d}{dt} \left[\int_{x_1(t)}^{x_2(t)} V(t,x) d\mu_-^t \right] + \int_{x_1(t)}^{x_2(t)} \frac{|c'(u)| \, \big| R^2 S - RS^2 \big|}{2c} \, dx.
\end{aligned}$$

Integrating over the interval $[t_1, t_2]$ and using the estimate (3.26) to control the total contribution of the source term, we obtain

$$\frac{|I_2|}{\varepsilon} \;\leq\; \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon} (1+R^2) \, dxdt \;\leq\; (t_2 - t_1) + \frac{1}{2c_0} (X_2 - X_1) + \mathcal{O}(1) \cdot (X_2 - X_1). \tag{4.13}$$

This proves the Lipschitz continuity of the map $X \mapsto u(X,Y)$.

The Lipschitz continuity of (t,x,u) as functions of Y is proved in exactly the same way. □

Remark 3. By Rademacher's theorem, the above result implies that the map

$$\Lambda : (X,Y) \mapsto (t(X,Y),\, x(X,Y)) \tag{4.14}$$

is a.e. differentiable. We can thus consider the set Ω of *critical points* and the set V of *critical values* of Λ, by setting

$$\Omega \doteq \Big\{(X,Y)\,;\quad \text{either } D\Lambda(X,Y) \text{ does not exists, or else } \det D\Lambda(X,Y) = 0\Big\}. \tag{4.15}$$

$$V \doteq \Big\{\Lambda(X,Y)\,;\quad (X,Y) \in \Omega\Big\}. \tag{4.16}$$

By the area formula [13], the 2-dimensional measure of V is zero.

In general, the map $\Lambda : I\!R^2 \mapsto I\!R^2$ is onto but not one-to-one. However, for each $(t_0, x_0) \notin V$, there exist a unique point (X,Y) such that $\Lambda(X,Y) = (t_0, x_0)$.

Next, consider a function $f(t,x)$, with $f \in \mathbf{L}^1(I\!R^2)$. Since f is defined up to a set of measure zero in the t-x plane, the composition $\tilde f(X,Y) = f(\Lambda(X,Y))$ is well defined at a.e. point $(X,Y) \in I\!R^2 \setminus \Omega$. Moreover, we have the change of variable formula

$$\int_{I\!R^2} f(t,x)\,dxdt \;=\; \int_{I\!R^2\setminus\Omega} \tilde f(X,Y) \cdot |\det D\Lambda(X,Y)|\,dXdY. \tag{4.17}$$

To compute the determinant of the Jacobian matrix $D\Lambda$, we observe that

$$x_X \;=\; c(u)\,t_X\,, \qquad\qquad x_Y \;=\; -\,c(u)\,t_Y\,, \tag{4.18}$$

$$D\Lambda \;=\; \begin{pmatrix} t_X & t_Y \\ x_X & x_Y \end{pmatrix} \;=\; \begin{pmatrix} \frac{x_X}{c(u)} & -\frac{x_Y}{c(u)} \\ x_X & x_Y \end{pmatrix}. \tag{4.19}$$

Hence

$$\big|\det D\Lambda\big| \;=\; \frac{2}{c(u)} x_X x_Y\,. \tag{4.20}$$

For future use, in the X-Y plane we define the "good set"

$$\mathcal{G} \;\doteq\; I\!R^2 \setminus \Omega\,. \tag{4.21}$$

5 An equivalent semilinear system

In this section we introduce further variables and show that, as functions of X, Y, these variables satisfy a semilinear system with smooth coefficients. In particular, their values are uniquely determined by the initial data. By showing that the map $(X,Y) \mapsto (t,x,u)(X,Y)$ is uniquely determined, we eventually obtain the uniqueness of the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2).

Recalling (2.2)-(2.3), for given initial values $\bar\alpha, \bar\beta$ let $t \mapsto \alpha(t, \bar\alpha)$ and $t \mapsto \beta(t, \bar\beta)$ be the unique solutions to

$$\begin{aligned} \alpha(t) &= \bar\alpha + \int_0^t \left(-c(0) + \int_{-\infty}^{x(t,\alpha(t))} \frac{c'(S - R + R^2 S - R S^2)}{2c}\, dx \right) dt\,, \\ \beta(t) &= \bar\beta + \int_0^t \left(c(0) - \int_{-\infty}^{y(t,\beta(t))} \frac{c'(S - R + R^2 S - R S^2)}{2c}\, dx \right) dt\,. \end{aligned}$$

The existence and uniqueness of these functions was proved in Section 3. We recall that $t \mapsto x^-(t) = x(t, \alpha(t))$ and $t \mapsto x^+(t) = y(t, \beta(t))$ are then the unique backward and forward characteristics starting from the points $x(0, \bar\alpha)$ and $y(0, \bar\beta)$, respectively. Define the new dependent variables $p(X, Y)$ and $q(X, Y)$ by setting

$$p(X,Y) \;=\; \frac{\partial}{\partial\bar\alpha}\alpha(\tau,\bar\alpha)\Big|_{\bar\alpha = X,\, \tau = t(X,Y)}, \qquad q(X,Y) \;=\; \frac{\partial}{\partial\bar\beta}\beta(\tau,\bar\beta)\Big|_{\bar\beta = Y,\, \tau = t(X,Y)}. \tag{5.1}$$

In addition, recalling the definitions of the maps $\alpha \mapsto x(t, \alpha)$ and $\beta \mapsto y(t, \beta)$ in (2.2)-(2.3), we define

$$\nu(X,Y) \doteq \frac{\partial x}{\partial\alpha}(t(X,Y), \alpha(t, x(X,Y))), \qquad \eta(X,Y) \doteq \frac{\partial x}{\partial\beta}(t(X,Y), \beta(t, x(X,Y)))\,. \tag{5.2}$$

Finally, observing that the functions c, p, q are strictly positive, we define ξ, ζ by setting

$$\xi(X,Y) \doteq \frac{2c(u(X,Y))}{p(X,Y)} u_X(X,Y)\,, \qquad \zeta(X,Y) \doteq \frac{2c(u(X,Y))}{q(X,Y)} u_Y(X,Y)\,. \tag{5.3}$$

By Rademacher's theorem, the above derivatives are a.e. well defined, because

(i) for any t, the functions $\bar\alpha \mapsto \alpha(t, \bar\alpha)$, $\bar\beta \mapsto \beta(t, \bar\beta)$, $\alpha \mapsto x(t, \alpha)$, and $\beta \mapsto y(t, \beta)$ are Lipschitz continuous, and

(ii) by Lemma 3, both x and u are Lipschitz continuous functions of X, Y.

Moreover,

$$p(X,Y) = q(X,Y) = 1 \qquad \text{if} \qquad t(X,Y) = 0.$$

Our main goal is to show that these variables satisfy the semilinear system with smooth coefficients

$$\begin{cases} u_X = \frac{1}{2c}\xi p, \qquad u_Y = \frac{1}{2c}\zeta q, \\ x_X = \frac{1}{2}\nu p, \qquad x_Y = -\frac{1}{2}\eta q, \\ t_X = \frac{1}{2c}\nu p, \qquad t_Y = \frac{1}{2c}\eta q, \\ p_Y = \frac{c'}{4c^2}[\zeta - \xi]pq, \\ q_X = \frac{c'}{4c^2}[\xi - \zeta]pq, \\ \nu_Y = \frac{c'}{4c^2}\xi(\nu - \eta)q, \\ \eta_X = -\frac{c'}{4c^2}\zeta(\nu - \eta)p, \\ \xi_Y = -\frac{c'}{8c^2}(\eta + \nu)q + \frac{c'}{4c^2}(\xi^2 + \eta\nu)q\,, \\ \zeta_X = -\frac{c'}{8c^2}(\eta + \nu)p + \frac{c'}{4c^2}(\zeta^2 + \eta\nu)p\,. \end{cases} \tag{5.4}$$

More precisely, we have

Theorem 4. *By possibly changing the functions $p, q, \nu, \eta, \xi, \zeta$ on a set of measure zero in the X-Y plane, the following holds.*

(i) For a.e. $X_0 \in I\!\!R$, the functions t, x, u, p, ν, ξ are absolutely continuous on every vertical segment of the form $S_0 \doteq \{(X_0, Y)\,; \quad a < Y < b\}$. Their partial derivatives w.r.t. Y satisfy a.e. the corresponding equations in (5.4).

(ii) For a.e. $Y_0 \in I\!\!R$, the functions t, x, u, q, η, ζ are absolutely continuous on every horizontal segment of the form $S_0 \doteq \{(X, Y_0)\,; \quad a < X < b\}$. Their partial derivatives w.r.t. X satisfy a.e. the corresponding equations in (5.4).

Toward a proof, we recall a standard result in the theory of Sobolev spaces.

Lemma 4. *Let $\Gamma =]a, b[\times]c, d[$ be a rectangle in the X-Y plane. Assume that $u \in \mathbf{L}^\infty(\Gamma)$ has a weak partial derivative w.r.t. X. That means*

$$\int_\Gamma (u\varphi_X + f\varphi)\, dXdY \;=\; 0 \tag{5.5}$$

for some $f \in \mathbf{L}^1(\Gamma)$ and all test functions $\varphi \in \mathcal{C}_c^\infty(\Gamma)$. Then, by possibly modifying u on a set of measure zero, the following holds. For a.e. $Y_0 \in]c, d[$, the map $X \mapsto u(X, Y_0)$ is absolutely continuous and

$$\frac{\partial}{\partial X}u(X, Y_0) \;=\; f(X, Y_0) \qquad\qquad \text{for a.e. } X \in]a, b[\,. \tag{5.6}$$

For a proof, see for example [2], p.159, or [13], p.44. To use the above result, it is convenient to replace the test functions φ with characteristic functions of arbitrary rectangles contained in Γ.

Lemma 5. *Let $\Gamma =]a,b[\times]c,d[$ be a rectangle in the X-Y plane. Assume that $u \in \mathbf{L}^\infty(\Gamma)$ and $f \in \mathbf{L}^1(\Gamma)$. Moreover assume that there exists null sets $\mathcal{N}_X \subset]a,b[$ and $\mathcal{N}_Y \subset]c,d[$ such that the following holds.*

For every $\overline{X}_1, \overline{X}_2 \notin \mathcal{N}_X$ and $\overline{Y}_1, \overline{Y}_2 \notin \mathcal{N}_Y$ with $\overline{X}_1 < \overline{X}_2$ and $\overline{Y}_1 < \overline{Y}_2$, one has

$$\int_{\overline{Y}_1}^{\overline{Y}_2} \Big[u(\overline{X}_2, Y) - u(\overline{X}_1, Y)\Big]\, dY \;=\; \int_{\overline{Y}_1}^{\overline{Y}_2} \int_{\overline{X}_1}^{\overline{X}_2} f(X,Y)\, dX dY\,. \tag{5.7}$$

Then the conclusion of Lemma 4 holds.

Proof. Consider any test function $\varphi \in \mathcal{C}_c^\infty(\Gamma)$. We need to show that (5.5) holds. Given $\varepsilon > 0$, we can find points

$$a = X_0 < X_1 < \cdots < X_N = b\,, \qquad\qquad c = Y_0 < Y_1 < \cdots < Y_N = d$$

with $X_i \notin \mathcal{N}_X$, $Y_i \notin \mathcal{N}_Y$ and

$$X_i - X_{i-1} \;<\; \varepsilon\,, \qquad Y_i - Y_{i-1} \;<\; \varepsilon\,, \qquad\qquad i = 1, \ldots, N\,.$$

Define the approximate function φ^ε by setting

$$\varphi^\varepsilon(X,Y) \;\doteq\; \varphi(X_{i-1}, Y_{i-1}) + (X - X_{i-1}) \cdot \big[\varphi(X_i, Y_{i-1}) - \varphi(X_{i-1}, Y_{i-1})\big].$$

Taking a sequence of these approximations with $\varepsilon \to 0$, we have the convergence

$$\|\varphi^\varepsilon - \varphi\|_{\mathbf{L}^\infty} \;\to\; 0, \qquad\qquad \|\varphi^\varepsilon_X - \varphi_X\|_{\mathbf{L}^\infty} \;\to\; 0.$$

Therefore,

$$\int_\Gamma (u\varphi_X + f\varphi)\, dX dY \;=\; \lim_{\varepsilon\to 0} \int_\Gamma (u\varphi^\varepsilon_X + f\varphi^\varepsilon)\, dX dY \;=\; 0. \tag{5.8}$$

Indeed, for every approximate function φ^ε one has

$$
\begin{aligned}
\int_\Gamma (u\varphi^\varepsilon_X + f\varphi^\varepsilon)\, dXdY \;&=\; \sum_{i,j=1}^N \int_{X_{i-1}}^{X_i}\int_{Y_{j-1}}^{Y_j} (u\varphi^\varepsilon_X + f\varphi^\varepsilon)\, dXdY \\
&= \sum_{i,j=1}^N \int_{Y_{j-1}}^{Y_j}\int_{X_{i-1}}^{X_i} \left[u(X_{i-1},Y) + \int_{X_{i-1}}^{X} f(X',Y)\, dX' \right] \varphi^\varepsilon_X \, dXdY \\
&\qquad + \sum_{i,j=1}^N \int_{Y_{j-1}}^{Y_j}\int_{X_{i-1}}^{X_i} f\varphi^\varepsilon \, dXdY \\
&= -\sum_{i,j=1}^N \int_{Y_{j-1}}^{Y_j}\int_{X_{i-1}}^{X_i} f\varphi^\varepsilon \, dXdY \\
&\qquad + \sum_{i,j=1}^N \int_{Y_{j-1}}^{Y_j} \Big[u(X_i,Y)\varphi(X_i,Y_{i-1}) - u(X_{i-1},Y)\varphi(X_{i-1}Y_{i-1}) \Big]\, dY \\
&\qquad + \sum_{i,j=1}^N \int_{Y_{j-1}}^{Y_j}\int_{X_{i-1}}^{X_i} f\varphi^\varepsilon \, dXdY \\
&= 0\,.
\end{aligned}
$$

□

Recalling the sets of regular and critical points $\mathcal{G}$, Ω, defined at (4.21) and (4.15) respectively, we now derive a representation for the variables ν, η, ξ, ζ in terms of R and S.

Lemma 6.

(i) If $(X,Y) \in \mathcal{G} \doteq I\!R^2 \setminus \Omega$ then

$$
\frac{p(X,Y)}{x_X(X,Y)} \;=\; 2(1+R^2)\,, \qquad\qquad \frac{q(X,Y)}{x_Y(X,Y)} \;=\; 2(1+S^2)\,, \tag{5.9}
$$

$$
\left\{ \begin{array}{rcl} \nu(X,Y) &=& \frac{1}{1+R^2}\,, \\ \eta(X,Y) &=& \frac{1}{1+S^2}\,, \end{array} \right. \qquad\qquad \left\{ \begin{array}{rcl} \xi(X,Y) &=& \frac{R}{1+R^2}\,, \\ \zeta(X,Y) &=& \frac{S}{1+S^2}\,, \end{array} \right. \tag{5.10}
$$

where the right hand sides are evaluated at the point $(t(X,Y), x(X,Y))$.

(ii) For a.e. $(X,Y) \in \Omega$, one has

$$
\nu(X,Y) \;=\; \eta(X,Y) \;=\; \xi(X,Y) \;=\; \zeta(X,Y) \;=\; 0\,. \tag{5.11}
$$

Proof. 1. Consider a regular value $(X, Y) \in \mathcal{G}$. To fix the ideas, let $t(X, Y) = \tau$. Recalling the definition (2.2) and the fact that the absolutely continuous part of μ_-^τ has density R^2, we conclude

$$\frac{\partial}{\partial \alpha} x(\tau, \alpha) \;=\; \frac{1}{1+R^2}. \tag{5.12}$$

On the other hand,

$$\frac{\partial}{\partial \bar{\alpha}} x(\tau, \alpha(\tau, \bar{\alpha})) \;=\; 2x_X\,. \tag{5.13}$$

Together, the above equalities yield the first identity in (5.9). The second one is proved similarly.

2. The first identity in (5.10) is precisely (5.12), and the second one is similar. To prove the third identity we observe that, at a point $(X, Y) \in \mathcal{G}$,

$$u_X \;=\; (u_t + c(u)u_x)\, t_X \;=\; R\,\frac{x_X}{c(u)}\,.$$

Therefore, by (5.9),

$$\xi \;\doteq\; \frac{2c(u)}{p} \cdot u_X \;=\; \frac{2c(u)}{p} \cdot R\,\frac{x_X}{c(u)} \;=\; \frac{R}{1+R^2}\,.$$

The last identity in (5.10) is proved similarly.

3. Finally, consider the set $\mathcal{G}_0$ of points (X, Y) where the map $\Lambda : (X, Y) \mapsto (t, x)$ is differentiable but either $x_X = 0$ or $x_Y = 0$. By Rademacher's theorem, the set $\mathbb{R}^2 \setminus (\mathcal{G} \cup \mathcal{G}_0)$ has measure zero.

Assume $(X, Y) \in \mathcal{G}_0$, with $x_X(X, Y) = 0$. Then

$$\nu(X, Y) \;=\; \frac{\partial x}{\partial \bar{\alpha}} \cdot \left(\frac{\partial \alpha}{\partial \bar{\alpha}}\right)^{-1} \;=\; 0.$$

In the same way one proves that $\eta = 0$.

Next, we claim that $t_X(X, Y) = x_X(X, Y) = 0$ implies $u_X = 0$. This will be proved by refining the estimates in step **3** of the proof of Lemma 3. Adopting the same construction, for any $\delta > 0$ we can replace the bound (4.12) with

$$|I_2| \;\leq\; \int_{\Gamma_\varepsilon} (C_\delta + \delta R^2)\, dxdt\,, \tag{5.14}$$

where $C_\delta = (4\delta)^{-1}$. In this way, the estimate (4.13) can be replaced by

$$\frac{|I_2|}{\varepsilon} \;\leq\; \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon} (C_\delta + \delta R^2)\, dxdt \;\leq\; C_\delta (t_2 - t_1) + \delta \left[\frac{1}{2c_0}(X_2 - X_1) + \mathcal{O}(1) \cdot (X_2 - X_1)\right]. \tag{5.15}$$

Letting $\varepsilon \to 0$ this implies

$$|u(X_2, Y) - u(X_1, Y)| \;\leq\; C_\delta \big|t(X_2, Y) - t(X_1, Y)\big| + \delta \left[\frac{1}{2c_0}(X_2 - X_1) + \mathcal{O}(1) \cdot (X_2 - X_1)\right]$$

In the above formula we can now take $X_1 = X$, $X_2 = X + \varepsilon$. Letting $\varepsilon \to 0$ one obtains

$$\big|u_X(X,Y)\big| \;=\; \left|\lim_{\varepsilon\to 0} \frac{u(X+\varepsilon,Y)-u(X,Y)}{\varepsilon}\right| \;\leq\; C_\delta t_X + \delta\cdot\mathcal{O}(1)\,.$$

Since $t_X(X,Y)=0$ and $\delta>0$ can be taken arbitrarily small, this proves $u_X(X,Y)=0$. From the definition (5.3) it thus follows $\xi(X,Y)=0$. Similarly, if $t_Y = x_Y = 0$, then $\zeta = 0$. □

Remark 4. By a direct calculation, one finds

$$\det D\Lambda \;=\; \frac{pq}{2c(1+R^2)(1+S^2)}. \tag{5.16}$$

5.1 Proof of Theorem 4.

To achieve a proof of Theorem 4, we will show that the assumptions of Lemma 5 apply to all the variables $t, x, u, p, q, \eta, \nu, \xi, \zeta$ in (5.4). For thus purpose, consider any rectangle $\mathcal{Q} \doteq [X_1, X_2]\times[Y_1,Y_2]$ in the X-Y plane.

1 - Equations for u. By Lemma 3, the function u is Lipschitz continuous w.r.t. the variables X, Y. The equations

$$u_X \;=\; \frac{1}{2c}\,\xi p\,, \qquad\qquad u_Y \;=\; \frac{1}{2c}\,\zeta q$$

are immediate consequences of the definitions (5.3).

2 - Equations for x and t. By Lemma 3, both functions x, t are Lipschitz continuous w.r.t. the variables X, Y. Recalling the definitions (5.2)-(5.3), we compute

$$\begin{aligned}
\frac{\partial}{\partial X}x(X,Y) &= \frac{1}{2}\frac{\partial}{\partial\bar\alpha}x(t(X,Y),\alpha(t(X,Y),\bar\alpha)) \\
&= \frac{1}{2}\frac{\partial x}{\partial\alpha}(t(X,Y),\alpha(t(X,Y),x(X,Y)))\frac{\partial\alpha}{\partial\bar\alpha}(t(X,Y),\bar\alpha)\bigg|_{\bar\alpha=X} \\
&= \frac{1}{2}\nu p.
\end{aligned} \tag{5.17}$$

The equation for x_Y is obtained in a similar way.

In turn, the equations for t are derived by

$$t_X \;=\; \frac{x_X}{c(u)} \;=\; \frac{1}{2c}\nu p, \qquad\qquad t_Y \;=\; -\frac{x_Y}{c(u)} \;=\; \frac{1}{2c}\eta q. \tag{5.18}$$

3 - Equations for p and q. By Lemma 3, the map $\Lambda(X,Y) \doteq (t(X,Y),\, x(X,Y))$ is Lipschitz continuous, hence its Jacobian matrix $D\Lambda$ is a.e. well defined. Consider the domain

$$\mathcal{D} \;\doteq\; \Big\{(X,Y)\,;\quad X\in[X_1,\,X_2],\quad Y\in[Y_1,Y_2],\qquad \det\, D\Lambda(X,Y)>0\Big\}. \tag{5.19}$$

Recalling (3.5) and using (4.17), we obtain

$$
\begin{aligned}
&\int_{X_1}^{X_2} p(X,Y_2) - p(X,Y_1) dX \\
&\quad = \int_{X_1}^{X_2} \left[\frac{\partial \alpha(\tau,X)}{\partial X}\bigg|_{\tau=\tau(X,Y_2)} - \frac{\partial \alpha(\tau,X)}{\partial X}\bigg|_{\tau=\tau(X,Y_1)} \right] dX \\
&\quad = \int_{X_1}^{X_2} \left[\frac{\partial}{\partial X} \int_{x(X,Y_1)}^{x(X,Y_2)} \int_{\tau(\widetilde{X},Y_1)}^{\tau(\widetilde{X},Y_2)} \frac{c'}{2c}(S - R + R^2S - S^2R) dt dx \right] d\widetilde{X} \\
&\quad = \iint_{\Lambda(\mathcal{D})} \frac{c'}{2c}(S - R + R^2S - S^2R) dx dt \qquad (5.20) \\
&\quad = \iint_{\mathcal{D}} \frac{c'}{2c}(S - R + R^2S - S^2R) \cdot \det D\Lambda(X,Y)\, dX dY \\
&\quad = \iint_{\mathcal{D}} \frac{c'}{4c^2}(S - R + R^2S - RS^2) \frac{1}{1+R^2} \frac{1}{1+S^2} pq\, dX dY \\
&\quad = \iint_{\mathcal{Q}} \frac{c'}{4c^2}(\zeta - \xi) pq\, dX dY.
\end{aligned}
$$

The last equality follows from Lemma 6, part (i) for the integral over $\mathcal{D}$ and part (ii) for the integral over $\mathcal{Q} \setminus \mathcal{D}$.

Thus by Lemma 4 and 5,

$$
p_Y = \frac{c'}{4c^2}(\zeta - \xi) pq. \qquad (5.21)
$$

Similarly,

$$
q_X = -\frac{c'}{4c^2}(\zeta - \xi) pq.
$$

4 - Equations for η and ν. We first observe that

$$
\int_{X_1}^{X_2} \Big[p\nu(X,Y_2) - p\nu(X,Y_1) \Big] dX = \int_{X_1}^{X_2} \left[\frac{\partial x(\tau,X)}{\partial X}\bigg|_{\tau=t(X,Y_2)} - \frac{\partial x(\tau,X)}{\partial X}\bigg|_{\tau=t(X,Y_1)} \right] dX
$$

$$
= \int_{X_1}^{X_2} \left[\frac{\partial}{\partial X} \int_{x(X,Y_1)}^{x(X,Y_2)} \int_{t(\widetilde{X},Y_1)}^{t(\widetilde{X},Y_2)} \frac{c'}{2c}(S - R) dt dx \right] d\widetilde{X} = \iint_{\Lambda(\mathcal{D})} \frac{c'}{2c}(S - R)\, dx dt\,.
$$

In turn, by Remark 3 we obtain

$$
\begin{aligned}
\int_{X_1}^{X_2}\Big[p\nu(X,Y_1)-p\nu(X,Y_2)\Big]\,dX &= \int_{\mathcal{D}}\frac{c'}{2c}(S-R)\cdot\det D\Lambda(X,Y)\,dXdY \\
&= \int_{\mathcal{D}}\frac{c'}{4c^2}(S-R)\frac{1}{1+R^2}\frac{1}{1+S^2}pq\,dXdY \qquad (5.22)\\
&= \int_{\mathcal{Q}}\frac{c'}{4c^2}(\nu\zeta-\xi\eta)pq\,dXdY
\end{aligned}
$$

By Lemma 4 and 5, the above implies

$$(p\nu)_Y \;=\; \frac{c'}{4c^2}(\nu\zeta-\xi\eta)pq. \qquad (5.23)$$

Recalling the equation (5.21) for p, we obtain

$$\nu_Y=\frac{c'}{4c^2}\xi(\nu-\eta)q. \qquad (5.24)$$

Similarly,

$$\eta_X=-\frac{c'}{4c^2}\zeta(\nu-\eta)p. \qquad (5.25)$$

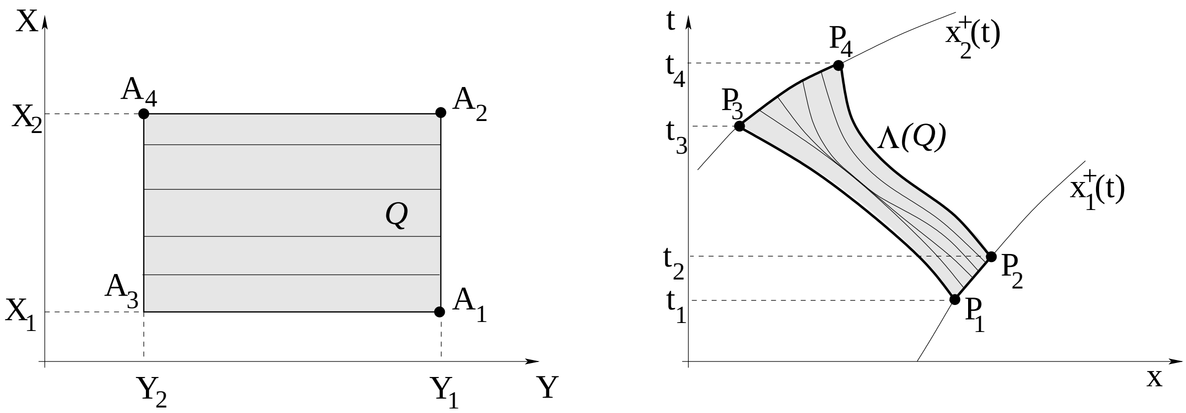

Figure 3: Left: the rectangle $\mathcal{Q}$ in the X-Y plane. Right: the image $\Lambda(\mathcal{Q})$ in the t-x plane. For $i=1,2,3,4$, the points $P_i=\Lambda(A_i)$ are defined as in (5.28).

5 - Equations for ξ and ζ. We observe that, by (1.7)-(1.8), R provides a weak solution to the balance law

$$R_t-(cR)_x \;=\; \frac{c'}{4c}(R^2-S^2)-c'u_xR \;=\; \frac{c'}{4c}(R^2-S^2)-\frac{c'}{2c}(R-S)R=-\frac{c'}{4c}(R-S)^2. \qquad (5.26)$$

Notice that, by definition of conservative solution, the right hand side is a function in $\mathbf{L}^1(\mathbb{R}^2)$, w.r.t. the variables t, x.

Next, we wish to characterize the distributional derivative u_{XY}. More precisely, we seek a function $f\in\mathbf{L}^1_{loc}(\mathbb{R}^2)$ such that the following holds. Consider any values $X_1<X_2$ and $Y_1>Y_2$. Then

$$[u(X_2,Y_1)-u(X_1,Y_1)]-[u(X_2,Y_2)-u(X_1,Y_2)] \;=\; \int_{X_1}^{X_2}\int_{Y_2}^{Y_1}f(X,Y)\,dXdY. \qquad (5.27)$$

Toward this goal, consider the image of these four points under the map Λ, in the t-x plane:

$$
\begin{array}{ll}
P_1 \doteq (t_1, x_1) = \Lambda(X_1, Y_1), & P_2 \doteq (t_2, x_2) = \Lambda(X_2, Y_1), \\
P_3 \doteq (t_3, x_3) = \Lambda(X_1, Y_2), & P_4 \doteq (t_4, x_4) = \Lambda(X_2, Y_2),
\end{array}
\tag{5.28}
$$

(see Fig. 3). We now construct a family of test functions ϕ^ϵ approaching the characteristic function of the set $\Lambda(\mathcal{Q})$, where $\mathcal{Q} \doteq [X_1, X_2] \times [Y_1, Y_2]$. More precisely:

$$
\phi^\epsilon(s, y) \doteq \min\{\varrho^\epsilon(s, y), \varsigma^\epsilon(s, y)\}, \tag{5.29}
$$

where

$$
\varrho^\epsilon(s, y) \doteq \begin{cases} 0 & \text{if } y \leq x_1^-(s) - \epsilon \\ 1 + \epsilon^{-1}(y - x_1^-(s)) & \text{if } x_1^-(s) - \epsilon \leq y \leq x_1^-(s) \\ 1 & \text{if } x_1^-(s) \leq y \leq x_2^-(s) \\ 1 - \epsilon^{-1}(y - x_2^-(s)) & \text{if } x_2^-(s) \leq y \leq x_2^-(s) + \epsilon \\ 0 & \text{if } y \geq x_2^-(s) + \epsilon, \end{cases} \tag{5.30}
$$

$$
\varsigma^\epsilon(s, y) \doteq \begin{cases} 0 & \text{if } y \leq x_1^+(s) - \epsilon \\ 1 + \epsilon^{-1}(y - x_1^+(s)) & \text{if } x_1^+(s) - \epsilon \leq y \leq x_1^+(s) \\ 1 & \text{if } x_1^+(s) \leq y \leq x_2^+(s) \\ 1 - \epsilon^{-1}(y - x_2^+(s)) & \text{if } x_2^+(s) \leq y \leq x_2^+(s) + \epsilon \\ 0 & \text{if } y \geq x_2^+(s) + \epsilon. \end{cases} \tag{5.31}
$$

Here $t \mapsto x_1^-(t)$ and $t \mapsto x_2^-(t)$ are the backward characteristics corresponding to $X = X_1$ and $X = X_2$ respectively. Similarly, $t \mapsto x_1^+(t)$ and $t \mapsto x_2^+(t)$ are the forward characteristics corresponding to $Y = Y_1$ and $Y = Y_2$ respectively.

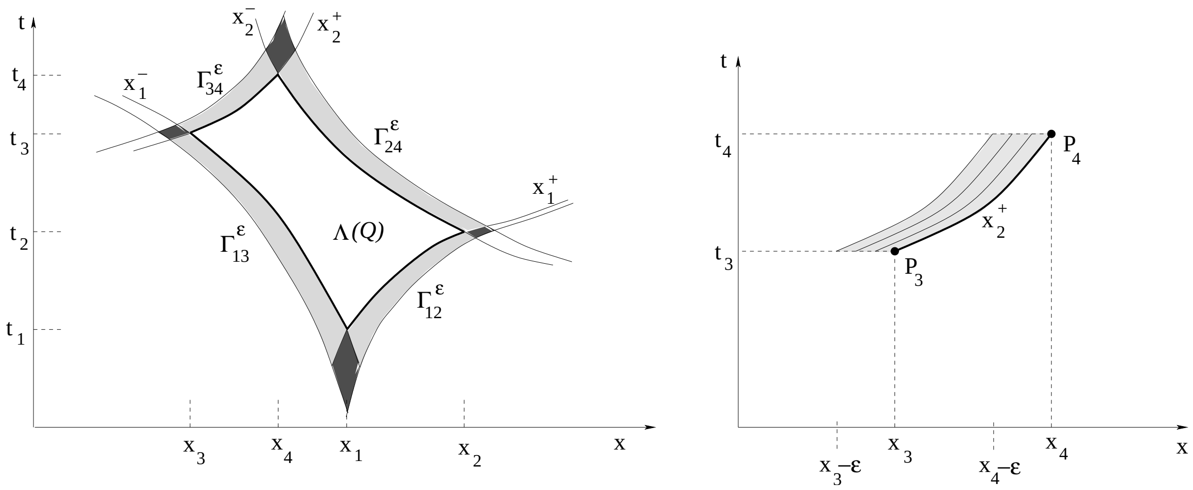

Figure 4: Left: the support of the test function ϕ^ϵ in (5.29). Right: the difference $u(P_4) - u(P_3)$ can be approximated by the integral of a directional derivative of u over the shaded area.

By (1.8) for every test function $\varphi \in \mathcal{C}_c^1(I\!R^2)$ we now have

$$
\iint R\Big[\varphi_t - (c\varphi)_x\Big]\, dxdt = -\iint \frac{c'}{4c}(R^2 - S^2)\varphi\, dxdt\,.
$$

Since

$$R \in \mathbf{L}^2_{loc}(I\!R^2), \qquad c(u) \in H^1_{loc}(I\!R^2), \qquad \frac{1}{c(u)} \in H^1_{loc}(I\!R^2),$$

we can take a sequence of test functions φ_n such that, as $n \to \infty$,

$$\varphi_n \;\to\; \frac{\phi^\epsilon}{c(u)} \qquad \text{in} \;\; H^1(I\!R^2).$$

Taking the limit, one obtains

$$\iint R\left[\left(\frac{\phi^\epsilon}{c}\right)_t - \phi^\epsilon_x\right] dxdt \;=\; -\iint \frac{c'}{4c^2}(R^2 - S^2)\phi^\epsilon\, dxdt. \tag{5.32}$$

By (1.7), this yields

$$\iint \left[\frac{R}{c}(\phi^\epsilon_t - c\phi^\epsilon_x) - \frac{c'}{2c^2}(R^2 + RS)\phi^\varepsilon\right] dxdt \;=\; -\iint \frac{c'}{4c^2}(R^2 - S^2)\phi^\epsilon\, dxdt\,, \tag{5.33}$$

and finally

$$\iint \frac{R}{c}(\phi^\epsilon_t - c\phi^\epsilon_x)\, dxdt \;=\; \iint \frac{c'}{4c^2}(R+S)^2\phi^\epsilon\, dxdt\,. \tag{5.34}$$

By the way the test function ϕ^ϵ has been defined at (5.29)–(5.31) the function $\phi^\epsilon_t - c\phi^\epsilon_x$ is supported on a small neighborhood of the boundary of $\Lambda(\mathcal{Q})$. More precisely, consider the four sets (Fig. 4, left)

$$\begin{aligned}
\Gamma^\epsilon_{12} &\doteq \Big\{(t,x)\,;\;\; x_1^+(t) \le x \le x_1^+(t) + \epsilon\,,\;\; x_1^-(t) - \epsilon \le x \le x_2^-(t) + \epsilon\Big\},\\
\Gamma^\epsilon_{13} &\doteq \Big\{(t,x)\,;\;\; x_1^-(t) - \epsilon \le x \le x_1^-(t)\,,\;\; x_1^+(t) - \epsilon \le x \le x_2^+(t) + \epsilon\Big\},\\
\Gamma^\epsilon_{24} &\doteq \Big\{(t,x)\,;\;\; x_2^-(t) \le x \le x_2^-(t) + \epsilon\,,\;\; x_1^-(t) - \epsilon \le x \le x_2^-(t) + \epsilon\Big\},\\
\Gamma^\epsilon_{34} &\doteq \Big\{(t,x)\,;\;\; x_2^+(t) - \epsilon \le x \le x_2^+(t)\,,\;\; x_1^-(t) - \epsilon \le x \le x_2^-(t) + \epsilon\Big\}.
\end{aligned} \tag{5.35}$$

Notice that these sets overlap near the points $P_i = (t_i, x_i)$, $i = 1, 2, 3, 4$. However, each of these intersections is contained in a ball of radius $\mathcal{O}(\varepsilon)$. For example,

$$\Gamma^\epsilon_{12} \cap \Gamma^\epsilon_{13} \;\subset\; B(P_1, K\epsilon),$$

for some constant K and all $\epsilon > 0$. Observing that $R \in \mathbf{L}^2_{loc}(I\!R^2)$, and $\|\phi^\epsilon_t + c(u)\phi^\epsilon_x\|_{\mathbf{L}^\infty} = \mathcal{O}(\epsilon)$, we obtain the estimate

$$\begin{aligned}
\left|\iint_{\Gamma^\epsilon_{12}\cap\Gamma^\epsilon_{13}} \frac{R}{c(u)}(\phi^\epsilon_t - c\phi^\epsilon_x)\, dxdt\right| &\le \frac{C_0}{\epsilon}\iint_{B(P_1,K\epsilon)} |R|\, dxdt\\
&\le \frac{C_0}{\epsilon}\int_{t_1-K\epsilon}^{t_1+K\epsilon}\left(\int_{x_1-2K\epsilon}^{x_1+2K\epsilon} R^2(t,x)dx\right)^{1/2}(2K\epsilon)^{1/2}\, dt \;\le\; \frac{C_0}{\epsilon}E_0^{1/2}(2K\epsilon)^{3/2},
\end{aligned} \tag{5.36}$$

for suitable constants C_0, K, and all $\epsilon > 0$. Repeating this argument for the other three intersections, we thus conclude

$$
\begin{aligned}
\lim_{\epsilon\to 0}\iint \tfrac{R}{c(u)}(\phi^\epsilon_t - c\phi^\epsilon_x)\,dxdt \quad &= \lim_{\epsilon\to 0}\iint_{\Gamma^\epsilon_{12}\cup\Gamma^\epsilon_{13}\cup\Gamma^\epsilon_{24}\cup\Gamma^\epsilon_{34}} \frac{R}{c(u)}(\phi^\epsilon_t - c\phi^\epsilon_x)\,dxdt \\
&= \lim_{\epsilon\to 0}\left(\iint_{\Gamma^\epsilon_{12}} + \iint_{\Gamma^\epsilon_{13}} + \iint_{\Gamma^\epsilon_{24}} + \iint_{\Gamma^\epsilon_{34}}\right)\frac{R}{c(u)}(\phi^\epsilon_t - c\phi^\epsilon_x)\,dxdt\,.
\end{aligned}
\tag{5.37}
$$

Since $c(u)$ is uniformly positive and bounded, the same argument used in (4.11) yields

$$
\lim_{\epsilon\to 0}\iint_{\Gamma^\epsilon_{13}\cup\Gamma^\epsilon_{24}} \frac{R}{c(u)}(\phi^\epsilon_t - c\phi^\epsilon_x)\,dxdt \;=\; 0\,. \tag{5.38}
$$

Concerning the integral over Γ^ϵ_{34}, by (5.31) we obtain

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\iint_{\Gamma^\epsilon_{34}} \frac{R(t,x)}{c(u(t,x))}\cdot\frac{c(u(t,x_2^+(t))) + c(u(t,x))}{\epsilon}\,dxdt \\
&= \lim_{\epsilon\to 0}\iint_{\Gamma^\epsilon_{34}} \frac{R(t,x)}{c(u(t,x))}\frac{2c(u(t,x))}{\epsilon}dxdt + \lim_{\epsilon\to 0}\iint_{\Gamma^\epsilon_{34}} \frac{R(t,x)}{c(u(t,x))}\,\frac{c(u(t,x_2^+(t))) - c(u(t,x))}{\epsilon}\,dxdt \\
&= \lim_{\epsilon\to 0}\frac{2}{\epsilon}\iint_{\Gamma^\epsilon_{34}} R(t,x)\,dxdt\,.
\end{aligned}
\tag{5.39}
$$

Since $(R+S)^2 \in \mathbf{L}^1_{loc}(I\!R^2)$, one has

$$
\lim_{\epsilon\to 0}\iint \frac{c'}{4c^2}(R+S)^2\phi^\epsilon\,dxdt \;=\; \iint_{\Lambda(\mathcal{Q})}\frac{c'}{4c^2}(R+S)^2dxdt\,. \tag{5.40}
$$

We thus conclude

$$
\lim_{\epsilon\to 0}\frac{1}{\epsilon}\iint_{\Gamma^\epsilon_{34}} R\,dxdt - \lim_{\epsilon\to 0}\frac{1}{\epsilon}\iint_{\Gamma^\epsilon_{12}} R\,dxdt \;=\; \iint_{\Lambda(\mathcal{Q})}\frac{c'}{8c^2}(R+S)^2dxdt\,. \tag{5.41}
$$

Next, since $u \in H^1_{loc}$, we can write (Fig. 4, right)

$$
\begin{aligned}
u(P_4) - u(P_3) \;&=\; \lim_{\varepsilon\to 0}\frac{1}{\epsilon}\left(\int_{x_4-\epsilon}^{x_4} u(t_4,y)\,dy - \int_{x_3-\epsilon}^{x_3} u(t_3,y)\,dy\right) \\
&=\; \lim_{\epsilon\to 0}\frac{1}{\epsilon}\iint_{\Gamma_{34}}\Big[u_t + c(u(t,x_1^+(t)))u_x\Big]\,dxdt \\
&=\; \lim_{\epsilon\to 0}\frac{1}{\epsilon}\iint_{\Gamma_{34}}\Big[u_t + c(u(t,x)u_x\Big]\,dxdt \\
&=\; \lim_{\epsilon\to 0}\frac{1}{\epsilon}\iint_{\Gamma_{34}} R\,dxdt\,.
\end{aligned}
\tag{5.42}
$$

This, and a similar estimate for $u(P_2)-u(P_1)$, yield

$$[u(P_4)-u(P_3)]-[u(P_2)-u(P_1)] \;=\; \iint_{\Lambda(\mathcal{Q})} \frac{c'}{8c^2}(R+S)^2\,dxdt\,. \tag{5.43}$$

Writing the right hand side of (5.43) as an integral w.r.t. the variables X, Y, by Remark 3

$$\begin{aligned}[u(X_2,Y_2)-u(X_1,Y_2)]-[u(X_2,Y_1)-u(X_1,Y_1)] \;&=\; \iint_{\Lambda(\mathcal{Q})} \frac{c'}{8c^2}(R+S)^2\,dxdt \\ =\; \iint_{\mathcal{Q}\cap\mathcal{G}} \frac{c'}{8c^2}(R+S)^2\cdot\frac{p}{(1+R^2)}\frac{q}{2c(1+S^2)}\,dXdY,&\end{aligned} \tag{5.44}$$

where the last equality follows from Lemma 6(ii). By Lemma 5, this shows that the weak derivative u_{XY} exists and is given by

$$u_{XY}(X,Y) \;=\; \begin{cases} -\dfrac{c'}{8c^2}(R+S)^2\cdot\dfrac{p}{(1+R^2)}\dfrac{q}{2c(1+S^2)} & \text{if} \quad \det D\Lambda(X,Y)>0, \\ \\ 0 & \text{if} \quad \det D\Lambda(X,Y)=0. \end{cases}$$

Recalling the definition (5.3), we can write this weak derivative as

$$u_{XY} \;=\; -\frac{c'}{16c^3}(\eta+\nu)pq+\frac{c'}{8c^3}(-\xi\zeta+\eta\nu)pq\,. \tag{5.45}$$

By Lemma 4, it follows that, for a.e. X, the map $Y\mapsto u_X(X,Y) \;=\; (\frac{1}{2c}\xi p)(X,Y)$ is absolutely continuous and its derivative is given by (5.45). Recalling the equations for p and u, since p remains uniformly positive on bounded sets, we conclude

$$\xi_Y \;=\; -\frac{c'}{8c^2}(\eta+\nu)q+\frac{c'}{4c^2}(\xi^2+\eta\nu)q\,. \tag{5.46}$$

Similarly,

$$\zeta_X \;=\; -\frac{c'}{8c^2}(\eta+\nu)p+\frac{c'}{4c^2}(\zeta^2+\eta\nu)p\,. \tag{5.47}$$

This completes the proof of Theorem 4. □

6 Uniqueness of conservative solutions

We can now give a proof of Theorem 3, showing that conservative solutions to the variational wave equation (1.1) are unique.

Let initial data $u_0\in H^1(I\!R)$, $u_1\in \mathbf{L}^2(I\!R)$ be given. These data uniquely determine a curve γ in the X-Y plane, parameterized by

$$X(x) \;\doteq\; x+\int_{-\infty}^{x} R^2(0,y)\,dy\,, \qquad\qquad Y(x) \;\doteq\; x+\int_{-\infty}^{x} S^2(0,y)\,dy.$$

Along γ, the values of the variables $(u,x,t,p,q,\nu,\eta,\xi,\zeta)$ are all determined by the data u_0, u_1. Indeed, at the point $(X(x),Y(x))\in\gamma$ we have

$$\begin{cases} t &= 0, \\ x &= x\,, \end{cases} \qquad\qquad \begin{cases} u &= u_0(x), \\ p &= q \;=\; 1\,, \end{cases}$$

$$\begin{cases} \nu &= \dfrac{1}{1+R^2(0,x)}\,, \\ \\ \eta &= \dfrac{1}{1+S^2(0,x)}\,, \end{cases} \qquad\qquad \begin{cases} \xi &= \dfrac{R(0,x)}{1+R^2(0,x)}\,, \\ \\ \zeta &= \dfrac{S(0,x)}{1+S^2(0,x)}\,. \end{cases}$$

We recall that, by (1.6),

$$R(0,x) \;=\; u_1(x) + c(u_0(x))\,u_{0,x}\,, \qquad\qquad S(0,x) \;=\; u_1(x) - c(u_0(x))\,u_{0,x}\,.$$

Since the right hand sides of the equations in (5.4) are smooth, given the above boundary data along γ, this semilinear system has a unique solution in the X-Y plane. In particular, the functions $(X,Y) \mapsto (x,t,u)(X,Y)$ are uniquely determined, up to a set of zero measure in the X-Y plane. Since the map $(x,t) \mapsto u(x,t)$ is continuous, we conclude that u is uniquely determined, pointwise in the x-t plane. This completes the proof of Theorem 3. □

Acknowledgment. This research was partially supported by NSF, with grant DMS-1411786: Hyperbolic Conservation Laws and Applications, and the AMS Simons Travel Grants Program.

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