Finite time singularities for hyperbolic systems

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Abstract

In this paper, we study the formation of finite time singularities in the form of super norm blowup for a spatially inhomogeneous hyperbolic system. The system is related to the variational wave equations as those in [18]. The system posses a unique C^1 solution before the emergence of vacuum in finite time, for given initial data that are smooth enough, bounded and uniformly away from vacuum. At the occurrence of blowup, the density becomes zero, while the momentum stays finite, however the velocity and the energy are both infinity.

1 Introduction

In this paper, we consider the following Cauchy problem of spatially inhomogeneous hyperbolic partial differential equations:

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + \left(\rho u^2 - c^2(x)\rho^{-1}\right)_x = -c(x)c'(x)\rho^{-1} \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \end{cases} \tag{1.1}$$

where $\rho(x,t) : \mathbb{R} \times [0, +\infty) \to [0, \infty)$ is the density, $u(x,t) : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is the velocity, ρ_0, u_0 are given initial data that will be specified later and $c(x) : \mathbb{R} \to \mathbb{R}^+$ is a given function satisfying

$$c(x) \in C^2, \quad 0 < c_0 \le c(x) \le C_0 < +\infty, \quad |c'(x)| \le C_1 < +\infty, \quad c'(x) \not\equiv 0, \tag{1.2}$$

for some constants c_0, C_0 and C_1. It is easy to verify that the smooth solutions of (1.1) satisfy an energy conservation law

$$E_t + q_x = 0 \tag{1.3}$$

with specific energy (entropy)

$$E = \frac{1}{2}\rho u^2 + \frac{1}{2}c^2\rho^{-1} \tag{1.4}$$

and entropy flux

$$q = \frac{1}{2}u^3\rho - \frac{1}{2}c^2\rho^{-1}u\,. \tag{1.5}$$

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1.1 Inhomogeneous linear elasticity

System (1.1) can be viewed as an Eulerian description of inhomogeneous linear elasticity.

Basic mechanics. For any smooth domain $\Omega_0^X \subset \mathbb{R}^n$ with $n = 1, 2$, or 3, let $X \in \Omega_0^X$ denotes the Lagrangian coordinates. The flow map $x(X,t) : \Omega_0^X \to \Omega_t^x$ in Figure 1 satisfies (see [1] for details)

$$\begin{cases} \dfrac{dx}{dt} &= u(x(X,t),t) \\ x(X,0) &= X\,. \end{cases} \tag{1.6}$$

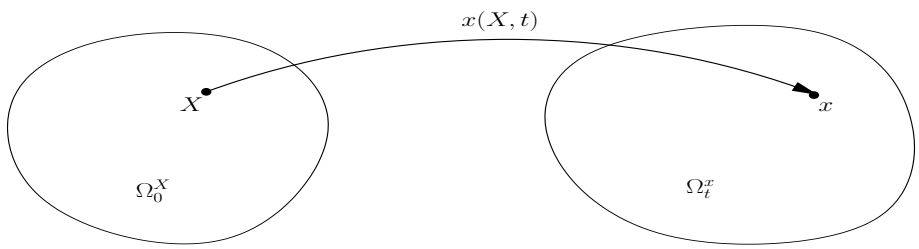

Figure 1: A flow map

Furthermore, let

$$\tilde{F}(X,t) = \frac{\partial x(X,t)}{\partial X} \tag{1.7}$$

be the deformation matrix associate with the flow map and define the Eulerian quantity (push forward)

$$F(x(X,t),t) = \tilde{F}(X,t). \tag{1.8}$$

In case of F preserving the sign, e.g. $\det \tilde{F} > 0$, not only $\tilde{F}$ is an invertible matrix, $x(X,t)$ preserves the orientation. By (1.7), (1.8) and direct calculation, we obtain that F satisfies the following kinematic relation (chain rule) (c.f. [31])

$$F_t + u \cdot \nabla_x F = \nabla_x u \cdot F. \tag{1.9}$$

Let $\rho(x,t) : \Omega_t^x \times [0,+\infty) \to \mathbb{R}^+$ be the density of mass with initial data $\rho_0(X) = \rho(x(X,0),0)$. The usual conservation of mass equation

$$\rho_t + \nabla_x \cdot (\rho u) = 0 \tag{1.10}$$

is equivalent to

$$\rho(x(X,t),t) = \frac{\rho_0(X)}{\det \tilde{F}(X,t)}. \tag{1.11}$$

Energetic variational approaches. By the first and second laws of thermodynamics, one can start with energy law for a conservative system:

$$\frac{d}{dt} \int_{\Omega_0^X} \mathcal{K}(X,t,x_t) + \mathcal{W}(X,t,x,\tilde{F})\, dX = 0, \tag{1.12}$$

where $\mathcal{K}(X,t,x_t)$ denotes the kinetic energy and $\mathcal{W}(X,t,x,\tilde{F})$ denotes the free energy. Specifically, one can consider the following simple forms of kinetic energy and internal energy (for inhomogeneous linear elasticity)

$$\frac{d}{dt}\frac{1}{2}\int_{\Omega_0^X}\rho_0(X)|x_t(X,t)|^2+c^2(x)|\tilde{F}(X,t)|^2\,dX=0, \tag{1.13}$$

where $c(x):\Omega_t^x\to\mathbb{R}^+$ is a given scalar function. The energy law (1.13) has been widely used to describe the elasticity in an inhomogeneous medium, that includes the coupling and competition between the kinetic energy and (linear) elastic energy (c.f [33], [31] and [28]). In particular, when the space dimension $n=1$ and the initial data $\rho_0(X)=1$, we have

$$\rho(x(X,t),t)=\frac{\rho_0(X)}{\det\tilde{F}(X,t)}=\frac{1}{\tilde{F}(X,t)}>0. \tag{1.14}$$

Therefore, to certain degree, in one space dimension one cannot distinguish elastic energy which depends only on $\tilde{F}$ and the conventional free energy of fluid which is a function of ρ (c.f. [15]).

System (1.1) can be derived from the energy law (1.13) by using the energetic variational methods under Eulerian coordinates (c.f. [20]). For completeness of our paper, we sketch main steps of derivation as those in [20].

For any $T>0$, the action of the system is

$$A(x)=\frac{1}{2}\int_0^T\int_\Omega\rho_0(X)|x_t(X,t)|^2-c^2(x)|\tilde{F}(X,t)|^2\,dXdt.$$

By the least action law, the variation of $A(x)$ with respect to x under Lagrangian coordinates can be calculated as follows:

$$\begin{aligned}0&=\frac{d}{d\epsilon}\bigg|_{\epsilon=0}A(x+\epsilon y)\\&=\int_0^T\int_{\Omega_0^X}\rho_0(X)x_t\cdot y_t-c(x)\,|\tilde{F}|^2\,(y\cdot\nabla_x)c(x)-c^2(x)\tilde{F}:\frac{\partial y}{\partial X}\,dXdt\\&=-\int_0^T\int_{\Omega_0^X}\rho_0(X)x_{tt}\cdot y+c(x)\,|\tilde{F}|^2\,(y\cdot\nabla_x)c(x)-\mathrm{div}_X\left(c^2(x)\tilde{F}\right)y\,dXdt,\end{aligned} \tag{1.15}$$

for any $y(X,t)\in C_0^\infty(\Omega_0^X\times(0,T))$. Here $A:B=A_{ij}B_{ij}$ denotes the inner product of two matrices. Therefore, the Euler-Lagrange equation of energy law (1.13) under Lagrangian coordinates for any $n=1,2$ or 3, is

$$\rho_0(X)x_{tt}^i+c(x)\tilde{F}^2\nabla_{x_i}c(x)-\nabla_{X_j}(c^2(x)\tilde{F}_{ij})=0. \tag{1.16}$$

Remark 1.1 *When the space dimension $n=1$ and the initial data $\rho_0(X)=1$, equation* (1.16) *can be written as nonlinear wave equation*

$$x_{tt}(X,t)-c(x)(c(x)\tilde{F}(X,t))_X=0, \tag{1.17}$$

which is exactly a special case of one dimensional variational wave equation modeling nematic liquid crystal dynamics. We will provide more details later.

Let $\tilde{y}(x(X,t),t)=y(X,t)$ be the pull-forward quantity of y from the Lagrangian to Eulerian coordinates. Integrating by parts with respect to t, changing variables and integrating by parts

with respect to x, one can obtain

$$
\begin{aligned}
0 &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} A(x+\epsilon y) \\
&= -\int_0^T \int_{\Omega_t} \rho(x,t)(u_t + u\cdot\nabla_x u)\tilde{y} + \frac{c(x)|F|^2(\tilde{y}\cdot\nabla_x)c(x)}{\det F} - \mathrm{div}_x\left(\frac{c^2(x)FF^T}{\det F}\right)\tilde{y}\,dxdt,
\end{aligned}
\tag{1.18}
$$

where (1.6) and (1.11) have been used in changing variables. Therefore, the Euler-Lagrange equation of energy law (1.13) in the Eulerian coordinates is

$$
\rho(x,t)(u_t + u\cdot\nabla_x u) + \frac{c(x)|F|^2\nabla_x c(x)}{\det F} - \mathrm{div}_x\left(\frac{c^2(x)FF^T}{\det F}\right) = 0. \tag{1.19}
$$

When the space dimension $n = 1$, combining (1.19), (1.10) and (1.9), we have the following coupled dynamic system

$$
\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
F_t + uF_x &= u_x F \\
\rho(u_t + uu_x) &= c(x)\left(c(x)F\right)_x.
\end{aligned}
\tag{1.20}
$$

Remark 1.2 *When the space dimension $n = 1$, $\Omega_t^x = \mathbb{R}$, the initial data $\rho_0(X) = 1$ $\left(i.e.\ F = \frac{1}{\rho}\right)$, system* (1.20) *becomes*

$$
\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
\rho(u_t + uu_x) &= c(x)\left(\frac{c(x)}{\rho}\right)_x,
\end{aligned}
\tag{1.21}
$$

which is exactly (1.1)*. And the corresponding energy law* (1.13) *becomes*

$$
\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} \rho(x,t)|u(x,t)|^2 + c^2(x)\rho^{-1}(x,t)\,dx = 0. \tag{1.22}
$$

From Remark 1.1 and Remark 1.2, when the space dimension $n = 1$, the initial data $\rho_0(X) = 1$, $\det F > 0$ and the solutions for both systems are smooth enough, (1.21) and variational wave equation (1.17) are formally equivalent systems under different coordinates. However, in general when one looks at (1.21) and (1.17) in weak form, they can be different since the deformation matrix F is not always invertible (when singularities occur).

The equation (1.17) is a special case of variational wave equation modeling nematic liquid crystal. In [2] such variational wave equation was first investigated in any dimensions when people were trying to find the minimal of the following energy

$$
\int_\Omega |\mathbf{n}_t|^2 - W(\mathbf{n},\nabla\mathbf{n})\,dy = 0, \tag{1.23}
$$

where $|\mathbf{n}| = 1$ and

$$
W(\mathbf{n},\nabla\mathbf{n}) = \alpha|\mathbf{n}\times(\nabla\times\mathbf{n})|^2 + \beta(\nabla\cdot\mathbf{n})^2 + \gamma(\mathbf{n}\cdot\nabla\times\mathbf{n})^2 + \eta\left[\mathrm{tr}(\nabla\mathbf{n})^2 - (\nabla\cdot\mathbf{n})^2\right].
$$

Here α, β, γ and η are all positive viscosity constants, and W is the Oseen-Frank potential for nematic liquid crystal (c.f. [2]). When $\mathbf{n}$ only depends on a single space variable X and

$$
\mathbf{n} = \cos\phi(X,t)\mathbf{e}_{y_1} + \sin\phi(X,t)\mathbf{e}_{y_2} \qquad \text{(planar deformation)},
$$

where $\mathbf{e}_{y_1}$ and $\mathbf{e}_{y_2}$ are the coordinate vectors in the y_1 and y_2 directions, respectively. The Euler-Lagrange equation of (1.23) was given in [2] as follows

$$\phi_{tt} - c(\phi)(c(\phi)\phi_X)_X = 0, \tag{1.24}$$

with

$$c^2(\phi) = \alpha\cos^2(\phi) + \beta\sin^2(\phi).$$

It is obvious that (1.24) is exactly (1.17) with ϕ replaced by x. In [5], Bressan and Zheng have established the global existence of energy conservative weak solutions for (1.17)) by introducing new energy-dependent coordinates (see also [19]). The solutions are locally Hölder continuous with exponent $\frac{1}{2}$. For general $W(\mathbf{n}, \nabla\mathbf{n})$ in one space dimension, the existence of weak solutions has been studied by a series of papers [40, 41, 10].

We really need to point out that the singularity formation for (1.17) has been first studied by Glassey, Hunter and Zheng in their seminal work [18], in which a gradient blowup example has been provided. When there is a damping term in (1.17), a similar gradient blowup example is provided in [11]. In [11, 18], the singularities they construct are "kink" solutions instead of shock waves constructed for systems of conservation laws including at least one genuinely nonlinear characteristic family [26, 23, 6, 7, 8, 9]. We will provide more details in Remark 1.4, Remark 1.6 and Remark 5.1.

Our first main result is for (1.1) with initial data $\rho_0 = 1$ describing the inhomogeneous elastic flow.

Theorem 1.3 *There exists a function $c(x) \in C^2$ (given in (4.1)) satisfying (1.2) and a finite time $T = O(1)\varepsilon^{-2} > 0$, such that the Cauchy problem of (1.1) with initial data $(\rho_0, u_0) = (1, c(x))$ has a unique C^1 solution $(\rho(x,t), u(x,t))$ on $(x,t) \in \mathbb{R} \times [0,T)$. Moreover, there exists a point (x^*, T) with $|x^*| < O(1)\varepsilon^{-1}$ at which the solution satisfies*

$$\lim_{t\to T^-} u(x^*,t) = \infty \quad \lim_{t\to T^-} \rho(x^*,t) = 0 \quad \lim_{t\to T^-} \rho u(x^*,t) = B \quad \text{and} \quad \lim_{t\to T^-} E(x^*,t) = \infty, \tag{1.25}$$

where B is a finite constant. The solutions $\rho(x,t)$ and $-u(x,t)$ have uniform upper bounds on $(x,t) \in \mathbb{R} \times [0,T)$.

Remark 1.4 *We have several remarks for Theorem 1.3:*

(1). In our example, the characteristic speeds of two families are uniformly away from each other, or in another word, system (1.1) is uniformly strictly hyperbolic (see Lemma 4.4 and Lemma 4.5).

(2). In our example, the L^∞ blowup happens at the same time of C^1 blowup, since two characteristic families for (1.1) are both linearly degenerate in the definition of Lax when $c(x)$ is constant. This is a rare *case for systems of hyperbolic conservation laws. In fact, for (1.32), this happens only when $\gamma = -1$.*

(3). By (1.11), (1.17) and the last remark, one could see that the C^1 blowup of the flow map essentially indicates the vacuum formation for (1.1) when the initial data $\rho_0 = 1$ and $F > 0$, through the transformation from Lagrangian coordinates to Eulerian coordinates.

The construction of our example is motivated by the pioneer work by Glassey, Hunter and Zheng [18] on the singularities of variational wave equation (1.17) (with $F = x_X$) in one dimension. However, the blowup example constructed in (1.17) does not satisfy the

restrictions $\rho_0 = 1$ (i.e. $x_X = 1$ initially) and $F = x_X > 0$, hence the system cannot be transformed to (1.1) *in this situation. By introducing new techniques, in Theorem 1.3, we construct a blowup example satisfying all these restrictions. We do adopt important ideas from [18], while the restrictions for inhomogeneous elastic flow make the construction much more complicated than the example in [18].*

(4). For any positive constant K, the total energy of the solution for any time until the blowup when $x \in [-K, K]$ is bounded by a constant depending on K, although at the time of blowup, the energy concentrates, i.e. energy density is infinity, somewhere.

(5). It is an interesting question to consider more general assumptions on c. The numerical experiments in [20] have indicated such blowup might happen for more general cases.

(6). When the initial data $\rho_0 \not\equiv 1$, one needs to investigate the system (1.20) *instead of* (1.1)*. It is also an interesting and challenging question.*

1.2 Isentropic duct flow for Chaplygin gas dynamics

System (1.1) has applications in various fields. We can reformulate the spatially inhomogeneous system (1.1) by setting

$$\rho = c(x)\bar{\rho}, \tag{1.26}$$

then the system (1.1) can be equivalently written as the isentropic flow for Chaplygin gas [13, 35] on varying cross-sectional area of the duct or with radially symmetry (see also equation (7.1.24) in [15]), which is used for the modeling of dark energy,:

$$\begin{cases} (c(x)\bar{\rho})_t + (c(x)\bar{\rho}u)_x = 0 \\ (c(x)\bar{\rho}u)_t + \left(c(x)\bar{\rho}u^2 - c(x)\bar{\rho}^{-1}\right)_x = -c'(x)\bar{\rho}^{-1} \end{cases} \tag{1.27}$$

with pressure

$$p(\bar{\rho}) = -\bar{\rho}^{-1},$$

where $\bar{\rho}$ is the density of gas at any point (x, t) and ρ is the density on a cross-section. For a duct flow, $c(x)$ is the cross-sectional area which is uniformly positive and bounded. We can also generally consider (1.1) as a model for the isentropic Chaplygin gas in an inhomogeneous medium.

Similar as Theorem 1.3, we construct blowups for the Cauchy problem of (1.27) (or equivalently (1.1) by the relation (1.26)). For simplicity, we only consider a very special example of $c(x)$:

$$c(x) = \begin{cases} 3 - \varepsilon^\alpha - \frac{1}{2}\eta\varepsilon^\alpha, & x \in (-\infty, -1-\eta), \\ \psi_1(x), & x \in [-1-\eta, -1), \\ 3 + \varepsilon^\alpha x, & x \in [-1, 1], \\ \psi_2(x), & x \in (1, 1+\eta], \\ 3 + \varepsilon^\alpha + \frac{1}{2}\eta\varepsilon^\alpha, & x \in (1+\eta, \infty), \end{cases} \tag{1.28}$$

where α is any constant in $[0, 1)$, $\psi_1(x)$ is an increasing function on $x \in [-1-\eta, -1)$ connecting $3 - \varepsilon^\alpha - \frac{1}{2}\eta\varepsilon^\alpha$ and $3 - \varepsilon^\alpha$ and $\psi_2(x)$ is an increasing function on $x \in [1, 1+\eta]$ connecting $3 + \varepsilon^\alpha$ and $3 + \varepsilon^\alpha + \frac{1}{2}\eta\varepsilon^\alpha$. The positive constant $\eta < \varepsilon^3$. Furthermore $0 < \varepsilon < 1$ is a small given number which will be provided in the proof of the theorem. By standard mollifier theory, we can find ψ_1 and ψ_2 such that $c(x)$ satisfies (1.2).

Now we list our first result on the singularity formation for the Cauchy problem of (1.27).

Theorem 1.5 *For any $\alpha \in [0,1)$ and $c(x)$ given in* (1.28)*, there exist uniformly bounded C^1 initial data $\rho_0(x)$ ($\rho_0 \not\equiv 1$ and has uniformly positive lower bound) and $u_0(x)$ (which will be given in the proof) and a finite time $T = O(\varepsilon^{-\alpha}) > 0$, such that the Cauchy problem of* (1.1) *with initial data (ρ_0, u_0) has a unique classical C^1 solution $(\rho(x,t), u(x,t))$ on $(x,t) \in \mathbb{R} \times [0,T)$. Moreover, there exists a point (x^*, T) with $|x^*| = O(\varepsilon^{\frac{1-\alpha}{2}}) < O(1)$ at which the solution satisfies*

$$\lim_{t\to T^-} u(x^*,t) = \infty \quad \lim_{t\to T^-} \rho(x^*,t) = 0 \quad \lim_{t\to T^-} \rho u(x^*,t) = B \quad \text{and} \quad \lim_{t\to T^-} E(x^*,t) = \infty, \tag{1.29}$$

where B is a finite constant. The solutions $\rho(x,t)$ and $-u(x,t)$ have uniform upper bounds on $(x,t) \in \mathbb{R} \times [0,T)$.

Remark 1.6 *We have several remarks for Theorem 1.5.*

(1). In this example, the characteristic speeds of two families are uniformly away from each other, or in another word, system (1.27) *is uniformly strictly hyperbolic (see Lemma 5.5 and Lemma 5.6).*

(2). The result is also motivated by the pioneer work by Glassey, Hunter and Zheng [18] on the singularities of variational wave equation (1.17) *in one space dimension. Although equations* (1.27) *and* (1.17) *are not equivalent when the initial $\rho_0 \neq 1$, initial data in Theorem 1.5 and in the example in [18] for* (1.17) *have a lot of similarities.*

(3). This blowup can happen on a very slowly varying duct, which means $\|c'\|_{L^\infty}$ and the total variation of c can be both arbitrarily small in Theorem 1.5. When the variation of c is larger ($c'(x)$ is larger) around $x = 0$, we show faster blowup. In fact, when α is decreasing, $c'(x)$ is increasing around $x = 0$, then the blowup time is shorter. When $\alpha = 0$, the blowup time is at most $O(1)$.

(4). For any positive constant K, the total energy of the solution for any time until the blowup when $x \in [-K, K]$ is bounded by a constant depending on K, although at the time of blowup, the energy concentrates, i.e. the energy density is infinity, somewhere.

When one looks for the radially symmetric solutions: $\bar{\rho}(\mathbf{y},t) = \bar{\rho}(x,t)$, $\mathbf{u}(\mathbf{y},t) = \mathbf{y}u(x,t)$ with radius $x \geq 0$ for

$$\begin{cases} \partial_t \bar{\rho} + \nabla_{\mathbf{y}} \cdot (\bar{\rho}\mathbf{u}) = 0 \\ \partial_t(\bar{\rho}\mathbf{u}) + \nabla_{\mathbf{y}} \cdot (\bar{\rho}\mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathbf{y}} p(\bar{\rho}) = 0, \end{cases} \tag{1.30}$$

with $p(\mathbf{y},t) = \bar{\rho}^{-1}(\mathbf{y},t)$, $\mathbf{u}(\mathbf{y},t) = (u_1, u_2, u_3)$ and $(\mathbf{y},t) \in \mathbb{R}^{m+1} \times \mathbb{R}^+$ with $m = 1$ or 2, the resulting system was in form of (1.27) with $c(x) = x^m$ ($m = 1$, cylindrical symmetric solution; $m = 2$, spherically symmetric solution). See [15]. It will be shown in Section 1.3.1 that (1.30) has strictly convex entropy.

Our next result concerns radially symmetric solutions for (1.30), that is the equation (1.27) with $c(x) = x^m$, where x denotes the radius. Without of loss of generality, we only consider the solutions with initial data given on a special interval $x \in [1,3]$.

Theorem 1.7 *For $m = 1, 2$ and some given C^1 initial data $(\rho_0(x), u_0(x))$ depending only on radius $x \in [1,3]$ (which will be prescribed in the proof), there exists a time $T = O(1) > 0$ such that the Cauchy problem of* (1.27) *with $c(x) = x^m$ has a unique classical C^1 solution in Ω_{symm}, where Ω_{symm} is the domain of dependence of the initial interval $x \in [1,3]$ for any time t in*

$(0,T)$. *Moreover, there exists a point* (x^*,T) *with* $|x^*| = O(\varepsilon^{\frac{1}{2}}) < O(1)$ *at which the solution satisfies*

$$\lim_{t\to T^-} u(x^*,t) = \infty \quad \lim_{t\to T^-} \rho(x^*,t) = 0 \quad \lim_{t\to T^-} \rho u(x^*,t) = B \quad \text{and} \quad \lim_{t\to T^-} E(x^*,t) = \infty, \tag{1.31}$$

where B *is a finite constant. The solutions* $\rho(x,t)$ *and* $-u(x,t)$ *have uniform upper bounds on* $(x,t) \in \mathbb{R}\times[0,T)$.

Remark 1.8 *Theorem 1.7 provides an example with finite time vacuum formation and* L^∞ *blowup for the radially symmetric solutions with radius varying in a finite closed interval away from zero. This example satisfies all properties as the example in Theorem 1.5 with* $\alpha = 0$.

1.3 Convex entropy and vacuum

Smooth solutions of the system (1.30) satisfy energy conservation law

$$\mathcal{E}_t + \nabla_{\mathbf{y}} \cdot \mathbf{Q} = 0$$

with entropy

$$\mathcal{E} = \frac{1}{2}\bar{\rho}|\mathbf{u}|^2 + \frac{1}{2}\bar{\rho}^{-1}$$

and entropy flux

$$\mathbf{Q} = (\frac{1}{2}\bar{\rho}|\mathbf{u}|^2 - \frac{1}{2}\bar{\rho}^{-1})\mathbf{u}.$$

The entropy $\mathcal{E}$ is a strictly convex function on conservative variables $(\bar{\rho},\mathbf{m})$, where $\mathbf{m} = \bar{\rho}\mathbf{u}$ is the momentum, i.e. $\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\bar{\rho}} + \frac{1}{2}\bar{\rho}^{-1}$ satisfies that $D^2\mathcal{E}$ is a positively defined matrix.

Concerning one dimensional case, the smooth solutions of equation (1.1) satisfy energy conservation law

$$E_t + q_x = 0,$$

with entropy

$$E = \frac{1}{2}\rho u^2 + \frac{1}{2}c^2\rho^{-1}$$

and entropy flux

$$q = \frac{1}{2}u^3\rho - \frac{1}{2}c^2\rho^{-1}u.$$

By direct calculation, we obtain

$$E = \frac{1}{2}\begin{pmatrix} u & \rho \end{pmatrix}\begin{pmatrix} \rho & 0 \\ 0 & c^2\rho^{-3} \end{pmatrix}\begin{pmatrix} u \\ \rho \end{pmatrix}.$$

This implies that the entropy E is strictly convex.

The L^∞ blowup in this paper is totally different from the previous L^∞ blowup results found first by Jenssen in his groundbreaking work [21], and then by several other authors [3, 22, 38, 39] by considering the shock interactions. To see the difference, more intuitively, one could still essentially consider that the L^∞ blowup constructed in our examples on $\frac{1}{\rho}$ are coming from the blowup on the gradient variable x_X through transformation between different coordinates. A key point worth mentioning is that after transformation from Lagrangian coordinates to Eulerian coordinates, one gets a linearly degenerate system, which *rarely* happen, on which C^1 solution exists before the L^∞ blowup.

Furthermore, except the L^∞ blowup, the more generic singularity: gradient blowup has been studied for systems of conservation laws. The gradient blowup in systems of conservation laws has been widely accepted as the most generic type of singularity related to the shock formation in [26, 23, 6, 7, 8, 9]. It is much harder to find the L^∞ blowup for systems of conservation laws.

Finally, we give a remark on the L^∞ blowup on u and vacuum formation. The isentropic hyperbolic systems with $p = \bar{\rho}^\gamma$ when the adiabatic constant $\gamma > 0$ and $p = -\bar{\rho}^\gamma$ when $\gamma < 0$ are given by

$$\begin{cases} (c(x)\bar{\rho})_t + (c(x)\bar{\rho}u)_x = 0 \\ (c(x)\bar{\rho}u)_t + \big(c(x)\bar{\rho}u^2 + c(x)p\big)_x = c'(x)p. \end{cases} \tag{1.32}$$

When $0 \le \gamma < 1$, the entropy is not strictly convex so these cases are not the interesting cases for us. When $\gamma \ge 1$ or $-1 < \gamma < 0$ or $\gamma < -1$, the system (1.32) is genuinely nonlinear when $c(x)$ is a constant, hence we expect shock formation and tend to believe that the shock could prevent the L^∞ blowup. For example, when $1 < \gamma \le \frac{5}{3}$, which is corresponding to gas dynamics, the L^∞ existence has already been proven in [12] for the duct flow and exterior radially symmetric flow, hence L^∞ blowup on u can not happen (see also [27] for the gas dynamics with $1 < \gamma < \infty$).

Furthermore, the nonisentropic compressible Euler equations with polytropic ideal gas are

$$\begin{cases} \rho_t + (\rho\, u)_x = 0 \\ (\rho\, u)_t + (\rho\, u^2 + p)_x = 0 \\ (\frac{1}{2}\rho\, u^2 + \rho\, e)_t + (\frac{1}{2}\,\rho\, u^3 + u\, p)_x = 0\,, \end{cases} \tag{1.33}$$

with equation of state

$$e = c_v T = \frac{p\,\tau}{\gamma - 1} \quad \text{and} \quad p\,\tau = \mathcal{R}\,T,$$

so that pressure

$$p = K \exp(\mathcal{S}/c_v)\,\tau^{-\gamma}, \tag{1.34}$$

where ρ is density, $\tau = 1/\rho$, u is velocity, e is specific internal energy, $\mathcal{S}$ is the entropy, T is the temperature, $\mathcal{R}$, K, c_v are positive constants, and $\gamma > 1$ is the adiabatic gas constant. In [9], the first author, R. Young and Q. Zhang have found uniform time-independent L^∞ bounds for ρ and $|u|$.

Whether the solution for (1.32) or (1.33) with $\gamma > 1$ has a finite time vacuum or not is still a major open problem for gas dynamics. If one only considers the smooth solutions for (1.32) with $\gamma > 1$ and constant c, one may conjecture, by a strong evidence from [32], that there will be no vacuum in finite time if there is no vacuum initially or instantaneously.

The rest of paper is organized as follows. In Section 2, we set up the Riemann coordinates and several lemmas for smooth solutions. In Section 3, we prove the existence of C^1 solution when $\|u\|_{L^\infty} + \|\rho\|_{L^\infty} < +\infty$ and $|\rho|$ is away from zero. In Section 4, we prove Theorem 1.3 for inhomogeneous elastic flow. In Section 5, we prove the Theorem 1.5 for isentropic Chaplygin gas and the Theorem 1.7 for the radially symmetric case.

2 Riemann coordinates

For smooth solutions, the system (1.1) can be written as

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0 \\ u_t + uu_x + c^2(x)\rho^{-3}\rho_x &= c(x)c'(x)\rho^{-2}. \end{aligned} \tag{2.1}$$

Hence

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + A \begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ c(x)c'(x)\rho^{-2} \end{pmatrix} \tag{2.2}$$

where

$$A = \begin{pmatrix} u & \rho \\ c^2\rho^{-3} & u \end{pmatrix}. \tag{2.3}$$

Direct calculation shows that the eigenvalues of A are

$$S = u + c(x)\rho^{-1}, \quad R = u - c(x)\rho^{-1}, \tag{2.4}$$

and the corresponding right eigenvectors are

$$v_1 = (1,\ c(x)\rho^{-2})^T, \quad v_2 = (1,\ -c(x)\rho^{-2})^T. \tag{2.5}$$

According to Lax [25], the two characteristic families for system (2.1) when c is a constant are both *linearly degenerate*. For any $(\bar{x}, \bar{t})$ with $\bar{t} > 0$, the plus and minus characteristics $x_\pm(t)$ through $(\bar{x}, \bar{t})$ are defined by

$$\frac{dx_+(t, \bar{x}, \bar{t})}{dt} = S(x_+, t) \qquad \text{and} \qquad \frac{dx_-(t, \bar{x}, \bar{t})}{dt} = R(x_-, t)\,. \tag{2.6}$$

For simplicity, we use $x_+(t)$ or $t_+(x)$ and $x_-(t)$ or $t_-(x)$ to denote the plus and minus characteristics respectively. For smooth solutions of equation (2.1), we obtain the following equation of $c(x)/\rho$, which will be used several times in the rest of paper.

$$c(x)(\rho^{-1})_t + u(c(x)\rho^{-1})_x - c(x)\rho^{-1}u_x = c'(x)u\rho^{-1}. \tag{2.7}$$

Lemma 2.1 *For smooth solutions of* (1.1)*, S and R satisfy*

$$\begin{aligned} S_t + RS_x = c'(x)u\rho^{-1} &= \frac{c'(x)}{4c(x)}\left(S^2 - R^2\right), \\ R_t + SR_x = -c'(x)u\rho^{-1} &= \frac{c'(x)}{4c(x)}\left(R^2 - S^2\right). \end{aligned} \tag{2.8}$$

Proof. By equation (2.1) and (2.7), we have

$$\begin{aligned} & S_t + RS_x \\ =&(u + c(x)\rho^{-1})_t + (u - c(x)\rho^{-1})(u + c(x)\rho^{-1})_x \\ =&u_t + c(x)(\rho^{-1})_t + uu_x - c(x)\rho^{-1}u_x + u(c(x)\rho^{-1})_x - c(x)\rho^{-1}(c(x)\rho^{-1})_x \\ =&u_t + uu_x - c(x)\rho^{-1}(c(x)\rho^{-1})_x + c(x)(\rho^{-1})_t - c(x)\rho^{-1}u_x + u(c(x)\rho^{-1})_x \\ =&c'(x)u\rho^{-1}. \end{aligned} \tag{2.9}$$

And

$$\begin{aligned} & R_t + SR_x \\ =&(u - c(x)\rho^{-1})_t + (u + c(x)\rho^{-1})(u - c(x)\rho^{-1})_x \\ =&u_t - c(x)(\rho^{-1})_t + uu_x + c(x)\rho^{-1}u_x - u(c(x)\rho^{-1})_x - c(x)\rho^{-1}(c(x)\rho^{-1})_x \\ =&u_t + uu_x - c(x)\rho^{-1}(c(x)\rho^{-1})_x - \left(c(x)(\rho^{-1})_t - c(x)\rho^{-1}u_x + u(c(x)\rho^{-1})_x\right) \\ =& - c'(x)u\rho^{-1}. \end{aligned} \tag{2.10}$$

□

Remark 2.2 *When $c(x)$ is a constant function, R and S are exactly two Riemann invariants along plus and minus characteristics with characteristic speeds S and R respectively by Lemma 2.1,*

$$\begin{cases} S_t + RS_x &= 0, \\ R_t + SR_x &= 0. \end{cases} \tag{2.11}$$

By the Theorem 2.3 in [29], system (2.11) admits a global-in-time unique C^1 solution if the initial data S_0 and R_0 have bounded C^1 norm.

Lemma 2.3 (Energy conservation law) *For smooth solutions of* (1.1)*, the energy density $E = \frac{1}{4}\rho(S^2 + R^2)$ satisfies*

$$\big(\rho(S^2 + R^2)\big)_t + \big(\rho(S^2R + R^2S)\big)_x = 0. \tag{2.12}$$

Proof. By(2.4), we have

$$\rho S = \rho u + c(x) \quad \text{and} \quad \rho R = \rho u - c(x). \tag{2.13}$$

Thus, multiplying the first equation of (2.8) by $2\rho S$ and the second equation of (2.8) by $2\rho R$, adding them up, and using (2.13) and the conservation of mass in (1.1), we obtain

$$\begin{aligned} &(\rho S^2 + \rho R^2)_t + \big(\rho(S^2R + R^2S)\big)_x \\ =&(S^2 + R^2)\rho_t + S^2(\rho R)_x + R^2(\rho S)_x + 2c'u(S - R) \\ =&(S^2 + R^2)\rho_t + S^2(\rho u - c)_x + R^2(\rho u + c)_x + c'(S + R)(S - R) \\ =&S^2(\rho_t + (\rho u)_x) + R^2(\rho_t + (\rho u)_x) + c'(R^2 - S^2) + c'(S^2 - R^2) \\ =&0. \end{aligned} \tag{2.14}$$

□

Finally, we give a key estimate for the proof of our main theorems. For any $(x_0, t_0) \in \mathbb{R} \times (0, +\infty)$, let γ_+ and γ_- be plus and minus characteristics through (x_0, t_0) and intersect $x-$axis at x_1 and x_2 respectively (see Figure 2). Integrating (2.12) over a characteristic triangle

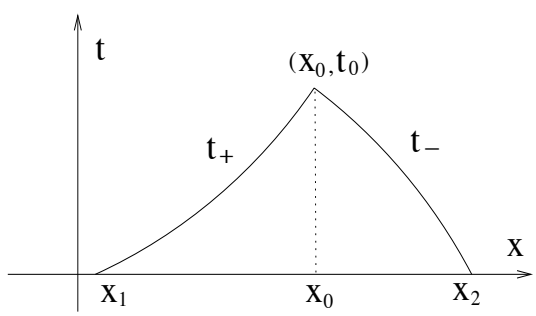

Figure 2: A characteristic triangle D.

D enclosed by γ_+, γ_- and $[x_1, x_2]$ as in Figure 2, we obtain an energy identity indicating the finite propagation of the waves.

Lemma 2.4 (Finite propagation) *For smooth solutions of* (1.1) *inside a characteristic triangle D in Figure 2,*

$$2\int_{x_1}^{x_0} cS(x, \gamma_+(x))\,dx - 2\int_{x_0}^{x_2} cR(x, \gamma_-(x))\,dx = \int_{x_1}^{x_2} \rho(S^2 + R^2)(x, 0)\,dx. \tag{2.15}$$

Proof. Integrating (2.12) over the region D and using the Green's theorem, we obtain

$$\begin{aligned} 0 &= \iint_D \left(\rho(S^2+R^2)\right)_t + \left(\rho(S^2R+R^2S)\right)_x \, dxdt \\ &= \int_{\partial D} \rho(S^2R+R^2S)\, dt - \rho(S^2+R^2)\, dx. \end{aligned} \tag{2.16}$$

On γ_+ which is the left boundary of D, by (2.4) and (2.6) we have

$$\begin{aligned} &\int_{\gamma_+} \rho(S^2R+R^2S)\, dt - \rho(S^2+R^2)\, dx \\ =&\int_{x_1}^{x_0} \rho(SR+R^2) - \rho(S^2+R^2)\, dx \\ =&\int_{x_1}^{x_0} \rho S(R-S)\, dx \\ =&-2\int_{x_1}^{x_0} cS(x,\gamma_+(x))\, dx. \end{aligned} \tag{2.17}$$

Similarly, on γ_- which is the right boundary of D, we have

$$\begin{aligned} &\int_{\gamma_-} \rho(S^2R+R^2S)\, dt - \rho(S^2+R^2)\, dx \\ =&\int_{x_0}^{x_2} \rho(S^2+RS) - \rho(S^2+R^2)\, dx \\ =&\int_{x_0}^{x_2} \rho R(S-R)\, dx \\ =&2\int_{x_0}^{x_2} cR(x,\gamma_-(x))\, dx. \end{aligned} \tag{2.18}$$

Putting (2.17) and (2.18) into (2.16), we obtain (2.15). □

3 Existence of C^1 solutions before L^∞ blowup

For smooth solutions of (1.1), let

$$v = \rho^{-1}S_x, \qquad w = \rho^{-1}R_x$$

be two gradient variables. Then we obtain the following lemma.

Lemma 3.1 *For smooth solutions of* (1.1)*, v and w satisfy*

$$\begin{aligned} v_t + Rv_x =& \frac{c'}{2c}\left[(2c\rho^{-1}+S)v - Rw\right] + \frac{c''c-c'^2}{4c^2}\rho^{-1}(S^2-R^2) \\ w_t + Sw_x =& \frac{c'}{2c}\left[(2c\rho^{-1}+R)w - Sv\right] + \frac{c''c-c'^2}{4c^2}\rho^{-1}(R^2-S^2)\,. \end{aligned} \tag{3.1}$$

Proof. By (2.8) and (2.4), we have

$$
\begin{aligned}
& v_t + Rv_x \\
=& S_x \left(\rho^{-1}\right)_t + S_x R \left(\rho^{-1}\right)_x + \rho^{-1} \left(S_x\right)_t + \rho^{-1} R \left(S_x\right)_x \\
=& S_x \left(\left(\rho^{-1}\right)_t + u \left(\rho^{-1}\right)_x - c\rho^{-1} \left(\rho^{-1}\right)_x\right) + \rho^{-1} \left(S_t + RS_x\right)_x - \rho^{-1} R_x S_x \\
=& S_x \left(u_x \rho^{-1} - c\rho^{-1} \left(\rho^{-1}\right)_x\right) + \rho^{-1} \left(S_t + RS_x\right)_x - \rho^{-1} R_x S_x \\
=& S_x \rho^{-1} \left(u_x - \left(c\rho^{-1}\right)_x + c'\rho^{-1}\right) + \rho^{-1} \left(S_t + RS_x\right)_x - \rho^{-1} R_x S_x \\
=& \rho^{-1} R_x S_x + c'\rho^{-2} S_x + \rho^{-1} \left(S_t + RS_x\right)_x - \rho^{-1} R_x S_x \\
=& c'\rho^{-2} S_x + \rho^{-1} \left(S_t + RS_x\right)_x \\
=& c'\rho^{-2} S_x + \rho^{-1} \left(\frac{c'(x)}{4c(x)} \left(S^2 - R^2\right)\right)_x,
\end{aligned}
\tag{3.2}
$$

which implies the first equation of (3.1). Similarly

$$
\begin{aligned}
& \left(\rho^{-1} R_x\right)_t + S \left(\rho^{-1} R_x\right)_x \\
=& R_x \left(\rho^{-1}\right)_t + R_x S \left(\rho^{-1}\right)_x + \rho^{-1} \left(R_x\right)_t + \rho^{-1} S \left(R_x\right)_x \\
=& R_x \left(\left(\rho^{-1}\right)_t + u \left(\rho^{-1}\right)_x - c\rho^{-1} \left(\rho^{-1}\right)_x\right) + \rho^{-1} \left(R_t + SR_x\right)_x - \rho^{-1} R_x S_x \\
=& R_x \left(u_x \rho^{-1} - c\rho^{-1} \left(\rho^{-1}\right)_x\right) + \rho^{-1} \left(R_t + SR_x\right)_x - \rho^{-1} R_x S_x \\
=& R_x \rho^{-1} \left(u_x - \left(c\rho^{-1}\right)_x + c'\rho^{-1}\right) + \rho^{-1} \left(R_t + SR_x\right)_x - \rho^{-1} R_x S_x \\
=& \rho^{-1} R_x S_x + c'\rho^{-2} R_x + \rho^{-1} \left(R_t + SR_x\right)_x - \rho^{-1} R_x S_x \\
=& c'\rho^{-2} R_x + \rho^{-1} \left(R_t + SR_x\right)_x \\
=& c'\rho^{-2} R_x + \rho^{-1} \left(\frac{c'(x)}{4c(x)} \left(R^2 - S^2\right)\right)_x,
\end{aligned}
\tag{3.3}
$$

which implies the second equation of (3.1). □

Remark 3.2 *System* (3.1) *indicates that the rates of change of v along the minus characteristic and w along the plus characteristic are both linear.*

Before we prove the C^1 existence result when the solution has L^∞ bounds, we first state an a priori condition.

(A) Suppose the initial data u_0 and ρ_0 are C^1 functions and uniformly bounded (ρ_0 is also uniformly bounded away from zero). Then for any C^1 solution $(\rho(x,t),\ u(x,t))$ of Cauchy problem of equation (1.1) with $(x,t) \in \mathbb{R} \times [0, T_*]$, for $0 < T_* < T$, there exists a positive constant L_*, only depending on T_* and initial data, such that

$$
\|\rho(x,t)\|_{L^\infty} + \|\rho^{-1}(x,t)\|_{L^\infty} + \|u(x,t)\|_{L^\infty} = L_* < \infty. \tag{3.4}
$$

Under the condition (A), by Lemma 3.1, observation in Remark 3.2 and C^∞ functions are dense in C^1, it is easy to obtain the following a priori estimates.

Lemma 3.3 *Assume that the condition (A) is satisfied, $u_0, \rho_0 \in C^1$ and uniformly bounded (ρ_0 is also uniformly bounded away from zero). Then any C^1 solutions $\rho(x,t)$, $u(x,t)$ for system* (1.1) *with $0 \le t \le T_*$ satisfy*

$$
\sup_{(x,t)\in\mathbb{R}\times[0,T^*]} \{|S_t|, |S_x|, |R_t|, |R_x|\} = M_*, \tag{3.5}
$$

for some positive constant M_* only depending on initial values, L_* and T_*.

Remark 3.4 *Lemma 3.3 implies that the C^1 blowups never occur before L^∞ blowup for* (1.1).

By the a priori estimate in Lemma 3.3, one can prove the existence of C^1 solution on $[0,T)$ under the condition (A).

Theorem 3.5 *If the condition (A) is satisfied for any $0 < T_* < T$, the initial value problem of* (1.1) *with uniformly bounded initial data $u_0, \rho_0 \in C^1$ (ρ_0 is also uniformly bounded away from zero) has a unique C^1 solution $(\rho(x,t),\ u(x,t))$ for $(x,t) \in \mathbb{R} \times [0,T)$.*

First we observe that (A) implies the uniformly strict hyperbolicity of (1.1). The local existence of the C^1 solution now can be obtained by standard argument in [30]. Under the condition (A), the local solution can be extended to $(x,t) \in \mathbb{R} \times [0,T)$ by Theorem 2.4 in [29] and Remark 2.20 in [29]. To make this paper self-contained, we sketch the proof here.

Proof. We first fix our consideration on $(x,t) \in \mathbb{R} \times [0,T_*]$ for some T_*.

By the local-in-time existence results for the quasi-linear first order hyperbolic systems in [30], for any strong determinate domain $\tilde{\Omega}_k$ corresponding to initial interval $x \in [-k,k]$, there exists some time $T_0 = T_0(k, L_*, M_*)$ such that (3.1) exists a unique C^1 solutions on $\tilde{\Omega}_k$ with $t \in [0,T_0]$. Here a domain

$$\tilde{\Omega}_{[a,b]} \equiv \tilde{\Omega}_{[a,b]}(\delta_0) = \big\{(x,t)\big|0 \le t \le \delta_0,\ x_1(t) \le x \le x_2(t)\big\}$$

is called a strong determinate domain of initial interval $[a,b]$ if

i. $x_1(t)$ and $x_2(t)$ are C^1 functions for $0 \le t \le \delta_0$.

ii. $x_1(0) = a$ and $x_2(0) = b$.

iii. For any C^1 solution in $\tilde{\Omega}(\delta_0)$, $x_1'(t) \ge M_S$ and $x_2'(t) \le M_R$,

where M_S is the upper bound of plus characteristic S and M_R is the lower bound of minus characteristic R on $\tilde{\Omega}(\delta_0)$. In our problem, M_S and M_R are only depending on L_*. Since T_0 is a constant, when k is large enough, one can prove the existence of C^1 solution on $\tilde{\Omega}_k$ with $t \in [0,T_*]$, by using the local existence proof finite many times.

Next, for any point $(x,t) \in \mathbb{R} \times [0,T_*]$, we could find $\tilde{\Omega}_k$ including this point with sufficiently large k, because M_S and M_R are only depending on L_*. Hence, we already proved the global existence on $(x,t) \in \mathbb{R} \times [0,T_*]$.

Furthermore, since T_* is any time before T, so we already proved the C^1 existence on $(x,t) \in \mathbb{R} \times [0,T)$, where the uniqueness of the local existence protects that we have one unique solution. □

4 Vacuum for the inhomogeneous elastic flow: proof of Theorem 1.3

We define an increasing C^2 smooth function

$$c(x) = \begin{cases} d_1, & x \in (-\infty, -\varepsilon^{-8} - \varepsilon^5), \\ \Psi_1(x), & x \in [-\varepsilon^{-8} - \varepsilon^5, -\varepsilon^{-8}), \\ \frac{\varepsilon}{1-\varepsilon x}, & x \in [-\varepsilon^{-8}, 1], \\ \Psi_2(x), & x \in (1, 1+\varepsilon^5], \\ d_2, & x \in (1+\varepsilon^5, \infty), \end{cases} \tag{4.1}$$

where d_1, d_2 are two positive constants and Ψ_1, Ψ_2 are increasing C^2 smooth functions such that

$$\frac{\varepsilon}{1-\varepsilon} \leq d_2 \leq \frac{\varepsilon}{1-\varepsilon} + \varepsilon^5, \tag{4.2}$$

and

$$\frac{\varepsilon}{1+\varepsilon(\varepsilon^{-8}+\varepsilon^5)}(1-\varepsilon^5) \leq d_1 \leq \frac{\varepsilon}{1+\varepsilon(\varepsilon^{-8}+\varepsilon^5)}. \tag{4.3}$$

The function $f(x) = \frac{\varepsilon}{1-\varepsilon x}$ on $x \in [-\varepsilon^{-8}, 1]$ satisfies

$$\frac{f'(x)}{f^2(x)} = 1,$$

hence we can find d_1, d_2, $\Psi_1(x)$ and $\Psi_2(x)$ such that

$$0 \leq \frac{c'(x)}{c^2(x)} \leq 1 + \varepsilon^8 \tag{4.4}$$

for any x. It is easy to see that there exists a function $c(x)$ such that (1.2), (4.1)$\sim$(4.4) are all satisfied. The positive constant $\varepsilon \ll 1$ will be given in the proof of the theorem.

Remark 4.1 *The construction of the initial data u_0 and $c(x)$ is motivated by the seminal work [18]. However, our restrictions are all on the function $c(x)$ since $\rho_0 \equiv 1$, $u_0 = c(x)$ and S is uniformly larger than R. It makes our construction much more involved than the one in [18].*

Throughout this paper, we use K_i and M_i to denote positive constants independent of ε. To prove Theorem 1.3, we show that there exists some time

$$T = M_0\varepsilon^{-2} \tag{4.5}$$

such that the a priori condition (A) is satisfied for any $t \in [0, T^*]$ with $T^* < T$ which indicates that C^1 solution exists for any $t \in [0, T^*]$. We also show that at $T = M_0\varepsilon^{-2}$, S blows up at somewhere while R is uniformly bounded.

Lemma 4.2 *For any C^1 solutions to* (1.1) *with $c(x)$ given in* (4.1) *and prescribed initial data $\rho_0 \equiv 1$, $u_0 = c(x)$, we have*

$$R \leq 0.$$

Proof. By Lemma 2.1, for any $x \in (-\infty, \infty)$

$$R_t + SR_x \leq M_1(\varepsilon, d_1)R^2, \tag{4.6}$$

where $M_1(\varepsilon, d_1)$ is a positive constant depending on ε and d_1, and left hand side is the derivative along a minus characteristic. By (4.6) and ODE comparison theorem,

$$R \leq 0$$

for any C^1 smooth solution. □

In Lemmas 4.3$\sim$4.5, we restrict our consideration on the C^1 solutions with $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0, T^*]$ with $T^* < T$ and T defined in (4.5). For these solutions, the plus and minus characteristics go in forward and backward directions respectively. We will show that the a priori condition on S is satisfied for any C^1 solutions with prescribed initial data in Lemma 4.6.

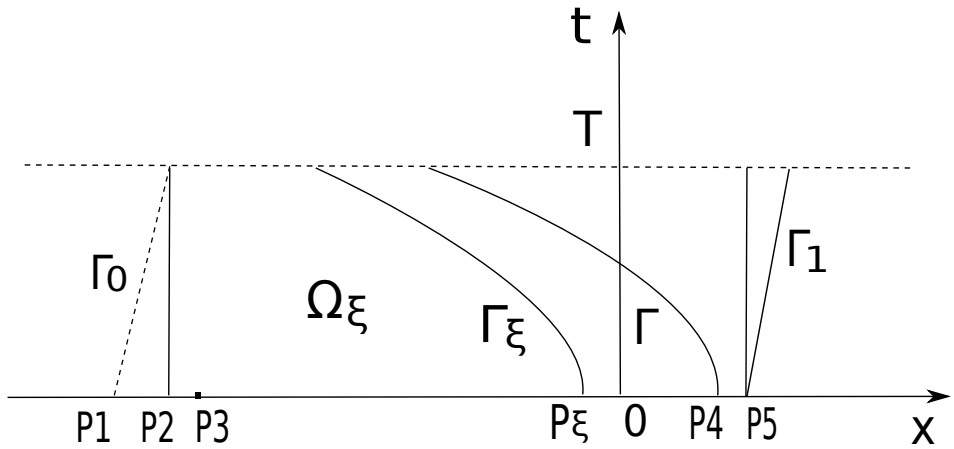

Figure 3: Proof of Theorem 1.3.

We use Figure 4 for the proof. Note in Figure 4, P_1 is the initial point of the forward characteristic Γ_0 intersecting with the vertical line passing P_2 at time T, $P_2 = (-\varepsilon^{-8} - \varepsilon^5, 0)$, $P_3 = (-\varepsilon^{-8}, 0)$, $P_4 = (1, 0)$ and $P_5 = (1 + \varepsilon^5, 0)$. The backward characteristic Γ_ξ starts from $P_\xi = (\xi, 0)$ and ends at $t = T$ with $x_{P_2} < \xi \le 1$. The backward characteristic Γ starts from P_4. The forward characteristic Γ_1 starts from P_5. Furthermore, we use Ω_ξ to denote the domain of dependence in the left of Γ_ξ when $t < T$.

Lemma 4.3 *Consider any C^1 solutions for* (1.1) *with $c(x)$ given in* (4.1) *and prescribed initial data $\rho_0 \equiv 1$, $u_0 = c(x)$. Assume $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0,T)$. Then the initial energy $E_\xi = \frac{1}{4}\int_{P_1P_\xi} \rho(S^2 + R^2)(x,0)\,dx$ in the interval P_1P_ξ with $x_{P_1} < \xi \le 1$ satisfies*

$$0 < (1+\varepsilon^8)E_\xi \le \frac{\varepsilon}{1-\varepsilon\xi}\,. \tag{4.7}$$

In the region Ω_ξ,

$$0 \ge R(x,t) \ge -\frac{\varepsilon}{2(1-\varepsilon\xi)} =: -a_\xi\,. \tag{4.8}$$

Also we know that the characteristic Γ will not interact with Γ_0 before T. Furthermore,

$$S_0(\xi) = \frac{2\varepsilon}{1-\varepsilon\xi}(1-\varepsilon^5) > a_\xi\,, \tag{4.9}$$

when $x_{P_2} < \xi \le 1$.

Proof. By Figure 4 and the initial data, the energy in the initial interval P_3P_ξ is

$$E_{P_3P_\xi} = \int_{-\varepsilon^{-8}}^{\xi} \Big(\frac{\varepsilon}{1-\varepsilon x}\Big)^2\,dx = \frac{\varepsilon}{1-\varepsilon\xi} - \frac{\varepsilon}{1-\varepsilon^{-7}}.$$

Since the length of P_2P_3 is ε^5 and the forward characteristic Γ_0 has speed less than $O(\varepsilon^7)$ which are both very small, so it is easy to see that energy $E_{P_1P_3}$ is omittable hence (4.7) is correct when ε is small enough, where we also use that $T = O(1)\varepsilon^{-2}$.

For any (x,t) in the left the forward characteristic Γ_0, $R(x,t) = 0$ since $c(x)$ is a constant when $x < -\varepsilon^{-8} - \varepsilon^5$ and (2.11), hence (4.8) is correct in this region.

By (2.8), we have

$$R_t + SR_x \geq -\frac{c'}{4c^2}S^2. \tag{4.10}$$

Integrating it along any forward characteristic in Ω_ξ starting from P_1P_4, by (2.15) and $\frac{c'}{c^2} \leq 1+\varepsilon^8$, for any (x,t) on this forward characteristic in Ω_ξ we have

$$R(x,t) \geq -\int_{\gamma_+\cap\Omega} \frac{c'}{8c^2}\, 2cS^2(t, x_+(t))dt \geq -\frac{1}{2}(1+\varepsilon^8)E_\xi \geq -\frac{\varepsilon}{2(1-\varepsilon\xi)} = -a_\xi. \tag{4.11}$$

Hence, (4.8) is always correct in Ω_ξ.

For any (x,t) on Γ with wave speed R, we have

$$\frac{1-x}{t} \leq \frac{\varepsilon}{2(1-\varepsilon)} = O(\varepsilon).$$

Since, $|P_1P_4| > \varepsilon^{-8}$, easy calculation shows that the characteristic Γ will not interact with Γ_0 before $T = O(\varepsilon^{-2})$.

Furthermore, it is easy to check that (4.9) is correct. Hence we proved this lemma. □

Lemma 4.4 *Consider C^1 solutions to* (1.1) *with $c(x)$ given in* (4.1) *and prescribed initial data $\rho_0 \equiv 1$, $u_0 = c(x)$. Assume $S(x,t) > 0$ for any $(x,t) \in \mathbb{R}\times[0,T)$. There exist constants κ_1 and κ_2 depending on ε such that*

$$-\kappa_1 < R(x,t) \leq 0, \tag{4.12}$$

and

$$0 < \kappa_2 < S(x,t), \tag{4.13}$$

for any $(x,t) \in \mathbb{R}\times[0,T)$.

Proof. We first estimate R. We have already proved (4.12) in $\Omega_{\{1\}}$, i.e. Ω_ξ with $\xi = 1$, in the left of Γ in Lemma 4.3. For (x,t) in the right of forward characteristic Γ_1, $R = 0$ since $c(x)$ is a constant when $x \geq x_{P_5} = 1+\varepsilon^5$ and (2.11). For any smooth solutions, by (2.15),

$$R_t + SR_x \geq -\frac{c'}{4c}S^2.$$

Using the same argument as in (4.11) and the initial energy in the domain of dependence including the region between Γ and Γ_1 is finite, we have

$$R > -\kappa_1.$$

for some positive constant κ_1.

Now we proceed to prove (4.13). To the right (or left) of the vertical line passing P_5 (or P_2) which are both backward characteristics, S equals to its initial constant data, hence (4.13) is correct. To consider S in the region $\Omega_{\{1\}}$, i.e. to the left of the characteristic Γ starting from P_4, and to the right of the vertical line passing P_2, by Lemmas 2.1 and Lemma 4.3, on any backward characteristic Γ_ξ when $t < T$ starting from the point $P_\xi(\xi, 0)$,

$$S_t + RS_x \geq \frac{c'}{4c^2}(S^2 - a_\xi^2).$$

with initial data $S_0(\xi) = \frac{2\varepsilon}{1-\varepsilon\xi} > a_\xi$. Hence $S \geq S_0(\xi)$ on Γ_ξ when $t < T$. So (4.13) is correct in this region for some κ_2.

On the backward characteristic starting from $[P_4, P_5]$, we also have that $S(x,t)$ has positive lower bound as the previous paragraph, since $S_0(\xi) \approx \frac{2\varepsilon}{1-\varepsilon} > a_{\{\xi=1\}} \approx a_\xi$ with differences at most in $O(\varepsilon^5)$ by (4.2) when $\xi \in (1, 1+\varepsilon^5)$. This completes the proof of (4.13), hence the proof of the lemma. □

Now we proceed to find the blowup of S.

Lemma 4.5 *Consider C^1 solutions to (1.1) with $c(x)$ given in (4.1) and prescribed initial data $\rho_0 \equiv 1$, $u_0 = c(x)$. Assume $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0,T)$. There exist positive constants M_0 such that*

$$0 < \kappa_2 < S(x,t) < +\infty, \tag{4.14}$$

for any $(x,t) \in \mathbb{R} \times [0,T)$ with $T = M_0\varepsilon^{-2}$ and

$$\lim_{(x,t)\to(x^*,T)} S(x,t) = +\infty$$

for some x^* such that $0 < 1 - x^* \le O(1)\varepsilon^{-1}$.

Proof. By Lemma 2.1, we have

$$S_t + RS_x \le \frac{d_2}{2} S^2. \tag{4.15}$$

where $d_2 = O(\varepsilon)$ and $0 < S_0 \le O(\varepsilon)$. So S stays finite until before

$$\bar{t} = M_2\varepsilon^{-2},$$

for some constant $M_2 > 0$.

Then we show the blowup happens at a time in $O(\varepsilon^{-2})$. For simplicity, we only consider the backward characteristic Γ starting from the $P_4(1,0)$ on (x,t)-plane. For any (x,t) on Γ, by the estimate of R in Lemma 4.4, we have

$$\frac{1-x}{t} \le a, \tag{4.16}$$

where we use

$$a = \frac{\varepsilon}{2(1-\varepsilon)}, \quad \text{to denote} \quad a_\xi = \frac{\varepsilon}{2(1-\varepsilon\xi)} \quad \text{at} \quad \xi = 1.$$

Before Γ interacts with Γ_0 which will happen not earlier than $O(\varepsilon^{-9})$ by (4.16), by Lemma 2.1, Lemma 4.4, (4.16) and definition of $c(x)$,

$$S_t + RS_x \ge \frac{c}{4}(S^2 - a^2) \ge \frac{\varepsilon}{4(1 + a\varepsilon t - \varepsilon)}(S^2 - a^2). \tag{4.17}$$

Studying the ODE

$$\frac{dg}{dt^-} = \frac{\varepsilon}{4(1 + a\varepsilon t - \varepsilon)}(g^2 - a^2) \tag{4.18}$$

with initial data

$$g(0) = S_0(1) = \frac{2\varepsilon}{1-\varepsilon} > a, \tag{4.19}$$

one has that g blows up at t^*, which satisfies

$$2\ln\left(\frac{S_0(1)+a}{S_0(1)-a}\right) + \ln(1-\varepsilon) = \ln(1 + a\varepsilon t^* - \varepsilon).$$

Therefore

$$t^* = \frac{16}{9a\varepsilon}(1-\varepsilon) = \frac{32(1-\varepsilon)^2}{9\varepsilon^2} = M_3\varepsilon^{-2}, \tag{4.20}$$

for constant $M_3 = \frac{32(1-\varepsilon)^2}{9}$. By comparison theorem of ODE, $S(x,t)$ blows up not later than $t^* = M_3\varepsilon^{-2}$.

Therefore, there exists $M_0 \in [M_2, M_3]$ such that for any $(x,t) \in \mathbb{R} \times [0,T)$ with $T = M_0\varepsilon^{-2}$

$$0 < S(x,t) < +\infty, \qquad \lim_{(x,t)\to(x^*,T)} S(x,t) = +\infty$$

for some x^*. By (4.16) and $T = M_0\varepsilon^{-2}$, we have $0 < 1 - x^* \le O(1)\varepsilon^{-1}$. Hence we complete the proof of the lemma by (4.13).

□

Next we show that the assumption that $S > 0$ is true for all C^1 solutions in our initial value problems, which implies that S is uniformly bounded away from zero by Lemma 4.4.

Lemma 4.6 *For any C^1 solutions to* (1.1) *with $c(x)$ given in* (4.1) *and prescribed initial data $\rho_0 \equiv 1$, $u_0 = c(x)$, one has $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0,T)$. Hence Lemmas 5.2∼5.6 are correct without the assumption that $S(x,t) > 0$ in the beginning.*

Proof. Note $R = 0$ and S are positive constants in the left of Γ_0 and in the right Γ_1 respectively, since $c(x)$ has constant value on each of these two regions. Denote the finite region between these two regions as Ω^* with $t < T$.

We prove the lemma by contradiction. Assume that $S = 0$ somewhere. Then $S = 0$ must first happen in Ω^*. We could find the minimum time such that $S = 0$. Assume that $S(\hat{x}, \hat{T}) = 0$ for some point $(\hat{x}, \hat{T})$ in Ω^* and $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0, \hat{T})$. Then running the proofs in Lemmas 4.2∼4.4, we can still get (4.14) for $(x,t) \in \mathbb{R} \times [0, \hat{T}]$ which contradicts to $S(\hat{X}, \hat{T}) = 0$. Hence, $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0,T)$. □

Finally we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Combining the a priori estimates in Lemma 4.4 and Lemma 4.5, using (2.4), we know the a priori condition (A) is true for any $T^* < T = M_0\varepsilon^{-2}$, where we use the fact: for any C^1 solution, S is bounded above in the closed set Ω^* defined in the previous lemma with $t \in [0, T^*]$ since S is not infinity when $t \in [0, T^*]$, and S has constant value in the left of Γ_0 or in the right of Γ_1.

As a conclusion, by Theorem 3.5, the initial value problem of (1.1) with the prescribed initial data exists a unique C^1 solution $(\rho(x,t), u(x,t))$ when $(x,t) \in \mathbb{R} \times [0,T)$. Furthermore, by Lemma 4.5, S is uniformly positive and

$$\lim_{t\to T^-} S(x^*, t) = \infty$$

for some $|x^*| = O(\varepsilon^{-1})$, while R is non-negative and uniformly bounded by Lemma 4.4. So by (2.4),

$$\lim_{t\to T^-} u(x^*,t) = \infty \quad \lim_{t\to T^-} \rho(x^*,t) = 0 \quad \lim_{t\to T^-} \rho u(x^*,t) = B \quad \text{and} \quad \lim_{t\to T^-} E(x^*,t) = \infty, \tag{4.21}$$

where B is a finite constant. Hence we complete the proof of Theorem 1.3.

□

5 Singularity formations in duct flow for Chaplygin gas: proof of Theorem 1.5 and 1.7

In this section, without any ambiguity, we still use several notations used in the previous section, such as ε, Ω, M_i etcs.

To prove Theorem 1.5, we need first set up the initial data ρ_0 and u_0 corresponding to $c(x)$ in (1.28)

$$\rho_0(x) = \begin{cases} \varepsilon^{-\alpha-1}, & x \in (-\infty, -2\varepsilon), \\ \phi_1(x), & x \in [-2\varepsilon, -\varepsilon), \\ 1, & x \in [-\varepsilon, \varepsilon], \\ \phi_2(x), & x \in (\varepsilon, 2\varepsilon], \\ \varepsilon^{-\alpha-1}, & x \in (2\varepsilon, \infty), \end{cases} \tag{5.1}$$

and

$$u_0(x) = \frac{c(x)}{\rho_0(x)} - \varepsilon^{\alpha+1}, \tag{5.2}$$

where the function ϕ_1 and ϕ_2 are C^1 increasing and decreasing functions on $[-2\varepsilon, -\varepsilon)$ and $(\varepsilon, 2\varepsilon]$ respectively. We collect some useful information on initial data here. First ρ_0 and $c(x)$ are C^2 functions bounded away from zero and infinity. $|c'(x)|$ is bounded above. u_0 is a C^1 function with finite upper and lower bounds. For any x,

$$2 \le c(x) \le 4, \tag{5.3}$$

$$c'(x) = \begin{cases} 0, & x \in (-\infty, -1-\eta), \\ \psi_1'(x) \in [0, \varepsilon^\alpha], & x \in [-1-\eta, -1), \\ \varepsilon^\alpha, & x \in [-1, 1], \\ \psi_2'(x) \in [0, \varepsilon^\alpha], & x \in (1, 1+\eta], \\ 0, & x \in (1+\eta, \infty), \end{cases} \tag{5.4}$$

$$\frac{1}{\rho_0(x)} = \begin{cases} \varepsilon^{\alpha+1}, & x \in (-\infty, -2\varepsilon), \\ \frac{1}{\phi_1(x)} \in [\varepsilon^{\alpha+1}, 1], & x \in [-2\varepsilon, -\varepsilon), \\ 1, & x \in [-\varepsilon, \varepsilon], \\ \frac{1}{\phi_2(x)} \in [\varepsilon^{\alpha+1}, 1], & x \in (\varepsilon, 2\varepsilon], \\ \varepsilon^{\alpha+1}, & x \in (2\varepsilon, \infty), \end{cases} \tag{5.5}$$

and

$$R_0(x) = -\varepsilon^{\alpha+1} \quad \text{and} \quad \varepsilon^{\alpha+1} < S_0(x) < +\infty. \tag{5.6}$$

The constants $\alpha \in [0, 1)$ and $0 < \eta < \varepsilon^3$. Furthermore $0 < \varepsilon \ll 1$ is a small given number which will be provided in the proof of the theorem.

Remark 5.1 *(i) The construction of the initial data ρ_0, u_0 and $c(x)$ is also motivated by the seminal work [18]. The initial data given by equations (1.4) and (1.5) in Theorem 1 of [18] were constructed for their unknown state ϕ (which is equivalent to $x(X,t)$ in our equation* (1.1) *if $\rho_0 = 1$).*
(ii) When $\rho_0 \not\equiv 1$, (1.1) *is not equivalent to* (1.17) *any more. This requires us to do extra constructions on c (as a function of $x \in (-\infty, \infty)$). While in [18], the wave speed c was a*

function of their unknown state ϕ (see (1.1) in [18]). They assumed c to be a uniformly positive and bounded smooth function.
(iii)The constructions on R_0, S_0 in (5.6) play an essential role in our proof.

Assuming that $\rho(x,t)$, $u(x,t)$ are C^1 solutions to Cauchy problem of (1.1) with given c, ρ_0 and u_0, we first do some analysis on the domains of dependence of different pieces of the initial data (see Figure 4), where $A_1 = -1-\eta$, $B_1 = -1$, $A_2 = 1$, $B_2 = 1+\eta$ and when $t \in [0, T^*]$ for any $T^* < T$ with

$$T = M_1 \varepsilon^{-\alpha}, \tag{5.7}$$

where M_1 is a fixed constant that will be provided later. In fact, T is the time of blowup.

We use Ω, Ω_L, Ω_R to denote three domains of dependence respect to $[-1,1]$, $(-\infty, -1-\eta)$, $(1+\eta, +\infty)$ respectively (boundaries are black solid lines). $\Pi 1$ and $\Pi 2$ are two regions in between Ω_L, Ω and Ω_R. The domains of dependence $\Omega_{\Pi 1}$ and $\Omega_{\Pi 2}$ including $\Pi 1$ and $\Pi 2$ respectively are regions with red dash lines as boundaries (with initial bases $[a_1, b_1]$ and $[a_2, b_2]$ respectively).

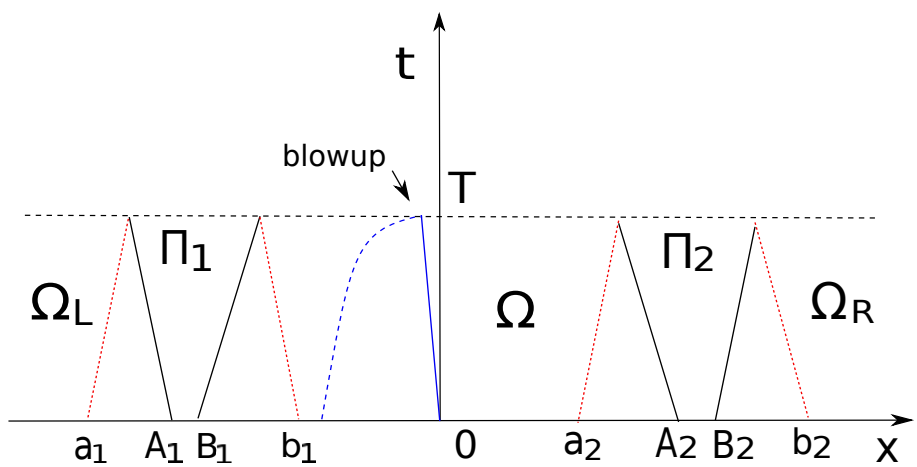

Figure 4: Proof of Theorem 1.5

We first have a lemma for C^1 solutions.

Lemma 5.2 *For any C^1 solutions with prescribed initial data,*

$$R < 0.$$

Proof. By Lemma 2.1, for any $x \in (-\infty, \infty)$

$$R_t + SR_x \leq \frac{1}{4} R^2, \tag{5.8}$$

where left hand side is derivative along a minus characteristic. By (5.6) and ODE comparison theorem,

$$R < 0$$

for any C^1 smooth solution. □

In Lemmas 5.3∼5.6, we restrict our consideration on the C^1 solutions with $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0, T)$ where T is defined in (5.7). For these solutions, the plus and minus characteristic go in forward and backward directions respectively. So we can always use Figure

2 to study the propagation of the solution along characteristics. We will show that this a priori condition on S is satisfied for any C^1 solutions in our initial value problems in Lemma 5.7.

Lemma 5.3 *Consider C^1 solutions with prescribed initial data and $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0,T)$. Let T_0 be the intersection time of forward characteristic from $(-1,0)$ and backward characteristic from $(1,0)$. Then there exists a constant $K_1 > 0$ such that*

$$T_0 \geq K_1 \varepsilon^{-1} > T. \tag{5.9}$$

And we also have the estimates

$$|b_1 - B_1| + |A_2 - a_2| \leq K_2 \varepsilon^{\frac{1-\alpha}{2}}, \tag{5.10}$$

where b_1, B_1, A_2 and a_2 are on Figure 4.

Proof. It can be calculate by (2.4), (1.28), (5.2) and (5.5) that the initial energy in Ω satisfies

$$\int_{-1}^{1} \rho_0 (S^2 + R^2)(x,0)\, dx = 2\int_{-1}^{1} \rho_0^{-1}(\rho_0^2 u_0^2 + c^2)\, dx = 2\int_{-1}^{1} \rho_0^{-1}\big(\big(c - \rho_0 \varepsilon^{\alpha+1}\big)^2 + c^2\big)\, dx \leq K_3 \varepsilon. \tag{5.11}$$

For any $x_1, x_2 \in [-1,1]$, let (x_0, t_0) be the intersection of forward and backward characteristics γ_+, γ_- starting $(x_1, 0)$ and $(x_2, 0)$ respectively. By Lemma 2.4 of finite propagation and (5.11), we know

$$\begin{aligned} &2\int_0^{t_0} cS^2(x_+(t),t)\, dt + 2\int_0^{t_0} cR^2(x_-(t),t)\, dt \\ =&2\int_{x_1}^{x_0} cS(x,t_+(x))\, dx - 2\int_{x_0}^{x_2} cR(x,t_-(x))\, dx \leq K_3\varepsilon. \end{aligned} \tag{5.12}$$

where K_i and M_i always mean positive constant in this paper.

Therefore

$$\begin{aligned} &x_2 - x_1 = x_0 - x_1 + x_2 - x_0 \\ =&\int_0^{t_0} S(x_+(t),t)\, dt - \int_0^{t_0} R(x_-(t),t)\, dt \\ \leq&\left(\int_0^{t_0} \frac{1}{c} dt\right)^{\frac{1}{2}} \left(2\int_0^{t_0} cS^2(x,x_+(t))\, dt\right)^{\frac{1}{2}} + \left(\int_0^{t_0} \frac{1}{c} dt\right)^{\frac{1}{2}} \left(2\int_0^{t_0} cR^2(x,x_-(t))\, dt\right)^{\frac{1}{2}} \\ \leq&K_4 t_0^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}, \end{aligned} \tag{5.13}$$

It is easy to see that (5.9) by (5.13).

Repeat the proof of (5.13), we can prove

$$|b_1 - B_1| + |A_2 - a_2| \leq 2K_5 T^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \leq K_2 \varepsilon^{\frac{1-\alpha}{2}}, \tag{5.14}$$

where in fact set $(x_1, t_1) = B_1$ or a_2 in Figure 2 then when $x_0 - x_1 = O(\varepsilon^{\frac{1-\alpha}{2}})$,

$$\begin{aligned}
x_0 - x_1 &= \int_{t_1}^{t_0} S(x_+(t), t)\, dt \\
&\leq \frac{1}{\sqrt{2}} \left(\int_{t_1}^{t_0} \frac{1}{c} dt \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_0} 2cS^2(x, x_+(t))\, dt \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}} \left(\int_{t_1}^{t_0} \frac{1}{c} dt \right)^{\frac{1}{2}} \left(2\int_{x_1}^{x_0} cS(x, t_+(x))\, dx \right)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2}} \left(\int_{t_1}^{t_0} \frac{1}{c} dt \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \rho_0 (S_0^2 + R_0^2)\, dx \right)^{\frac{1}{2}} \\
&\leq M_4 \cdot (t_0 - t_1)^{\frac{1}{2}} (x_0 - x_1)^{\frac{1}{2}} \varepsilon^{\frac{1+\alpha}{2}}
\end{aligned}$$

so $t_0 - t_1 \geq O(\varepsilon^{-\alpha})$.

Hence we complete the proof of Lemma 5.3. □

Remark 5.4 *(i) By* (5.10)*, it is easy to get that the initial energy in Ω_{Π_1} or Ω_{Π_2} is not greater than $O(\varepsilon^{2+\alpha})$ by* (5.10)*.*
(ii) On regions to the left or right of Ω, Ω_{Π_1} and Ω_{Π_2}, the wave speeds S and R are constants by (2.11) *($c(x)$ has constant values in those regions) and S and R are constants initially in each of these two regions.*

Lemma 5.5 *Consider C^1 solutions with prescribed initial data and $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0, T)$. There exist constants δ_2 and δ_3 such that*

$$-\delta_3 < R(x,t) < -\delta_2 < 0. \tag{5.15}$$

Proof. By Remark 5.4 (ii) and (5.6), we only need to consider the solution on Ω, Ω_{Π_1} and Ω_{Π_2}. By Lemma 2.1, for any $x \in (-\infty, \infty)$

$$R_t + SR_x \leq \frac{1}{8} R^2. \tag{5.16}$$

By (5.6) and ODE comparison theorem, there exists δ_2 such that

$$R < -\delta_2 < 0.$$

For any forward characteristic in $\Omega \cup \Omega_{\Pi_1} \cup \Omega_{\Pi_2}$, by (2.8), (1.28) and (5.4) we have

$$R_t + SR_x \geq -\frac{\varepsilon^\alpha}{16} cS^2. \tag{5.17}$$

Integrating it along any forward characteristic γ_+, by Lemma 2.15, Remark 5.4 (i) and (5.6), we have

$$\begin{aligned}
R(t_0, x_0) &\geq -\frac{\varepsilon^\alpha}{16} \int_0^{t_0} cS^2(t, x_+(t)) dt + R_0(x_1) \\
&= -\frac{\varepsilon^\alpha}{16} \int_{x_1}^{x_0} cS(t_+(x), x) dx - \varepsilon^{\alpha+1} \\
&\geq -\frac{\varepsilon^\alpha}{16} \int_{x_1}^{x_2} \rho_0 (S_0^2 + R_0^2) dx - \varepsilon^{\alpha+1} \\
&\geq -K_6 \varepsilon^{1+\alpha} := -\delta_3.
\end{aligned} \tag{5.18}$$

□

Now we proceed to estimate S.

Lemma 5.6 *Consider C^1 solutions with prescribed initial data and $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0,T)$. There exist positive constants M_1 and δ_1 such that*

$$0 < \delta_1 < S(x,t) < +\infty, \tag{5.19}$$

for any $(x,t) \in \mathbb{R} \times [0,T)$ with $T = M_1 \varepsilon^{-\alpha}$ and

$$\lim_{(x,t)\to(x^*,T)} S(x,t) = +\infty$$

for some $x^* = O(1)\varepsilon^{\frac{1-\alpha}{2}}$.

Proof. By Remark 5.4 (ii) and (5.6), we only need to consider the solution on Ω, Ω_{Π_1} and Ω_{Π_2}.

By Lemma 2.1 and the estimate of R in Lemma 5.5, we have

$$S_t + RS_x \le \frac{\varepsilon^\alpha}{8} S^2. \tag{5.20}$$

So S stays finite until before

$$\bar{t} = M_2 \varepsilon^{-\alpha},$$

where $M_2 = \frac{8}{S_0(x)}$.

When $x \notin [-2\varepsilon, 2\varepsilon]$, we have $S_0(x) = O(\varepsilon^{1+\alpha})$. So by the comparison theorem of ODE, $S(x,t)$ along any backward characteristic starting from an initial point with $x \notin [-2\varepsilon, 2\varepsilon]$, will not blowup until $\bar{t} = O(\varepsilon^{-1-2\alpha})$. Comparing to the blowup time in $O(\varepsilon^{-\alpha})$ proved later, we see that the blowup can only happen on a backward characteristic starting from the initial interval $[-2\varepsilon, 2\varepsilon]$.

Then we show the blowup happens at a time in $O(\varepsilon^{-\alpha})$. For simplicity, we only consider the backward characteristic Γ starting from the origin on (x,t)-plane. On Γ, by Lemma 2.1 and the estimate of R in Lemma 5.5,

$$S_t + RS_x \ge \frac{\varepsilon^\alpha}{16} S^2 - \frac{\varepsilon^\alpha}{8} \delta_3^2. \tag{5.21}$$

Studying the ODE

$$\frac{dg}{dt^-} = \frac{\varepsilon^\alpha}{16} g^2 - \frac{\varepsilon^\alpha}{8} \delta_3^2 \tag{5.22}$$

with initial data

$$g(0) = S_0(0) = 6 - \varepsilon^{\alpha+1} \in (5,6), \tag{5.23}$$

one has that g blows up at

$$t^* = \frac{8}{K_7 \varepsilon^{1+2\alpha}} \ln \left| \frac{S_0(0) + 2\delta_3}{S_0(0) - 2\delta_3} \right| = M_3 \varepsilon^{-\alpha}, \tag{5.24}$$

for some constant $M_3 > 0$. By comparison theorem of ODE, $S(x,t)$ blows up not later than $\bar{t} = M_3 \varepsilon^{-\alpha}$.

Therefore, there exists $M_1 \in [M_2, M_3]$ such that for any $(x,t) \in \mathbb{R} \times [0,T)$ with $T = M_1 \varepsilon^{-\alpha}$

$$0 < S(x,t) < +\infty, \qquad \lim_{(x,t)\to(x^*,T)} S(x,t) = +\infty$$

for some x^*. By (5.13) and $T = M_1\varepsilon^{-\alpha}$, we have $x^* = O(1)\varepsilon^{\frac{1-\alpha}{2}}$.

For any $(x,t) \in \mathbb{R} \times [0,T)$, by (5.21), we have

$$S_t + RS_x \geq -\frac{\varepsilon^\alpha}{8}\delta_3^2,$$

then by (5.6) ($S_0(x) > \varepsilon^{\alpha+1}$) and comparison theorem of ODE, we obtain

$$0 < \delta_1 < S(x,t) < +\infty \tag{5.25}$$

for some constant δ_1, which completes the proof of the lemma.

□

Next we show that the assumption that $S > 0$ is true for all C^1 solutions in our initial value problems.

Lemma 5.7 *For any C^1 solutions with prescribed initial data, $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0,T)$. Hence Lemmas 5.2∼5.6 are correct without the assumption that $S(x,t) > 0$ in the beginning.*

Proof. We prove it by contradiction. Assume that $S = 0$ somewhere.

Note R and S are non-zero negative and positive constants respectively when $(x,t) \notin \Omega \cup \Omega_{\Pi_1} \cup \Omega_{\Pi_2}$. So if $S = 0$ then it must first happen in $\Omega \cup \Omega_{\Pi_1} \cup \Omega_{\Pi_2}$. We could find the minimum time such that $S = 0$. Assume that $S(\hat{x}, \hat{T}) = 0$ for some $(\hat{x}, \hat{T}) \in \Omega \cup \Omega_{\Pi_1} \cup \Omega_{\Pi_2}$ and $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0, \hat{T})$. Then running the proofs in Lemmas 5.2∼5.6, we can still get (5.25) for $(x,t) \in \mathbb{R} \times [0, \hat{T}]$ which contradicts to $S(\hat{X}, \hat{T}) = 0$. Hence, $S(x,t) > 0$ for any $(x,t) \in \mathbb{R} \times [0,T)$.
□

Finally we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Combining the a priori estimates in Lemma 5.5 and Lemma 5.6, using (2.4), we know the a priori condition (A) is true for any $T^* < T = M_1\varepsilon^{-\alpha}$, where we use the fact: for any C^1 solution, S is bounded above in the closed set $\Omega \cup \Omega_{\Pi_1} \cup \Omega_{\Pi_2}$ with $t \in [0, T^*]$ since S is not infinity when $t \in [0, T^*]$, then uniformly bounded from above for any $(x,t) \in \mathbb{R} \times [0, T^*]$ by Remark 5.4 (ii).

As a conclusion, by Theorem 3.5, there exists a unique C^1 solution $(\rho(x,t), u(x,t))$ for equation (1.1) when $(x,t) \in \mathbb{R} \times [0,T)$ with the prescribed initial data.

Furthermore, by Lemma 5.6

$$\lim_{t \to T^-} S(x^*, t) = \infty$$

while R is uniformly bounded and negative by Lemma 5.5. So by (2.4),

$$\lim_{t \to T^-} u(x^*, t) = \infty \quad \lim_{t \to T^-} \rho(x^*, t) = 0 \quad \lim_{t \to T^-} \rho u(x^*, t) = B \quad \text{and} \quad \lim_{t \to T^-} E(x^*, t) = \infty, \tag{5.26}$$

where B is a finite constant. Here $|x^*| = O(\varepsilon^{\frac{1-\alpha}{2}}) < O(1)$ and the blowup must be on some characteristic starting form the initial interval $x \in [-2\varepsilon, 2\varepsilon]$. Hence we complete the proof of Theorem 1.5.

□

Finally we prove the Theorem 1.7 for the Cauchy problem of (1.30) with radially symmetry, i.e. (1.27) or (1.1) with $c(x) = x^m$ ($m = 1$, cylindrical symmetric solution; $m = 2$, spherically

symmetric solution) and the radius $x \in [1,3]$. This theorem is also correct when $x \in [a,b]$ with $0 < a < b$.

Proof of Theorem 1.7. When $x \in [1,3]$, we consider initial data $\rho_0(x)$ and $u_0(x)$ satisfying

(1).

$$\rho_0(x) = \begin{cases} \varepsilon^{-1}, & x \in [1, 2-2\varepsilon), \\ \phi_3(x), & x \in [2-2\varepsilon, 2-\varepsilon), \\ 1, & x \in [2-\varepsilon, 2+\varepsilon], \\ \phi_4(x), & x \in (2+\varepsilon, 2+2\varepsilon], \\ \varepsilon^{-1}, & x \in (2+2\varepsilon, 3], \end{cases} \tag{5.27}$$

(2). $u_0(x) = \frac{c(x)}{\rho_0(x)} - \varepsilon$.

The function ϕ_3 and ϕ_4 are increasing and decreasing smooth functions on $[2-2\varepsilon, 2-\varepsilon)$ and $(2+\varepsilon, 2+2\varepsilon]$ respectively. Furthermore $\varepsilon < 1$ is a small given number which will be provided in the proof of the theorem.

In order to directly use the proof for Theorem 1.5, we extend the definition of initial data from $x \in [1,3]$ to $x \in (-\infty, \infty)$ by

(1').

$$\tilde{c}(x) = \begin{cases} 1-\delta, & x \in (-\infty, 1-\eta), \\ \psi_3(x), & x \in [1-\eta, 1), \\ x^m, & x \in [1,3], \\ \psi_4(x), & x \in (3, 3+\eta], \\ 3^m+\delta, & x \in (3+\eta, \infty), \end{cases} \tag{5.28}$$

where $\psi_3(x)$ is an increasing C^2 convex positive function on $x \in [1-\eta, 1)$ and $\psi_4(x)$ is an increasing C^2 concave positive function on $x \in (3, 3+\eta]$. The positive constants

$$\delta \ll \eta \ll \varepsilon \ll 1.$$

(2').

$$\tilde{\rho}_0(x) = \begin{cases} \varepsilon^{-1}, & x \in (-\infty, 2-2\varepsilon), \\ \phi_3(x), & x \in [2-2\varepsilon, 2-\varepsilon), \\ 1, & x \in [2-\varepsilon, 2+\varepsilon], \\ \phi_4(x), & x \in (2+\varepsilon, 2+2\varepsilon], \\ \varepsilon^{-1}, & x \in (2+2\varepsilon, \infty), \end{cases} \tag{5.29}$$

(3'). $\tilde{u}_0(x) = \frac{\tilde{c}(x)}{\tilde{\rho}_0(x)} - \varepsilon$.

The initial data in Theorem 1.7 are very similar to the initial data in Theorem 1.5 with $\alpha = 0$. In fact, we only need to change $x \in [-1,1]$ to radius $x \in [1,3]$ and slightly change the values of c, then we can prove Theorem 1.7 by an entirely same way as the proof in Theorem 1.5, and finally we only have to use the piece of solution on Ω_{symm} with $t \in [0,T)$ after finding the C^1 solution for $(x,t) \in \mathbb{R} \times [0,T)$. We leave the details to the reader.

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