Near Crude Block Finite Representability of ℓ_1 in PL-space Bases

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1 Preliminary Information

The notation and terminology used in this work matches that which is used in [2]. A basic sequence $(v_i)_{i=1}^\infty$ in a Banach space $(V, \|\cdot\|_V)$ is said to be spreading if, for all $N \in \mathbb{N}$,

$$\left\| \sum_{i=1}^N \lambda_i v_i \right\|_V = \left\| \sum_{k=1}^N \lambda_k v_{i_k} \right\|_V$$

for all scalar sequences $(\lambda_i)_{i=1}^N$ and for all positive integers $(i_k)_{i=1}^N$ such that $i_1 < \ldots < i_N$. This spreading sequence is then said to be a spreading model of X if it is Schreier almost isometric to a normalized basic sequence $(x_i)_{i=1}^\infty$ in X. In other words, there exists a sequence $(\varepsilon_n)_{n=1}^\infty$ of positive real numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ such that

$$\left| \left\| \sum_{i=1}^N \lambda_i v_i \right\|_V - \left\| \sum_{k=1}^N \lambda_k x_{i_k} \right\| \right| \leq \varepsilon_{i_1}$$

for all positive integers $N \leq i_1 < \ldots < i_N$ and for all real sequences $(\lambda_i)_{i=1}^N$ that satisfy $|\lambda_i| \leq 1$. It is worth noting that the definitions of spreading sequences and spreading models given here are artificially restricted as in [4], whereas $(V, \|\cdot\|_V)$ can in general be a semi-normed vector space and neither $(v_i)_{i=1}^\infty$ nor $(x_i)_{i=1}^\infty$ needs to be basic (see [1, 2]). An unpublished result attributed in [4] to A. Pelczyński and G.C. da Rocha Filho and proved therein is that every spreading model (as defined in this work) of a PL-space is equivalent to the canonical ℓ_1 unit vector basis $(e_i)_{i=1}^\infty$.

Note that $(\bar{x}_i)_{i=1}^\infty$ where $\bar{x}_i = \frac{x_i}{\|x_i\|}$ is a normalized basis for the Banach space X if $(x_i)_{i=1}^\infty$ is a given basis. It then follows by a standard result [3, Theorem 1.3] due to Brunel and Sucheston that $(\bar{x}_i)_{i=1}^\infty$ has a subsequence that admits a spreading model $(v_i)_{i=1}^\infty$ and, if X is a PL-space, that this spreading model is equivalent to $(e_i)_{i=1}^\infty$. This implies a weak form of block finite representability of $(e_i)_{i=1}^\infty$ in $(\bar{x}_i)_{i=1}^\infty$. Namely, there exists a constant $M \geq 1$ such that for all $N \in \mathbb{N}$ and for all $\delta > 0$, there exist block vectors $z_1 < \ldots < z_N$ of $(\bar{x}_i)_{i=1}^\infty$ so that

$$-\delta + \frac{1}{M} \sum_{i=1}^N |\lambda_i| \leq \left\| \sum_{k=1}^N \lambda_k z_k \right\| \leq \delta + M \sum_{i=1}^N |\lambda_i| \tag{1.1}$$

for all real sequences $(\lambda_i)_{i=1}^N$ that satisfy $|\lambda_i| \leq 1$. The relationship (1.1) is automatically valid if $(e_i)_{i=1}^\infty$ is crudely block finitely represented in $(\bar{x}_i)_{i=1}^\infty$ so the former will be defined to be nearly crudely block finitely reprsented (NCBFR) in the latter if (1.1) holds. It is hopefully possible to characterize PL-spaces with bases in terms of the block finite representability properties of $(e_i)_{i=1}^\infty$ in their normalized bases.

2 A Psuedo Conjecture

The following is an example of the type of conjecture that might ultimately be true.

Conjecture 2.1. *Let X be a Banach space with a basis $(x_i)_{i=1}^\infty$. Then, X is a PL-space if and only if $(e_i)_{i=1}^\infty$ is NCBFR(?) in $(\bar{x}_i)_{i=1}^\infty$.*

If X is a PL-space, then there exists a normalized basic subsequence $(\bar{x}_{i_k})_{k=1}^\infty$ that is by means of the above results Schreier almost isometric to a basic sequence $(v_k)_{k=1}^\infty$ that is M-equivalent for some $M \geq 1$ to $(e_k)_{k=1}^\infty$. Let $\delta > 0$ be given and let $N \in \mathbb{N}$. Choose integers $N \leq i_1 < \ldots < i_N$ such that $\varepsilon_{i_1} \in (0, \delta)$ and note that

$$-\delta + \frac{1}{M}\sum_{k=1}^{N}|\lambda_k| \leq -\varepsilon_{i_1} + \left\|\sum_{k=1}^{N}\lambda_k v_k\right\|_V \leq \left\|\sum_{k=1}^{N}\lambda_k \bar{x}_{i_k}\right\| \leq \varepsilon_{i_1} + \left\|\sum_{k=1}^{N}\lambda_k v_k\right\|_V \leq \delta + M\sum_{k=1}^{N}|\lambda_k|$$

for all real sequences $(\lambda_k)_{k=1}^N$ that satisfy $|\lambda_k| \leq 1$. It follows that $(e_k)_{k=1}^\infty$ is NCBFR in $(\bar{x}_i)_{i=1}^\infty$ with respect to the block vectors $\bar{x}_{i_1} < \ldots < \bar{x}_{i_N}$. The precise converse statement for Conjecture 2.1 unfortunately does not appear to be true because NCBFR offers no control over the location or the size of the blocks $z_1 < \ldots < z_N$ of $(\bar{x}_i)_{i=1}^\infty$. This freedom is allowed in the definition of an asymptotic-ℓ_1 (with respect to a basis) Banach space and is required for proving that every such space is a PL-space (see [2, 4]). A relevant question is then how one might replicate this freedom and this proof (that a bounded X-valued function on $[0,1]$ whose set of discontinuities has positive Lebesgue measure is not Riemann-integrable) in the context of crude block finite representability or some variant thereof.

References

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