Prelim Semigroup Notes

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1 C_0-SEMIGROUPS

Unless otherwise stated, we will assume throughout that X is a complex Banach space, with $\mathscr{L}(X)$ denoting the Banach space of continuous linear operators on X.

Definition. (Semigroups) By a semigroup, we mean a one-parameter family of continuous linear operators, $\{T(t)\}_{t\geq 0} \subset \mathscr{L}(X)$, such that,

1. $T(0) = I$, where I is the identity map on X.
2. For all $s, t \geq 0$, we have $T(s+t) = T(s) \circ T(t)$.

Additionally, if one has $\lim_{t\to 0^+} T(t)x = x$ for every $x \in X$, then the semigroup is said to be "strongly continuous," or C_0, for short.

To every C_0 semigroup, we associate another linear map called its generator, which describes how the semigroup is affecting initial data in X.

Definition. (Infinitesimal Generator) Given a C_0-semigroup, $\{T(t)\}_{t\geq 0}$, its infinitesimal generator is the map $A : D(A) \to X$, where

$$D(A) = \left\{ x \in X \;\middle|\; \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and Ax is defined to be the value of this found limit for $x \in D(A)$.

It is quite clear that $D(A)$ is a subspace of X, and further, that A is a (not necessarily continuous) linear map. C_0-semigroups and their generators enjoy some rather pleasant basic properties, which we will now elucidate.

Lemma. *(Continuity of Orbits) Given a C_0-semigroup, $\{T(t)\}_{t\geq 0}$, we have that the orbit map $t \mapsto T(t)x$ is continuous for each $x \in X$.*

Proof. Apply the semigroup property and then strong continuity. □

Lemma. *(Exponential Bounds) Given a C_0-semigroup, $\{T(t)\}_{t\geq 0}$, there exists constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that for every $t \geq 0$, $||T(t)|| \leq Me^{\omega t}$.*

Proof. Suppose for a contradiction that for every $n \in \mathbb{N}$, there exists a point $t_n \in [0, \frac{1}{n}]$ such that $||T(t_n)|| \geq n$. Now, defining

$$\{T(t_n) \mid n \in \mathbb{N}\} \subset \mathscr{L}(X)$$

we see that for each $x \in X$, $\sup_{n\in\mathbb{N}} ||T(t_n)x|| \leq \sup_{[0,1]} ||T(t)x|| < \infty$ by the continuity of the orbit map. Then, by uniform boundedness, it follows that $\sup_{n\in\mathbb{N}} ||T(t_n)|| < \infty$, but this is an obvious contradiction.

Specifically, we are now afforded an $n_0 \in \mathbb{N}$ such that $||T(t)|| < n_0$ for every $t \in [0, \frac{1}{n_0}]$. Then, fixing $t_* \geq 0$ arbitrarily, we may write $t_* = k\frac{1}{n_0} + r$ for some $k \in \mathbb{N}_0$ and $r \in [0, \frac{1}{n_0})$, which yields that,

$$\begin{aligned} ||T(t_*)|| &= \left|\left|T\Big(k\frac{1}{n_0} + r\Big)\right|\right| \leq M^{k+1} \qquad & M &= \sup\left\{||T(t)|| \;\Big|\; t \in \left[0, \frac{1}{n_0}\right]\right\} \\ &= Me^{\ln(M^k)} \leq Me^{\omega t_*} & \omega &= n_0 \ln(M) \end{aligned}$$

and since $M \geq 1$ and $\omega \in \mathbb{R}$ do not depend on t_*, we are done. □

The next lemma rounds out our basic "toolbox" of techniques for manipulating C_0-semigroups and their generators.

Lemma. *(Toolbox) Given a C_0-semigroup, $\{T(t)\}_{t\geq 0}$, and its generator $A : D(A) \to X$, we may conclude the following:*

1. *For every $x \in X$ and $t \geq 0$, we have*
   - *(a)* $\frac{1}{h}\int_t^{t+h} T(s)xds \to T(t)x$ *as* $h \to 0^+$
   - *(b)* $\int_0^t T(s)xds \in D(A)$ *and* $A(\int_0^t T(s)xds) = T(t)x - x$
2. *For every $x \in D(A)$ and $t \geq 0$, we have*
   - *(a)* $T(t)x \in D(A)$ *and* $\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$
   - *(b) For every* $0 \leq s \leq t$, *we have* $\int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau = T(t)x - T(s)x$
3. *A is densely-defined and closed.*

Proof. Continuity of the orbit map implies *1 (a)*, and *1 (b)* is a straightforward application of *1 (a)*. Next, *2 (a)* follows from the semigroup property, and *2 (b)* is simply the fundamental theorem of calculus. Lastly, *3* requires just a little bit of unpacking. First, fix $x \in X$, and define,

$$x_n = n\int_0^{n^{-1}} T(t)xdt \qquad (n \in \mathbb{N})$$

so that $x_n \in D(A)$ by *1 (b)*, and $x_n \to x$ by *1 (a)* as $n \to \infty$, meaning that A is densely-defined. Then, supposing

$$x_n \to y \in X \text{ and } Ax_n \to z$$

it follows by *2 (b)* that for $h > 0$, we have

$$\frac{T(h)x_n - x_n}{h} = \frac{1}{h}\int_0^h T(t)Ax_n dt$$

and since $||T(t)Ax_n - T(t)z|| \le Me^{\omega h}||Ax_n - z|| \to 0$ as $n \to \infty$, it follows by uniform convergence on $[0, h]$ that,

$$\frac{T(h)y - y}{h} = \frac{1}{h}\int_0^h T(t)z dt$$

so taking $h \to 0^+$, it follows by *1 (a)* that $y \in D(A)$ and $Ay = z$, meaning that A is a closed operator, as required. □

The main idea behind semigroup theory is that it allows us to solve the so-called "Abstract Cauchy Problem" (ACP), which is the initial value problem,

$$\begin{cases} u'(t) = Au(t) & t \ge 0 \\ u(0) = u_0 & u_0 \in D(A) \end{cases}$$

for some linear map A and unknown function $u : [0, \infty) \to X$. We are always hoping to show that A generates a C_0-semigroup, in which case the ACP is well-posed, and the solution is given via a "semigroup representation,"

$$u(t) = T(t)u_0 \qquad (t \ge 0)$$

Recall from ODE theory that we should have $T(t) \approx e^{tA}$, and it is easy to see that $A \in \mathscr{L}(X)$ will generate $\{e^{tA}\}_{t\ge0}$. Conversely, it is straightforward to show that given an exponential semigroup, its generator must be the exponentiated bounded operator. More generally, the collection of exponential semigroups constitute a subset of strongly continuous semigroups which are said to be *uniformly continuous.*

Definition. (UC Semigroups) A strongly continuous semigroup, $\{T(t)\}_{t\ge0}$, is called uniformly continuous if $\lim_{t\to0^+} ||T(t) - I|| = 0$.

The following lemma is an important general fact, and we will use it here in the context of showing that the exponential semigroups and UC semigroups coincide.

Lemma. *(Von Neumann) If $B_1, B_2 \in \mathscr{L}(X)$ where B_1 is an invertible map, and $||B_2|| < ||B_1^{-1}||^{-1}$, then $B_1 - B_2 \in \mathscr{L}(X)$ is an invertible operator.*

Proof. We'll split this proof into two steps. The first step is to consider the case where $B_1 \equiv I$ and $||B_2|| < 1$. In this case,

$$(I - B_2)\sum_{j=0}^{n} B_2^j = \sum_{j=0}^{n} B_2^j - \sum_{j=0}^{n} B_2^{j+1} = I - B_2^{n+1} \qquad (n \in \mathbb{N})$$

and taking $n \to \infty$, it follows that $(I - B_2)\sum_{j=0}^{\infty} B_2^j = I$, and as this sum is also clearly a left inverse of $I - B_2$, we have that $I - B_2$ is invertible and $\sum_{j=0}^{\infty} B_2^j$ is its inverse. We next consider the general case,

$$B_1 - B_2 = B_1(I - B_1^{-1}B_2)$$

and since $||B_1^{-1}B_2|| < 1$, the LHS must be invertible, so $B_1 - B_2$ is invertible with inverse given by, $(B_1 - B_2)^{-1} = (I - B_1^{-1}B_2)^{-1}B_1^{-1}$. □

Theorem. ($EXP \leftrightarrow UC$) *The collection of exponential semigroups exactly coincides with the collection of UC semigroups, that is*

$$\{\{e^{tA}\}_{t\geq 0} \mid A \in \mathscr{L}(X)\} = \{UC \ semigroups\}$$

Proof. The $\subseteq$ containment is obvious, and for the $\supseteq$ containment, suppose that $\{T(t)\}_{t\geq 0}$ is UC and note that for a sufficiently small $r > 0$, we have

$$\Big|\Big|I - \frac{1}{r}\int_0^r T(t)dt\Big|\Big| \leq \frac{1}{r}\int_0^r ||I - T(t)||dt < 1$$

so by the Von Neumann lemma, it follows that $\int_0^r T(t)dt$ must be invertible. Then, fixing $x \in X$ and $h \in (0, r)$, we have

$$\begin{aligned} \frac{T(h)x - x}{h} &= \Big[\int_0^r T(t)dt\Big]^{-1}\Big[\int_0^r T(t)dt\Big]\frac{T(h)x - x}{h} \\ &= \Big[\int_0^r T(t)dt\Big]^{-1}\Big[\frac{1}{h}\int_r^{r+h} T(t)xdt - \frac{1}{h}\int_0^h T(t)xdt\Big] \end{aligned}$$

so letting $G : D(G) \to X$ generate $\{T(t)\}_{t\geq 0}$, one has $X \subseteq D(G)$, and hence, $G = [\int_0^r T(t)dt]^{-1}[T(r) - I] \in \mathscr{L}(X)$. □

Our results about UC semigroups now allow us to characterize the generators of arbitrary C_0-semigroups, and this is hugely important because it tells us when the ACP is well-posed.

Definition. (Resolvent and Spectrum) The resolvent set of a closed operator, $A : D(A) \to X$, is the set given by

$$\rho(A) = \{\lambda \in \mathbb{C} \mid \lambda - A : D(A) \to X \text{ is bijective}\}$$

and its complement, $\sigma(A) = \mathbb{C} \setminus \rho(A)$, is said to be the spectrum of A. Note that we do not assume the density of $D(A)$ in X - only the closedness of A is needed for a reasonable spectral theory. For each $\lambda \in \rho(A)$, we may form the operator,

$$R(\lambda, A) = (\lambda - A)^{-1}$$

and since this operator is defined on X and closed (by the closedness of A), the closed graph theorem implies its boundedness. Lastly, $R(\lambda, A)$ is called the resolvent operator of A at the point λ.

We will return later to the spectral theory of semigroups and their generators, but in fact, this definition is the last necessary prerequisite for giving a complete characterization of C_0-semigroup generators.

Theorem. *(Hille-Yosida for Contraction Semigroups) Given a linear map, A, we can say that A will generate a contraction semigroup on X iff it is closed, densely-defined, and for every $\lambda \in (0, \infty)$, we have $||R(\lambda, A)|| \leq \lambda^{-1}$.*

Proof. For necessity, suppose that A generates a strongly continuous (contraction) semigroup, $\{T(t)\}_{t \geq 0}$, so $||T(t)|| \leq 1$ for every $t \geq 0$. The toolbox lemma already tells us that A is closed and densely-defined, so we need only check the third condition. To that end, fix $\lambda > 0$, and consider the operator,

$$R_\lambda x := \int_0^\infty e^{-\lambda t} T(t) x dt \qquad (x \in X)$$

which is clearly bounded in operator norm by λ^{-1}, so we need only prove $\lambda - A$ to be a bijection onto X with inverse R_λ. For $x \in X$, consider

$$\begin{aligned} \frac{T(h)R_\lambda x - R_\lambda x}{h} &= \frac{1}{h} \int_h^\infty e^{-\lambda(t-h)} T(t) x dt - \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t) x dt \\ &= \frac{e^{\lambda h} - 1}{h} \int_h^\infty e^{-\lambda t} T(t) x dt - \frac{1}{h} \int_0^h e^{-\lambda t} T(t) x dt \end{aligned}$$

and since the RHS converges to $\lambda R_\lambda x - x$ as $h \to 0^+$, it follows that for $x \in X$,

$$(\lambda - A) R_\lambda x = \lambda R_\lambda x - A R_\lambda x = x$$

and conversely, since the toolbox lemma shows that A and $T(t)$ commute when restricted to $D(A)$, it follows that for $x \in D(A)$, we must have,

$$R_\lambda (\lambda - A) x = \lambda R_\lambda x - R_\lambda A x = \lambda R_\lambda x - A R_\lambda x = x$$

where we have also used the fact that A is a closed operator, and this completes the proof of necessity.

For sufficiency, we begin with a closed, densely-defined operator, $A : D(A) \to X$, whose resolvent set contains the positive real line, and whose resolvent operator at the point $\lambda > 0$ is bounded in operator norm by λ^{-1}. Then, setting

$$A_\lambda = \lambda A R(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda \qquad (\lambda > 0)$$

we have that,

1. Since A and $R(\lambda, A)$ commute for $x \in D(A)$, we may write,

$$||\lambda R(\lambda, A) x - x|| = ||A R(\lambda, A) x|| \leq \frac{||Ax||}{\lambda}$$

   and by density, this implies $\lim_{\lambda \to \infty} \lambda R(\lambda, A) x = x$ for every $x \in X$.

2. Since $A_\lambda \in \mathscr{L}(X)$, it must generate the UC semigroup $\{e^{tA_\lambda}\}_{t\geq 0}$, and as

$$||e^{tA_\lambda}|| = ||e^{t(\lambda^2 R(\lambda,A)-\lambda)}|| \leq e^{-t\lambda}e^{t\lambda^2||R(\lambda,A)||} \leq 1$$

this semigroup is also a contraction.

Now, fixing $t \geq 0$, we consider $x \in D(A)$ with $\lambda, \mu > 0$ to write,

$$\begin{aligned} ||e^{tA_\lambda}x - e^{tA_\mu}x|| &= \left|\left| \int_0^1 \frac{d}{ds}\left[e^{tsA_\lambda}e^{t(1-s)A_\mu}\right] x ds \right|\right| \\ &= \left|\left| \int_0^1 e^{tsA_\lambda}e^{t(1-s)A_\mu}(tA_\lambda - tA_\mu)xds \right|\right| \leq t||A_\lambda x - A_\mu x|| \end{aligned}$$

and since the RHS converges to zero as $\lambda, \mu \to \infty$ (by 1.), the density of $D(A)$ in X allows us to define,

$$T(t)x := \lim_{\lambda\to\infty} e^{tA_\lambda}x \qquad (x \in X)$$

and in particular, the one-parameter family $\{T(t) \mid t \geq 0\} \subset \mathscr{L}(X)$ must be a contraction semigroup, as strong continuity results from $T(t)x$ being the uniform limit of continuous functions on any bounded interval. Assuming $G : D(G) \to X$ generates this semigroup, we'll let $x \in D(A)$, and consider

$$\begin{aligned} \frac{T(h)x - x}{h} &= \frac{1}{h}(\lim_{\lambda\to\infty} e^{hA_\lambda}x - x) \\ &= \frac{1}{h}\lim_{\lambda\to\infty}\int_0^h e^{tA_\lambda}A_\lambda x dt \\ &= \frac{1}{h}\int_0^h T(t)Axdt \qquad \text{uniform convergence on } [0,h] \end{aligned}$$

and this implies $A \subseteq G$ as operators, and since $1 \in \rho(A) \cap \rho(G)$, we have

$$(1-G)[D(A)] = (1-A)[D(A)] = X$$

meaning that $D(A) = R(1,G)[X] = D(G)$, which completes the proof. □

The above theorem is the workhorse of basic semigroup theory in the sense that a full characterization of strongly continuous semigroup generators can now be (more or less) obtained through rescaling/renorming arguments alone.

Theorem. *(Full Hille-Yosida) Given a linear map, A, we can say that A will generate a strongly continuous semigroup, $\{T(t)\}_{t\geq 0}$, satisfying $||T(t)|| \leq Me^{\omega t}$ for each $t \geq 0$, iff A is closed, densely-defined, and for every $\Re(\lambda) \in (\omega, \infty)$, we have that $||R(\lambda, A)^n|| \leq M(\Re(\lambda) - \omega)^{-n}$ for each $n \in \mathbb{N}$.*

The full Hille-Yosida result is often difficult to apply as the resolvent bound must be checked for each $n \in \mathbb{N}$. Of more practical importance is the so-called

Lumer-Philips Theorem, which relates the Hille-Yosida theorem for contractions to the concept of a dissipative operator.

Briefly speaking, writing X' for the dual space of X, the complex version of Hahn-Banach shows that for each $x \in X$,

$$F(x) = \{x' \in X' \mid ||x'||_{X'}^2 = x'(x) = ||x||^2\}$$

must be nonempty, and then, given an operator, $A : D(A) \to X$, dissipativity is expressed by the condition,

$$\forall x \in D(A)\ \exists x' \in F(x) \text{ s.t. } \Re[x'(Ax)] \leq 0$$

which may be equivalently expressed as $||(\lambda - A)x|| \geq \lambda ||x||$ for every $x \in D(A)$ and $\lambda > 0$. This equivalence is not trivial (sufficiency requires Banach-Alaoglu), but it leads to the following characterization of C_0-semigroup generators.

Theorem. *(Lumer-Philips) Let A be a densely-defined operator. If A generates a contraction semigroup, then it must be dissipative, and for each $\lambda > 0$, we must have $(\lambda - A)[D(A)] = X$. Conversely, if A is dissipative and there is a $\lambda_0 > 0$ such that $(\lambda_0 - A)[D(A)] = X$, then A must generate a contraction semigroup.*

As an example, assume $X = L^2([0,1], \mathbb{R})$ is equipped with the usual L^2 inner product, and consider the subset,

$$D(A) = \{u \in H^1([0,1], \mathbb{R}) \mid u(1) = 0\}$$

which is dense in X, with $Au = u'$. Next, fixing $u \in D(A)$, we'll have

$$||u||^2 = \langle u, u \rangle = x'(u)$$

for some $x' \in X'$, where $||x'|| = ||u||$, so it follows that $x' \in F(u)$, and then,

$$\begin{aligned} x'(Au) = \langle Au, u \rangle &= \int_0^1 u' u dx \\ &= -\frac{1}{2} u(0)^2 \leq 0 && \text{integration by parts} \end{aligned}$$

and since $\Re[x'(Au)] \leq 0$, we have that A is dissipative. Lastly, for a fixed $\lambda_0 > 0$, one can show that the equation,

$$(\lambda_0 - A)u = f$$

is solvable for any $f \in X$ by some $u \in D(A)$, so it follows by Lumer-Philips that A generates a contraction semigroup on X. In hindsight, this result should not be terribly surprising, given that A is a derivative operator.

2 Asymptotics of Semigroup Solutions

The previous section was concerned with introducing the concept of a strongly continuous semigroup, and investigating the well-posedness of the ACP by characterizing semigroup generators. We will now explore a different question: given a well-posed ACP, what is the long-term behavior of the semigroup solution?

Definition. (Growth Bound) Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup with $||T(t)|| \leq Me^{\omega t}$ for each $t \geq 0$. Since strongly continuous semigroups may be uniquely associated with their generators, we suppose A generates $\{T(t)\}_{t\geq 0}$, and we define the growth bound of the semigroup to be the number,

$$\omega(A) = \inf\{\omega \in \mathbb{R} \mid \exists M_\omega \text{ s.t. } ||T(t)|| \leq M_\omega e^{\omega t} \ \forall t \geq 0\}$$

and in the event that this set has no greatest lower bound in $\mathbb{R}$, we allow the case $\omega(A) = -\infty$.

When presented with a well-posed ACP, our goal is usually to show that the growth bound of the semigroup solution is strictly negative, as this will imply the exponential decay of the solution.

Lemma. *Suppose that $\eta : [0, \infty) \to \mathbb{R}$ is subadditive and bounded on compact intervals, then we must have,*

$$\inf_{t>0} \frac{\eta(t)}{t} = \lim_{t\to\infty} \frac{\eta(t)}{t}$$

where the existence of the limit is part of the claim.

Proof. Fixing $p > 0$, we may, for any $t_* > p$, write $t_* = kp + r$ for some positive natural number k, and $r \in [0, p)$. Then,

$$\begin{aligned} \frac{\eta(t_*)}{t_*} &= \frac{\eta(kp+r)}{kp+r} \leq \frac{\eta(kp)+\eta(r)}{kp+r} && \text{subadditivity} \\ &\leq \frac{k\eta(p)+\eta(r)}{kp+r} && \text{subadditivity, again} \\ &\leq \frac{k\eta(p)}{kp+r} + \frac{\eta(r)}{kp+r} \leq \frac{\eta(p)}{p} + \frac{\eta(r)}{kp} \end{aligned}$$

and since $\eta(r) \leq \sup\{|\eta(x)| \mid x \in [0, p]\} < \infty$, it follows that,

$$\limsup_{t_*\to\infty} \frac{\eta(t_*)}{t_*} \leq \frac{\eta(p)}{p} \qquad \text{as } k \to \infty \text{ when } p, r \text{ stay fixed/bdd}$$

and as this inequality holds for an arbitrary $p > 0$, we may write,

$$\limsup_{t_*\to\infty} \frac{\eta(t_*)}{t_*} \leq \inf_{p>0} \frac{\eta(p)}{p} \leq \liminf_{t_*\to\infty} \frac{\eta(t_*)}{t_*}$$

which proves the claim (and note that this limit/inf may equal $-\infty$). □

There are several different ways to represent the growth bound of a semigroup, and the above lemma links them together.

Theorem. *(Representation of the Growth Bound) Let* $\{T(t)\}_{t\geq 0}$ *be the strongly continuous semigroup generated by* A*, with growth bound* $\omega(A)$*. Then,* $\omega(A) = \inf_{t>0} \frac{\log ||T(t)||}{t} = \lim_{t\to\infty} \frac{\log ||T(t)||}{t} = \frac{\log r(T(t_*))}{t_*}$ *for every* $t_* > 0$*, where* $r(T(t_*))$ *denotes the spectral radius of* $T(t_*)$*.*

Proof. First, note that $\log ||T(t)|| : [0, \infty) \to \mathbb{R}$ is subadditive and bounded on compact intervals, so by the above lemma, we have

$$\inf_{t>0} \frac{\log ||T(t)||}{t} = \lim_{t\to\infty} \frac{\log ||T(t)||}{t}$$

and calling this (extended) real number ν, it follows that for a fixed $t_* > 0$ and for any $\omega > \omega(A)$,

$$\nu \leq \frac{||T(t_*)||}{t_*} \leq \frac{\log Me^{\omega t_*}}{t_*} = \frac{\log M + \omega t_*}{t_*} = \frac{\log M}{t_*} + \omega$$

implying (by the arbitrariness of t_*) that $\nu \leq \omega$ for any $\omega > \omega(A)$, and hence, that $\nu \leq \omega(A)$ as well. For the reverse inequality, we may fix $\mu > \nu$, and find a $t_* > 0$ such that,

$$\nu \leq \frac{\log ||T(t)||}{t} < \mu \qquad (t \geq t_*)$$

but this implies that $||T(t)|| \leq e^{\mu t}$ for $t \geq t_*$, and since $t \mapsto ||T(t)||$ is bounded on $[0, t_*]$, we may find an $M_* \geq 1$ such that $||T(t)|| \leq M_* e^{\mu t}$ for all $t \geq 0$, which shows that $\omega(A) \leq \mu$. Now, as $\nu \leq \omega(A) \leq \mu$, the reverse inequality will follow by the arbitrariness of $\mu > \nu$, and lastly, the spectral radius representation is derived by an application of Gelfand's formula. □

Given the multitude of ways to represent the growth bound of a strongly continuous semigroup, it should come as no surprise that there are multiple reprentations for the negativity of this quantity. This condition is called (uniform) stability of the semigroup.

Theorem. *(Uniform Stability) Let* $\{T(t)\}_{t\geq 0}$ *be the strongly continuous semigroup generated by* A*. Then, the following condtions are equivalent.*

1. $\omega(A) < 0$
2. $\lim_{t\to\infty} ||T(t)|| = 0$
3. *There exists* $t > 0$ *such that* $||T(t)|| < 1$
4. *There exists* $t > 0$ *such that* $r(T(t)) < 1$

Proof. First, if $\omega(A) < 0$, then we may find $\omega \in (\omega(A), 0)$ such that $||T(t)|| \leq Me^{\omega t}$ and since the RHS converges to zero as $t \to \infty$, we have that $1 \implies 2$.

Next, 2 $\implies$ 3 is trivial, and supposing we may find $t_* > 0$ so that $||T(t_*)|| < 1$, the growth bound theorem tells us that

$$\frac{\log r(T(t_*))}{t_*} \leq \frac{\log ||T(t_*)||}{t_*} < 0$$

which implies $1 > e^{\frac{\log r(T(t_*))}{t_*}} = e^{\log r(T(t_*))^{\frac{1}{t_*}}} = r(T(t_*))^{\frac{1}{t_*}}$, so raising both sides to the t_* power, it follows that 3 $\implies$ 4. Lastly, if t_* is such that $r(T(t_*)) < 1$, then the growth bound theorem tells us that

$$\omega(A) = \frac{\log r(T(t_*))}{t_*} < 0$$

so we have that 4 $\implies$ 1, which completes the proof. □

We have already seen some uniformly stable semigroups. For instance, any UC-semigroup will clearly satisfy condition 3, and more generally, this condition holds for any so-called "eventually norm continuous" semigroup (e.g. it is UC after some finite time). The above theorem is not an exhuastive list of uniform stability criteria, as we have omitted a related condition called the Datko-Pazy Theorem, but it is more than sufficient for a proof of the Gearhart-Prüss Theorem, which is our next goal.

Gearhart-Prüss characterizes the uniform stability of a strongly continuous semigroup on a Hilbert space. We'll need to introduce several preliminary ideas before attempting a proof, and we return first to the notion of the resolvent and spectrum of a closed operator.

Lemma. *(Resolvent Properties) Given a closed operator, $A : D(A) \to X$, the following properties hold:*

1. *The resolvent set, $\rho(A)$, is open in $\mathbb{C}$, and for each $\lambda_0 \in \rho(A)$, one has*

$$R(\lambda, A) = \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R(\lambda_0, A)^{j+1}$$

   for all $\lambda \in \mathbb{C}$ with $|\lambda_0 - \lambda| \leq \frac{1}{||R(\lambda_0, A)||}$.

2. *The map $\lambda \mapsto R(\lambda, A)$ is analytic, and moreover, we have,*

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1} \qquad (n \in \mathbb{N})$$

3. *Given $\lambda_0 \in \mathbb{C}$ and a sequence $(\lambda)_n \subset \rho(A)$ which converges to λ_0, we have that $\lambda_0 \in \sigma(A)$ iff $\lim_{n\to\infty} ||R(\lambda_n, A)|| = \infty$.*

Proof. Fix $\lambda_0 \in \rho(A)$ and consider that for any $\lambda \in \mathbb{C}$, we have,

$$\lambda - A = \lambda_0 - A + \lambda - \lambda_0 = \underbrace{[I - (\lambda_0 - \lambda)R(\lambda_0, A)]}_{\text{inverse will be in } \mathscr{L}(X)}(\lambda_0 - A)$$

which shows that if $|\lambda_0 - \lambda| < \frac{1}{||R(\lambda_0, A)||}$, then the RHS is invertible, and further, must be a bijection from $D(A)$ *onto* X, meaning that $\lambda \in \rho(A)$. Hence, $\rho(A)$ is open in $\mathbb{C}$, and we may again apply Von Neumann to conclude that,

$$R(\lambda, A) = R(\lambda_0, A) \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R(\lambda_0, A)^j = \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R(\lambda_0, A)^{j+1}$$

for each λ in the prescribed open ball centered at λ_0. Now, since the resolvent operator has a local power series representation, it must be an analytic function, and differentiating through the sum yields the desired formula for the n^{th}-order derivative of $R(\lambda, A)$. Finally, consider a point $\lambda_0 \in \mathbb{C}$ and a sequence $(\lambda_n)_n \subset \rho(A)$ which converges to λ_0. Assuming $\lambda_0 \in \sigma(A)$, the first claim of this lemma shows that,

$$\text{dist}(\lambda_0, \lambda_n) \geq \frac{1}{||R(\lambda_n, A)||} \qquad (n \in \mathbb{N})$$

so rearranging and observing that $\text{dist}(\lambda_0, \lambda_n)$ converges to zero as $n \to \infty$,

$$\infty = \liminf_{n\to\infty} \frac{1}{\text{dist}(\lambda_0, \lambda_n)} \leq \liminf_{n\to\infty} ||R(\lambda_n, A)||$$

and this implies $\lim_{n\to\infty} ||R(\lambda_n, A)|| = \infty$, as desired. For the converse implication, assume that the resolvent norms tend to ∞, but suppose for a contradiction that $\lambda_0 \in \rho(A)$. Then, we have,

$$\{\lambda_n \mid n \in \mathbb{N}\}$$

is closed and bounded, and hence compact in $\mathbb{C}$. Now, as the resolvent map is continuous, it must be bounded on this set, which yields a contradiction. □

The next preliminary concept is that of the so-called "adjoint semigoup." In general, given $\{T(t)\}_{t\geq 0}$, the adjoint semigroup is the collection of operators,

$$\{T^*(t) \in \mathscr{L}(X') \mid t \geq 0\} \qquad T^*(t) := [T(t)]^*$$

which is necessarily a semigroup, but is <u>not</u> necessarily strongly continuous. As an example, the semigroup of left translations on $L^1(\mathbb{R})$ is strongly continuous, but its adjoint consists of the right translations of $L^\infty(\mathbb{R})$, which is not a strongly continuous semigroup. Specifically, if the underlying space is reflexive, then the adjoint semigroup must be strongly continuous, and what's more, its generator, A^*, must be the adjoint of the original semigroup generator, A, and we will also have that $\sigma(A^*) = \sigma(A)$.

Lastly, recall that in the Hille-Yosida theorem for contraction semigroups, we represented the resolvent of A at the point λ in terms of the Laplace Transform of the semigroup. This yields two important ideas,

1. We can represent the resolvent along a vertical line in the complex plane in terms of the Fourier Transform of the semigroup.

2. Representation of the resolvent in terms of the semigroup begs the question of whether the semigroup has a representation in terms of the resolvent.

Both 1 and 2 arise naturally in the proof Gearhart-Prüss, and specifically, 2 is the notion of "semigroup inversion." This area is rather technical, and we will simply need one result, whose proof can be found in [2].

Lemma. *(Semigroup Inversion) Let $\{T(t)\}_{t\geq 0}$ be the strongly continuous semigroup generated by A. Then, for any $x \in D(A^2)$, we have that*

$$T(t)x = \frac{j-1}{t^{j-1}} \frac{1}{2\pi i} \lim_{n\to\infty} \int_{\omega-in}^{\omega+in} e^{zt} R(z,A)^j x dz$$

will hold for any $j \in \mathbb{N}$ and $t > 0$, whenever $\omega > \omega(A)$. Additionally, the integral converges absolutely for any fixed positive t, and as a function of t, it converges uniformly on compact subintervals of $(0, \infty)$.

Finally, we note that whenever A generates a strongly continuous semigroup on X, then $D(A^2)$ must be dense in X as well, and this brings us to the Gearhart-Prüss Theorem. As a notational convention, we set $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \Re(z) > 0\}$.

Theorem. *(Gearhart-Prüss) Suppose that A generates the strongly continuous semigroup, $\{T(t)\}_{t\geq 0}$, on the Hilbert space, $\mathscr{H}$. Then, $\{T(t)\}_{t\geq 0}$ is a uniformly stable semigroup iff $\mathbb{C}_+ \subseteq \rho(A)$ and $M = \sup_{\lambda\in\mathbb{C}_+} ||R(\lambda, A)|| < \infty$.*

Proof. Assume first that $\omega(A) < 0$, and pick $\omega \in (\omega(A), 0)$. Then,

$$\mathbb{C}_+ \subset \{\lambda \in \mathbb{C} \mid \Re(\lambda) > \omega\} \subseteq \rho(A)$$

and also, there exists an $M_\omega \geq 1$ such that for every $\Re(\lambda) > \omega$ and $n \in \mathbb{N}$,

$$||R(\lambda, A)^n|| \leq \frac{M_\omega}{(\Re(\lambda) - \omega)^n}$$

Thus, taking $\lambda \in \mathbb{C}_+$ and $n = 1$, we arrive at the estimate,

$$||R(\lambda, A)|| \leq \frac{M_\omega}{\Re(\lambda) - \omega} \leq \frac{M_\omega}{-\omega}$$

which implies that $M = \sup_{\lambda\in\mathbb{C}_+} ||R(\lambda, A)|| \leq \frac{M_\omega}{-\omega} < \infty$. Now, for the converse implication, we first note that by the third claim from the "resolvent properties" lemma, we must have $\mathbb{C}_+ \cup i\mathbb{R} \subseteq \rho(A)$, and moreover, the uniform estimate on the norm of the resolvent operator will extend to this set by continuity. Second, setting $\omega > |\omega(A)| + 1$, we may rescale our strongly continuous semigroup by,

$$T_{-\omega}(t) = e^{-\omega t} T(t)$$

so that $\{T_{-\omega}(t)\}_{t\geq 0}$ is a uniformly stable, strongly continuous semigroup generated by $A - \omega$. We can now represent the resolvent operator of our original semigroup along the vertical line $\omega + is \subset \mathbb{C}_+$ in terms of the resolvent of our rescaled semigroup, that is, for a fixed $x \in \mathscr{H}$ and for $s \in \mathbb{R}$,

$$\begin{aligned} R(\omega + is, A)x &= \int_0^\infty e^{-(\omega+is)t} T(t)x dt \\ &= \int_0^\infty e^{-ist} T_{-\omega}(t)x dt = R(is, A-\omega)x \end{aligned}$$

specifically, setting $T_{-\omega}(t) = 0$ for all $t < 0$, we may then realize $R(\omega+is, A)x$ as the Fourier Transform of the function $T_{-\omega}(t)x \in L^2(\mathbb{R}, \mathscr{H})$, so by Plancharel's Theorem, we have,

$$\int_{\mathbb{R}} ||R(\omega + is, A)x||^2 ds = 2\pi \int_{\mathbb{R}} ||T_{-\omega}(t)x||^2 dt \leq L \cdot ||x||^2$$

where we may find $L > 0$ by the uniform stability of the rescaled semigroup. Next, applying the resolvent identity, we observe,

$$\begin{aligned} R(is, A) &= R(\omega + is, A) + (is - (\omega + is))R(is, A)R(\omega + is, A) \\ &= [I - \omega R(is, A)]R(\omega + is, A) \end{aligned}$$

so for any $x \in \mathscr{H}$, we must have $||R(is, A)x|| \leq (1 + M\omega)||R(\omega + is, A)x||$, and this yields the estimate,

$$\begin{aligned} \int_{\mathbb{R}} ||R(is, A)x||^2 ds &\leq (1 + M\omega)^2 \int_{\mathbb{R}} ||R(\omega + is, A)x||^2 ds \\ &\leq (1 + M\omega)^2 \cdot L \cdot ||x||^2 \end{aligned}$$

Now, since $||T(t)|| = ||T^*(t)||$ for all $t \geq 0$, since $\mathbb{C}_+ \cup i\mathbb{R} \subseteq \rho(A) \cap \rho(A^*)$ (as the spectra of A and A^* coincide), and as the adjoint resolvent operator must also be uniformly bounded in norm on this set by M, we may perform the same estimate for the adjoint semigroup, that is, for each $y \in \mathscr{H}$,

$$\int_{\mathbb{R}} ||R(is, A^*)||^2 ds \leq (1 + M\omega)^2 \cdot L \cdot ||y||^2$$

Finally, applying the inversion formula in the case where $j = 2$, we may fix $x \in D(A^2)$, $y \in \mathscr{H}$, and for any $t > 0$, write that,

$$\begin{aligned} |\langle tT(t)x, y\rangle| &= \Big|\Big\langle \frac{1}{2\pi i} \lim_{n\to\infty} \int_{-n}^{n} e^{(\omega+is)t} R(\omega + is, A)^2 x ds, y\Big\rangle\Big| \\ &= \frac{1}{2\pi} \lim_{n\to\infty} \Big| \int_{-n}^{n} e^{(\omega+is)t} \langle R(\omega + is, A)^2 x, y\rangle ds\Big| \end{aligned}$$

where viewing the inner product as a continuous linear functional on $\mathscr{H}$ lets us first move the limit outside the inner product, and then move the inner product under the integral. By Cauchy's Theorem, we may then derive the upper bound,

$$\leq \frac{1}{2\pi}\limsup_{n\to\infty}\Big|\int_{-n}^{n} e^{ist}\langle R(is,A)^2x,y\rangle ds\Big|$$
$$+\frac{1}{2\pi}\limsup_{n\to\infty}\Big|\int_{0}^{\omega} e^{(r+in)t}\langle R(r+in,A)^2x,y\rangle dr\Big|$$
$$+\frac{1}{2\pi}\limsup_{n\to\infty}\Big|\int_{0}^{\omega} e^{(r-in)t}\langle R(r-in,A)^2x,y\rangle dr\Big| = (\#)$$

and noting that for any $0\neq\lambda\in\mathbb{C}_+\cup i\mathbb{R}$ and $x\in D(A^2)\subseteq D(A)$, we must have

$$||R(\lambda,A)x|| = \frac{1}{|\lambda|}||\lambda R(\lambda,A)x|| = \frac{1}{|\lambda|}||R(\lambda,A)Ax-x|| \leq \frac{1}{|\lambda|}\Big(M||Ax||+||x||\Big)$$

which means that we may upper bound (#) by,

$$\begin{aligned}
(\#) &\leq \frac{1}{2\pi}\limsup_{n\to\infty}\int_{-n}^{n}|\langle R(is,A)^2x,y\rangle|ds + \limsup_{n\to\infty}\frac{\omega Me^{\omega t}||y||(M||Ax||+||x||)}{\pi n^2}\\
&= \frac{1}{2\pi}\limsup_{n\to\infty}\int_{-n}^{n}|\langle R(is,A)x,R(-is,A^*y\rangle|ds \qquad \Big([R(is,A)]^*=R(-is,A^*)\Big)\\
&\leq \frac{1}{2\pi}\Big|\Big|\,||R(is,A)x||\cdot||R(-is,A^*)y||\,\Big|\Big|_1\\
&\leq \frac{1}{2\pi}\Big|\Big|\,||R(is,A)x||\,\Big|\Big|_2\cdot\Big|\Big|\,||R(-is,A^*)y||\,\Big|\Big|_2 \qquad \text{Hölder's Inequality}\\
&= \frac{1}{2\pi}\Big[\int_{\mathbb{R}}||R(is,A)x||^2ds\Big]^{1/2}\Big[\int_{\mathbb{R}}||R(is,A^*)y||^2ds\Big]^{1/2}\\
&\leq \frac{1}{2\pi}\cdot\sqrt{(1+M\omega)^2\cdot L\cdot||x||^2}\cdot\sqrt{(1+M\omega)^2\cdot L\cdot||y||^2}\\
&= \frac{L(1+M\omega)^2||x||\cdot||y||}{2\pi}
\end{aligned}$$

and lastly, since $D(A^2)$ is dense in $\mathscr{H}$, we have that,

$$\begin{aligned}
||tT(t)|| &= \sup\{|\langle tT(t)x,y\rangle| \mid x,y\in D(A^2),\ ||x||=||y||=1\}\\
&\leq \frac{L(1+M\omega)^2}{2\pi}
\end{aligned}$$

which implies $\lim_{t\to\infty}||T(t)||=0$, so by the uniform stability theorem, we must have $\omega(A)<0$, as required. □

3 Spectral Theory of Semigroups and Generators

When A generates a strongly continuous semigroup on some Hilbert space, the location of $\sigma(A)$ is important in terms of Gearhart-Prüss. More precisely, $\sigma(A)$ needs to be contained by the strict left-half plane, $\mathbb{C}_-$, if there is to be any hope for uniform stability.

Definition. (Spectral Bound) Letting $A : D(A) \to X$ be a closed operator, it's spectral bound is the number $s(A) = \sup\{\Re(\lambda) \mid \lambda \in \sigma(A)\}$.

In the event that A generates a strongly continuous semigroup, the relation $s(A) \leq \omega(A)$ always holds, and we are especially interested in when equality persists, as this will allow us to investigate the stability of semigroup solutions in terms of the spectrum of A.

The spectrum of any closed operator can be decomposed according to exactly how $\lambda - A : D(A) \to X$ fails to be bijective. Specifically, when equality between the spectral and growth bounds fails, these decompositions help us to pinpoint precisely what has gone wrong.

Definition. (Point Spectrum) Letting $A : D(A) \to X$ be a closed operator, the subset of the spectrum given by,

$$P\sigma(A) = \{\lambda \in \sigma(A) \mid \lambda - A \text{ is not injective}\}$$

is called the point spectrum of A.

For every $\lambda \in P\sigma(A)$, we must have that $N[\lambda - A] \neq \emptyset$, so each such λ is, in this sense, a true eigenvalue of the linear map, A. Sometimes, we need to extend the notion of a true eigenvalue to the collection of complex numbers which are "almost eigenvalues."

Definition. (Approximate Point Spectrum) Letting $A : D(A) \to X$ be a closed operator, the subset of the spectrum given by,

$$A\sigma(A) = \{\lambda \in \sigma(A) \mid \lambda - A \text{ is not injective or } R[\lambda - A] \text{ is not closed in } X\}$$

is called the approximate point spectrum of A.

Clearly, $P\sigma(A) \subseteq A\sigma(A)$ always holds, and the following lemma makes it clear why the term "approximate" should be used for the larger set.

Lemma. *(Approximate PS Characterization) Letting $A : D(A) \to X$ be a closed operator, we have that $\lambda \in A\sigma(A)$ iff there exists a unit-norm sequence $(x_n)_n \subset D(A)$ such that $\lim_{n\to\infty} ||(\lambda - A)x_n|| = 0$.*

For any $\lambda \in \sigma(A)$ which is not an approximate eigenvalue, we must have that $\lambda - A$ is injective <u>and</u> $R[\lambda - A]$ is closed in X. This forces the range not to be a dense subset of X (else λ would be in the resolvent set), so we may collect the non-approximate members of the spectrum as follows.

Definition. (Residual Spectrum) Letting $A : D(A) \to X$ be a closed operator, the subset of the spectrum given by,

$$R\sigma(A) = \{\lambda \in \sigma(A) \mid R[\lambda - A] \text{ is not dense in } X\}$$

is called the residual spectrum of A.

Note that the approximate and residual spectra need not be disjoint, as any $\lambda \in R\sigma(A)$ may give rise to a non-injective $\lambda - A$, or a range, $R[\lambda - A]$, which is neither closed, nor dense in X. However, we do have $\sigma(A) = A\sigma(A) \cup R\sigma(A)$.

Theorem. *(Resolvent Spectral Mapping) Let $A : D(A) \to X$ be a closed operator with a nonempty resolvent set, $\rho(A)$, and fix $\lambda_0 \in \rho(A)$. Then,*

$$\sigma(R(\lambda_0, A)) \setminus \{0\} = \left\{ \frac{1}{\lambda_0 - \mu} \;\middle|\; \mu \in \sigma(A) \right\}$$

and moreover, this relation holds individually for the point, approximate point, and residual spectra of A and $R(\lambda_0, A)$.

Proof. Fixing $\lambda_0 \in \rho(A)$, we have that for each nonzero $\lambda \in \mathbb{C}$, and any $x \in X$,

$$\begin{aligned}
[\lambda - R(\lambda_0, A)]x &= \lambda\Big[1 - \frac{1}{\lambda}R(\lambda_0, A)\Big]x \\
&= \lambda\Big[\Big(\lambda_0 - \frac{1}{\lambda}\Big) - A\Big]R(\lambda_0, A)x
\end{aligned}$$

implying that if $\lambda \in P\sigma(R(\lambda_0, A))$, then $\lambda_0 - \frac{1}{\lambda} \in P\sigma(A)$, and hence,

$$\lambda = \frac{1}{\lambda_0 - (\lambda_0 - \frac{1}{\lambda})} \in \left\{ \frac{1}{\lambda_0 - \mu} \;\middle|\; \mu \in P\sigma(A) \right\}$$

which shows the $\subseteq$ containment for the point spectra. Conversely, given $\lambda_0 - \frac{1}{\lambda} \in P\sigma(A)$, then we must have $\lambda \in P\sigma(R(\lambda_0, A))$, so since

$$\frac{1}{\lambda_0 - (\lambda_0 - \frac{1}{\lambda})} = \lambda \in P\sigma(R(\lambda_0, A))$$

which shows the $\supseteq$ containment, so the desired relation will hold for the point spectra. Similar arguments show that the nonzero approximate point and residual spectra must satisfy similar relations, and this completes the proof. $\square$

Indeed, so-called "spectral mapping" results will be the key to showing the opposite inequality, $\omega(A) \leq s(A)$, which will allow us to investigate the stability of semigroup solutions through the lens of generator spectra. There are two main spectral mapping results.

Theorem. *(Spectral Inclusion One) Let $A : D(A) \to X$ generate the strongly continuous semigroup, $\{T(t)\}_{t\geq 0}$, on the Banach space X. Then, for each $t \geq 0$, the inclusion given by*

$$e^{t\sigma(A)} = \{e^{t\lambda} \mid \lambda \in \sigma(A)\} \subseteq \sigma(T(t))$$

holds, and further, the above inclusion holds when restricted to the point, approximate point, or residual parts of the spectrum.

Proof. First, we note that for any $\lambda \in \mathbb{C}$, the shifted semigroup $\{e^{-\lambda t}T(t)\}_{t\geq 0}$ is generated by $A - \lambda$. Then, by the Toolbox Lemma, we have that,

$$\begin{aligned}(e^{\lambda t} - T(t))x &= -(A-\lambda)\int_0^t e^{\lambda(t-s)}T(s)x\,ds && (x \in X)\\ &= \int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)x\,ds && (x \in D(A))\end{aligned}$$

and we now make the following observations:

1. Clearly, $\lambda \in P\sigma(A)$ implies $e^{t\lambda} \in P\sigma(T(t))$, so $e^{tP\sigma(A)} \subseteq P\sigma(T(t))$ holds.
2. In general, if $\lambda \in A\sigma(A)$, then the sequential characterization of approximate eigenvalues shows $e^{tA\sigma(A)} \subseteq A\sigma(A)$, and the point spectrum inclusion holds within this larger inclusion.
3. One can show that the norm-closure of $R[e^{t\lambda} - T(t)]$ is contained by the norm-closure of $R[\lambda - A]$, so if $\lambda \in R\sigma(A)$, then $e^{t\lambda} \in R\sigma(T(t))$ as well.

and hence,

$$\begin{aligned}e^{t\sigma(A)} &= e^{tA\sigma(A)} \cup e^{tR\sigma(A)}\\ &\subseteq A\sigma(T(t)) \cup R\sigma(T(t)) = \sigma(T(t))\end{aligned}$$

where we have that $e^{tP\sigma(A)} \subseteq P\sigma(T(t))$ within the approximate point spectrum inclusion as well, and this completes the proof. ☐

It turns out that once zero is removed from consideration (since e raised to any power is nonzero), the spectral inclusions for the point and residual spectra will actually hold with equality.

Theorem. *(Spectral Inclusion Two) For the generator, $A : D(A) \to X$, of the strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on the Banach space X, the identities,*

$$\begin{aligned}e^{tP\sigma(A)} &= P\sigma(T(t)) \setminus \{0\}\\ e^{tR\sigma(A)} &= R\sigma(T(t)) \setminus \{0\}\end{aligned}$$

will hold for each $t \geq 0$.

These spectral mapping results point towards our goal of showing that for the generator, A, of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, the relation

$$e^{t\sigma(A)} = \sigma(T(t)) \setminus \{0\} \qquad \text{(SMP)}$$

holds for every $t \geq 0$. When the SMP holds, we see that for $t_* > 0$,

$$\begin{aligned}r(T(t_*)) &= \max\left\{\sup\left\{|\lambda| \;\middle|\; \lambda \in \sigma(T(t_*)) \setminus \{0\}\right\}, 0\right\}\\ &\leq \sup\left\{e^{t_*|\mu|} \;\middle|\; \mu \in \sigma(A)\right\} \leq e^{t_* s(A)}\end{aligned}$$

where the penultimate inequality follows by the SMP. Now, taking the log of both sides and dividing by t_*, we get,

$$\omega(A) = \frac{\log r(T(t_*))}{t_*} \leq s(A)$$

and recalling that $\omega(A) \geq s(A)$ always holds, the SMP implies that the spectral bound and growth bound of the generator A must coincide. Lastly, one has, by Spectral Inclusion Two, that any failure in the SMP must be due to a nonzero approximate eigenvalue, $\lambda \in A\sigma(A) \setminus P\sigma(A)$.

4 Additional Topics: Analytic Semigroups, Fractional Powers, and Asymptotics Revisited

So far, we have focused primarily on the properties of generic strongly continuous semigroups, briefly touching on uniformly continuous semigroups as necessary. In terms of regularity, the so-called analytic semigroups fall between strong and uniform continuity, and arising naturally from the concept of sectorial operators, they are of independent interest with respect to asymptotics.

Definition. (Sectorial Operators) A closed and densely-defined linear operator, $A : D(A) \to X$, is said to be sectorial (of angle δ) if there exists a $\delta \in (0, \frac{\pi}{2}]$ such that

$$\Sigma_{\frac{\pi}{2}+\delta} := \left\{ \lambda \in \mathbb{C} \;\middle|\; 0 \leq |\arg(\lambda)| < \frac{\pi}{2} + \delta \right\} \setminus \{0\}$$

is a subset of the resolvent set, $\rho(A)$, and if, for every $\varepsilon \in (0, \delta)$, the estimate

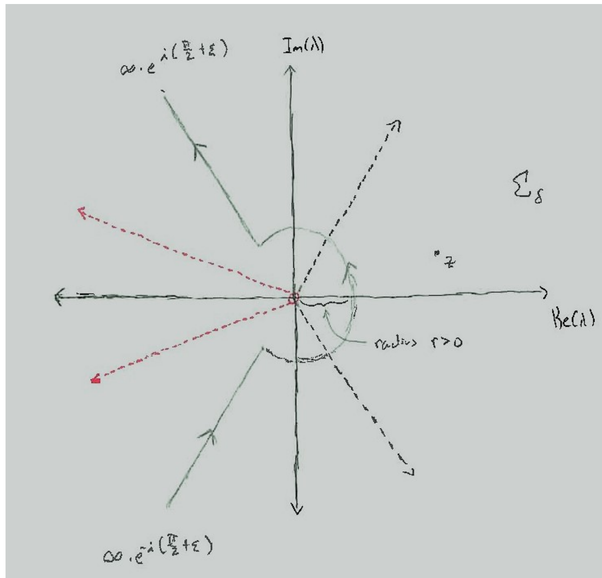
$$||R(\lambda, A)|| \leq \frac{M_\varepsilon}{|\lambda|}$$

holds for some $M_\varepsilon \geq 1$ and every $0 \neq \lambda \in \overline{\Sigma}_{\frac{\pi}{2}+\varepsilon}$.

Given a sectorial operator of angle δ, we may define a one-parameter family of operators on $\Sigma_\delta \cup \{0\}$ as follows:

$$\mathcal{T} := \begin{cases} T(0) := I \\ T(z) := \frac{1}{2\pi i} \int_\gamma e^{\lambda z} R(\lambda, A) d\lambda \qquad z \in \Sigma_\delta \end{cases}$$

where, for a fixed $\varepsilon \in (|\arg(z)|, \delta)$, the piecewise smooth curve γ is as depicted on the next page,



and where we observe that for any $z \in \Sigma_\delta$, there will always exist such an ε, and where among all suitable ε, we may, by analyticity, choose the radius $r > 0$ to be any number we wish. Indeed, the value of $T(z)$, if it exists, will be invariant among suitable ε and positive radii as a consequence of Cauchy's Theorem. This is now the task at hand - to show that $T(z)$ is well-defined for each $z \in \Sigma_\delta$. We estimate the three components of γ as follows, first:

$$\begin{aligned} \left|\left|\frac{1}{2\pi i}\int_{\gamma_1} e^{\lambda z} R(\lambda, A) d\lambda\right|\right| &\leq \frac{1}{2\pi}\int_{-\infty}^{-r} \left|e^{-s|z|e^{i(\arg(z)-(\frac{\pi}{2}+\varepsilon))}}\right| \cdot \frac{M_\varepsilon}{|s|} ds \\ &= \frac{M_\varepsilon}{2\pi}\int_{r}^{\infty} \left|e^{u|z|\sin(\arg(z)-\varepsilon)}\right| \cdot \frac{1}{|u|} du \qquad u = -s \\ &= \frac{M_\varepsilon}{2\pi}\int_{r}^{\infty} \frac{1}{|u| e^{u|z|\sin(\varepsilon - \arg(z))}} du < \infty \end{aligned}$$

where the convergence happens because $0 < \varepsilon - \arg(z) < \pi$, so it follows that $0 < \sin(\varepsilon - \arg(z)) < 1$. Now, the integral along the other unbounded portion of γ will converge absolutely by a similar argument, and the integral along the circle of radius r is just a proper Riemann integral (so it is absolutely convergent already). To summarize, we have not only shown that $T(z)$ is well-defined for each $z \in \Sigma_\delta$, but that $\mathcal{T} \subset \mathscr{L}(X)$ as well.

Theorem. *(Properties of $\mathcal{T}$) Let $A : D(A) \to X$ be a sectorial operator of angle δ and let $\mathcal{T} \subset \mathscr{L}(X)$ be the family of operators defined as above. Then, we have that $z \mapsto ||T(z)||$ is analytic on Σ_δ, uniformly bounded on $\Sigma_{\delta'} \cup \{0\}$ for any $\delta' \in (0, \delta)$, and strongly continuous for any such δ' in the sense that,*

$$\lim_{\substack{z \to 0 \\ z \in \Sigma_{\delta'} \cup \{0\}}} T(z)x = x \qquad (x \in X)$$

and lastly, that $T(z+w) = T(z)T(w)$ for all $z, w \in \Sigma_\delta$.

Proof. Fixing $\delta' \in (0, \delta)$, we see that $z \mapsto T(z)$ is continuous on the compact set $\overline{\Sigma}_{\delta'} \cap \overline{B}_1(0)$, and by the form of the estimate for absolute convergence, uniformly bounded on $\overline{\Sigma}_{\delta'} \cap B_1(0)^c$. In other words, $z \mapsto T(z)$ must be uniformly bounded on $\overline{\Sigma}_{\delta'} \supset \Sigma_{\delta'} \cup \{0\}$, which proves the second claim.

Now, fixing $z_0 \in \Sigma_\delta$, there exists a closed ball $\overline{B}_\rho(z_0) \subset \Sigma_\delta$ so that, for any $z \in \overline{B}_\rho(z_0)$, we may express the (analytic) integrand as a power series to get,

$$\begin{aligned} T(z) &= \frac{1}{2\pi i}\int_\gamma \Big[\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}(z-z_0)^i \sum_{j=0}^{\infty}(z_0 - z)^j R(z_0, A)^{j+1}\Big] d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma \sum_{k=0}^{\infty} \Big[\sum_{l=1}^{k} \frac{\lambda^l}{l!} R(z_0, A)^{k-1}\Big](z-z_0)^k d\lambda \end{aligned}$$

and since the integrand converges uniformly on $\overline{B}_\rho(z_0)$, we may pass the integral through both summations to see that $T(z)$ may be locally expanded as a power series about z_0, and hence, must be analytic on Σ_δ.

For strong continuity, it suffices to show the required identity on $D(A)$, which is dense in X by definition. Fixing $x \in D(A)$, suppose that $(z_n)_n \subset \Sigma_{\delta'} \cup \{0\}$ with $z_n \to 0$ as $n \to \infty$, and consider

$$T(z_n)x - x = \frac{1}{2\pi i}\int_\gamma e^{z_n\lambda} R(\lambda, A) x d\lambda - \frac{1}{2\pi i}\int_\gamma \frac{e^{z_n \lambda}}{\lambda} x d\lambda = (\#)$$

where we have used the identity, $1 = \int_\Gamma \frac{e^{z_n\lambda}}{\lambda} d\lambda$, for the closed curve,

$$\begin{aligned} \Gamma := \{-se^{-i(\frac{\pi}{2}+\varepsilon)} \mid -N \le s \le 1\} &\cup \{e^{i\theta} \mid -(\frac{\pi}{2}+\varepsilon) \le \theta \le \frac{\pi}{2}+\varepsilon\} \\ &\cup \{se^{i(\frac{\pi}{2}+\varepsilon)} \mid 1 \le s \le N\} \cup \{\Re(Ne^{i(\frac{\pi}{2}+\varepsilon)}) + is \mid -N \le s \le N\} \end{aligned}$$

and taking $N \to \infty$, we are left with $\int_\gamma \frac{e^{z_n\lambda}}{\lambda} d\lambda = 1$. Returning to our original problem, we may then write,

$$\begin{aligned} (\#) &= \frac{1}{2\pi i}\int_\gamma e^{z_n\lambda}\Big(R(\lambda, A) - \frac{1}{\lambda}\Big) x d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma \frac{e^{z_n\lambda}}{\lambda} R(\lambda, A) A x d\lambda \end{aligned}$$

where we are recalling (from Hille-Yosida) that $\lambda R(\lambda, A)x - x = AR(\lambda, A)x = R(\lambda, A)Ax$ for $x \in D(A)$. We may then estimate,

$$\Big|\Big|\frac{e^{z_n\lambda}}{\lambda} R(\lambda, A)Ax\Big|\Big| \le \frac{M_\varepsilon ||Ax||}{|\lambda|^2}|e^{z_n\lambda}| < \infty$$

where, taking $r = 1$ along γ, this integrand will be bounded above by the same integrand showing $T \in \mathcal{T}$ to be well-defined. Regarding these integrals as being of Bochner type, we may then apply dominated convergence to get,

$$\lim_{n\to\infty}[T(z_n)x - x] = \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda}R(\lambda, A)Axd\lambda = 0$$

where the equality on the RHS follows by a similar argument to our earlier use of the residue theorem, except we are now closing sections of γ on the right with circles of increasing radius, instead of vertical lines on the left. As the sequence $(z_n)_n \subset \Sigma_{\delta'} \cup \{0\}$ was arbitrary, the desired strong continuity of $\mathcal{T}$ now follows. Lastly, the semigroup property of $\mathcal{T}$ follows by arguments of a similar flavor. □

Having established the strongly continuous one-parameter family $\mathcal{T} \subset \mathscr{L}(X)$ which satisfies the semigroup property, it should be noted that a "usual" strongly continuous semigroup is embedded within this family. Perhaps not surprisingly, the sectorial operator $A : D(A) \to X$, from which $\mathcal{T}$ is defined, is actually the generator of this embedded semigroup. This yields two important ideas:

1. Any sectorial operator of angle $\delta \in (0, \frac{\pi}{2}]$ generates a strongly continuous semigroup which can be extended analytically to a sector containing the non-negative real numbers.

2. Verifying the sectoriality of an operator does not require checking infinitely-many conditions, as would be required by Hille-Yosida.

The family of operators $\mathcal{T}$ can be called an analytic semigroup in the sense that it satisfies the conditions of the above theorem.

Definition. (Analytic Semigroup) For a fixed $\delta \in (0, \frac{\pi}{2}]$, a family of bounded linear operators, $\mathcal{T} := \{T(z)\}_{z\in\Sigma_\delta} \subset \mathscr{L}(X)$, is called an analytic semigroup (of angle δ) if,

1. $T(0) = I$ and for all $z, w \in \Sigma_\delta$, $T(z + w) = T(z)T(w)$.

2. The map $z \mapsto T(z)$ is analytic on Σ_δ

3. For any $\delta' \in (0, \delta)$, the strong continuity condition, given by,

$$\lim_{\substack{z\to 0 \\ z\in\Sigma_{\delta'}\cup\{0\}}} T(z)x = x$$

   holds for every $x \in X$.

4. For any $\delta' \in (0, \delta)$, the map $z \mapsto T(z)$ is uniformly bounded on $\Sigma_{\delta'} \cup \{0\}$.

To summarize: any sectorial operator will give rise to an analytic semigroup of the form $\mathcal{T}$, and generate the embedded strongly continuous semigroup. As a final note, there are a variety of conditions which are equivalent to the sectoriality of an operator, see [2, 4].

The connection between analytic semigroups and asymptotics is established by the notion of fractional powers of a closed operator. Indeed, if $A : D(A) \to X$

is a sectorial operator of angle $\delta \in (0, \frac{\pi}{2}]$ and $0 \in \rho(A)$, then for any $\alpha > 0$, we define the operator,

$$A^{-\alpha} := \frac{1}{2\pi i} \int_{\gamma} \lambda^{-\alpha} R(\lambda, A) d\lambda$$

where γ runs from $\infty e^{-i\delta'}$ to $\infty e^{i\delta'}$ for a given $\delta' \in (0, \delta)$, and avoids $\mathbb{R}_+ \cup \{0\}$ by traversing a circle of sufficiently small radius to the left of zero. Additionally, for this definition to be unambiguous, we specify,

$$\lambda^{-\alpha} := e^{-\alpha \log(\lambda)}$$

where the logarithm is the principal value logarithm of $\lambda \neq 0$, that is, $\log(\lambda) = \log|\lambda| + i \arg(\lambda)$.

This definition of $A^{-\alpha}$ actually holds for a general class of closed operators which satsify certain conditions on their resolvent sets and on the norms of their resolvent operators. In our present situation (where A is sectorial), we have the equivalent representation,

$$B^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} T(t) dt \qquad B := -A$$

where $T(t)$ is the embedded strongly continuous semigroup within the analytic semigroup defined by A (see [4, 2.6]).

Let us now turn to the useful asymptotics results one may find in [1]. To be specific, the following result will be our starting point.

Theorem. *Let $\{T(t)\}_{t\geq 0}$ be a bounded and strongly continuous semigroup on a Banach space, X, with an invertible generator, $A : D(A) \to X$. If, in addition, $i\mathbb{R} \subset \rho(A)$, then we have*

$$||T(t)A^{-1}|| \to 0 \qquad (t \to \infty)$$

In other words, for any $y \in D(A)$, with $y = Ax$ for some $x \in D(A)$, the solution

$$u(t) := T(t)y$$

to the ACP converges to 0 as $t \to \infty$.

We are interested in examining the rate of this solution decay. In particular, let us define the following functions,

$$M(\eta) := \max_{t\in[-\eta,\eta]} ||R(it, A)|| \qquad (\eta \geq 0)$$

which is continuous, and obviously non-decreasing. Then, we write an associated function,

$$M_{\log}(\eta) := M(\eta)\Big[\log(1 + M(\eta)) + \log(1 + \eta)\Big] \qquad (\eta \geq 0)$$

which is clearly also continuous, and strictly increasing as well. To be specific, $M_{\log}$ satisfies these properties on $[0, \infty)$, and accordingly, has an inverse, $M_{\log}^{-1}$, defined on $[M_{\log}(0), \infty)$. This leads to the Batty-Duyckaerts Theorem, which puts a specific rate on the solution decay from the above theorem.

Theorem. *(Batty-Duyckaerts) Let* $\{T(t)\}_{t\geq 0}$ *be a bounded and strongly continuous semigroup on a Banach space,* X*, with an invertible generator,* $A : D(A) \to X$*. If, in addition,* $i\mathbb{R} \subset \rho(A)$*, and* M *and* $M_{\log}$ *are defined as above, then there must exist constants* $B, C \geq 0$ *such that*

$$||T(t)A^{-1}|| \leq \frac{C}{M_{\log}^{-1}(\frac{t}{C})} \qquad (t \geq B)$$

and in particular, given some $\alpha > 0$*, with* $M(\eta) \leq C(1 + \eta^{\alpha})$*, then (via some algebra) the above estimate becomes*

$$||T(t)A^{-1}|| \leq C\Big(\frac{\log(t)}{t}\Big)^{\frac{1}{\alpha}} \qquad (t \geq B)$$

and this estimate is essentially sharp, in the sense that it cannot be improved in arbitrary Banach spaces, but it can be made better if the underlying space happens to be a Hilbert space.

Let us now discuss this type of semigroup decay in more detail, for which we will need a few lemmas found in [1].

Lemma. *(Bounded Semigroups on a Hilbert Space) Let* $\{T(t)\}_{t\geq 0}$ *be a strongly continuous semigroup on a Hilbert space,* $\mathscr{H}$*, with generator* $A : D(A) \to \mathscr{H}$*. Then,* $\{T(t)\}_{t\geq 0}$ *is bounded (i.e.* $||T(t)|| \leq M$*) if and only if* $\mathbb{C}_+ \subset \rho(A)$ *and*

$$\sup_{\zeta>0} \zeta \int_{\mathbb{R}} \Big(||R(\zeta + i\eta, A)x||^2 + ||R(\zeta + i\eta, A^*x||^2\Big) d\eta < \infty$$

for every $x \in \mathscr{H}$*.*

Proof. Let us assume first that $\{T(t)\}_{t\geq 0}$ is bounded so $||T(t)|| \leq M$ for each $t \geq 0$. Then, as $\omega(A) \leq 0$, we must have $\mathbb{C}_+ \subset \rho(A)$ automatically, and for $x \in \mathscr{H}$, we may write

$$\begin{aligned} R(\zeta + i\eta, A)x &= \int_0^{\infty} e^{-(\zeta+i\eta)t} T(t)x dt \\ &= \int_{\mathbb{R}} e^{-i\eta t} T_{\zeta}(t) x dt \end{aligned}$$

where $T_{\zeta}(t)$ is defined as in $T_{\omega}(t)$ in the proof of Gearhart-Prüss. Then, by Plancharel's Theorem, it follows that

$$\int_{\mathbb{R}} ||R(\zeta + i\eta, A)x||^2 d\eta = 2\pi \int_0^{\infty} ||e^{-\zeta} T(t)x|| dt \leq \frac{C_1}{\zeta}$$

and a similar estimate holds so that

$$\int_{\mathbb{R}} ||R(\zeta + i\eta, A^*)x||^2 d\eta \leq \frac{C_2}{\zeta}$$

and hence for any $\zeta > 0$ and $x \in \mathscr{H}$, we have

$$\zeta \int_{\mathbb{R}} \Big(||R(\zeta + i\eta, A)x||^2 + ||R(\zeta + i\eta, A^*)x||^2 \Big) d\eta \le \zeta \Big(\frac{C_1 + C_2}{\zeta} \Big) = C_1 + C_2 < \infty$$

which proves the desired result upon taking the supremum over positive ζ.

Conversely, let us assume that $\mathbb{C}_+ \subset \rho(A)$ and that

$$\sup_{\zeta > 0} \zeta \int_{\mathbb{R}} \Big(||R(\zeta + i\eta, A)x||^2 + ||R(\zeta + i\eta, A^*)x||^2 \Big) d\eta < \infty$$

for every $x \in \mathscr{H}$, and we'll try to show that $\{T(t)\}_{t \ge 0}$ is a bounded semigroup. Indeed, for $x \in \mathscr{H}$ and $\lambda \in \mathbb{C}_+$, let us write

$$\lambda \mapsto R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t) x dt$$

Then, we may (complex) differentiate both sides with respect to λ to get

$$\begin{aligned} -R(\lambda, A)^2 x &= \frac{d}{d\lambda} \int_0^\infty e^{-\lambda t} T(t) x dt \\ &= -\int_0^\infty t e^{-\lambda t} T(t) x dt = -\mathcal{L}(tT(t)x) \end{aligned}$$

where $\mathcal{L}$ is the Laplace Transform. Applying the inverse Laplace Transform, $\mathcal{L}^{-1}$, and setting $\zeta := \Re(\lambda)$, we get that

$$\begin{aligned} t \mapsto tT(t)x &= \mathcal{L}^{-1}(R(\lambda, A)^2 x) \\ &= \frac{1}{2\pi i} \int_{-\infty}^\infty e^{(\zeta + is)t} R(\zeta + is, A)^2 x ds \end{aligned}$$

so for a fixed $t \ge 0$, let us now divide by t and for an arbitrary $h \in \mathscr{H}$, consider

$$\begin{aligned} \langle T(t)x, h \rangle &= \Big\langle \frac{1}{2\pi i t} \int_{-\infty}^\infty e^{(\zeta + is)t} R(\zeta + is, A)^2 x ds, h \Big\rangle \\ &= \frac{1}{2\pi i t} \int_{-\infty}^\infty e^{(\zeta + is)t} \langle R(\zeta + is, A)^2 x, h \rangle ds \end{aligned}$$

where we may move the inner product under the integral sign just as in the proof of Gearhart-Prüss. Now, by Cauchy-Schwarz and a special case of Young's Inequality ($2ab \le a^2 + b^2$), we get that

$$\begin{aligned} |\langle T(t)x, h \rangle| &\le \frac{e^{\zeta t}}{2\pi t} \int_{-\infty}^\infty |\langle R(\zeta + is, A)x, R(\zeta - is, A^*)h \rangle| ds \\ &\le \frac{e^{\zeta t}}{2\pi t} \int_{-\infty}^\infty ||R(\zeta + is, A)x|| \cdot ||R(\zeta - is, A^*)h|| ds \\ &\le \frac{e^{\zeta t}}{4\pi t} \int_{-\infty}^\infty \Big(||R(\zeta + is, A)x||^2 + ||R(\zeta - is, A^*)h||^2 \Big) ds \end{aligned}$$

Then, since we were free to choose $\zeta := \Re(\lambda) = \frac{1}{t}$, we may write

$$|\langle T(t)x, h\rangle| \leq C\zeta \int_{-\infty}^{\infty} \Big(||R(\zeta + is, A)x||^2 + ||R(\zeta - is, A^*)h||^2 \Big) ds$$

which, by some algebra and our supremum assumption is finite. Thus, $\{T(t)\}_{t\geq 0}$ is a bounded semigroup by the uniform boundedness principle. □

Rather than assuming the invertibility of A as in the previous two theorems of this section, we may just define its fractional powers as a closed operator, and this yields the following idea.

Lemma. *(Bounded Semigroups on a Hilbert Space 2) Let $\{T(t)\}_{t\geq 0}$ be a bounded, strongly continuous semigroup on a Hilbert space, $\mathscr{H}$, with generator $A : D(A) \to \mathscr{H}$. Then, if $i\mathbb{R} \subset \rho(A)$, we have that for a fixed $\alpha > 0$,*

$$||R(\lambda, A)(-A)^{-\alpha}|| \leq C \qquad (\Re(\lambda) > 0)$$

if and only if

$$||R(is, A)|| = O(|s|^{\alpha})$$

Proof. It has already been shown (see [1] references) that

$$||R(\lambda, A)|| \leq C(1 + |\lambda|^{\alpha}) \qquad (0 < \Re(\lambda) < 1)$$

holds if and only if the condition

$$||R(\lambda, A)(-A)^{\alpha}|| < C_1 \qquad (0 < \Re(\lambda) < 1)$$

holds as well. Then, let us define the function

$$F(\lambda) := R(\lambda, A)\lambda^{-\alpha}\Big(1 - \frac{\lambda^2}{B^2}\Big) \qquad B \gg 1$$

where $\lambda^{-\alpha} = e^{-\alpha \log(\lambda)}$ with the complex logarithm taking its principle value, and with λ being a member of the set,

$$D := \{\lambda \in \mathbb{C} \mid \Re(\lambda) \geq 0 \text{ and } 1 \leq |\lambda| \leq B\}.$$

We observe that ∂D consists of three individual pieces, namely,

1. $D_1 = \{e^{i\theta} \mid -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$
2. $D_2 = \{is \mid 1 \leq s \leq B\}$
3. $D_3 = \{Be^{i\theta} \mid -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$

and then, applying the maximum principle to F, we need only worry about it's value on the boundary, which, by the form of F, implies the desired result. □

Let us now state and prove the central result of Borichev and Tomilov (as stated in [1])

Theorem. *(Borichev and Tomilov) Let* $\{T(t)\}_{t\geq 0}$ *be a bounded, strongly continuous semigroup on a Hilbert space,* $\mathscr{H}$ *with generator* A *such that* $i\mathbb{R} \subset \rho(A)$*. Then, for a fixed* $\alpha > 0$*, the following conditions are equivalent:*

1. $||R(is, A)|| = O(|s|^{\alpha})$ *as* $s \to \infty$
2. $||T(t)(-A)^{-\alpha}|| = O(t^{-1})$ *as* $t \to \infty$
3. $||T(t)(-A)^{-\alpha}|| = o(t^{-1})$ *as* $t \to \infty$
4. $||T(t)A^{-1}|| = O(t^{-1/\alpha})$ *as* $t \to \infty$
5. $||T(t)A^{-1}x|| = o(t^{-1/\alpha})$ *as* $t \to \infty$

Proof. Every statement besides *1.* $\implies$ *3.* has been previously shown (see [1] references), so let us tackle this remaining implication. Write

$$\mathcal{H} := \mathscr{H} \oplus \mathscr{H}$$

and define the operator

$$\mathcal{A} := \begin{pmatrix} A & (-A)^{-\alpha} \\ O & A \end{pmatrix}$$

on the diagonal domain $D(\mathcal{A}) := D(A) \oplus D(A)$. Then, it is not hard to see that $\sigma(A) = \sigma(\mathcal{A})$, and for each $\lambda \in \rho(\mathcal{A})$, that the resolvent operator takes the form

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, A) & R^2(\lambda, A)(-A)^{-\alpha} \\ O & R(\lambda, A) \end{pmatrix} \qquad \lambda \in \rho(\mathcal{A}) = \rho(A)$$

Next, let us define the one-parameter family of operators

$$\mathcal{T}(t) := \begin{pmatrix} T(t) & tT(t)(-A)^{\alpha} \\ O & T(t) \end{pmatrix}$$

which, as it is easy to see, is a strongly continuous semigroup on $\mathcal{H}$. More interesting is its generator. Indeed, consider $x := (x_1, x_2) \in D(\mathcal{A})$, and write

$$\begin{aligned}
\frac{\mathcal{T}(h)x - x}{h} &= \frac{\begin{pmatrix} T(h) & hT(h)(-A)^{-\alpha} \\ O & T(h) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}{h} \\
&= \frac{\begin{pmatrix} T(h)x_1 + hT(h)(-A)^{-\alpha}x_2 \\ T(h)x_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}{h} \\
&= \begin{pmatrix} \frac{T(h)x_1 - x_1}{h} + T(h)(-A)^{-\alpha}x_2 \\ \frac{T(h)x_2 - x_2}{h} \end{pmatrix} \to \begin{pmatrix} Ax_1 + (-A)^{-\alpha}x_2 \\ Ax_2 \end{pmatrix} \qquad (h \to 0^+)
\end{aligned}$$

In other words, $\{\mathcal{T}(t)\}_{t\geq 0}$ is a strongly continuous semigroup whose generator takes the form of $\mathcal{A}$ acting on the diagonal domain $D(\mathcal{A})$ as defined above. By our assumption *1.* and the fact that $i\mathbb{R} \subset \rho(A) = \rho(\mathcal{A})$, it follows by the second lemma that

$$||R(\lambda, A)(-A)^{-\alpha}|| \leq C \qquad (\Re(\lambda) > 0)$$

which then implies for $x = (x_1, x_2) \in \mathcal{H}$ and $\lambda \in \mathbb{C}_+$ that

$$\begin{aligned}
||R(\lambda, \mathcal{A})x||^2 &= \left\| \begin{pmatrix} R(\lambda, A) & R^2(\lambda, A)(-A)^{-\alpha} \\ O & R(\lambda, A) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|^2 \\
&\leq ||R(\lambda, A)x_1 + R^2(\lambda, A)(-A)^{-\alpha}x_2||^2 + ||R(\lambda, A)x_2||^2 \\
&\leq (||R(\lambda, A)x_1|| + ||R(\lambda, A)(-A)^{\alpha}x_2||)^2 + ||R(\lambda, A)x_2||^2 \\
&\leq (||R(\lambda, A)x_1|| + C||R(\lambda, A)x_2||)^2 + ||R(\lambda, A)x_2||^2 = (\#)
\end{aligned}$$

where the $C \geq 0$ follows by commutativity and the bound for $||R(\lambda, A)(-A)^{\alpha}||$. Then, we have

$$\begin{aligned}
(\#) &= ||R(\lambda, A)x_1||^2 + 2C||R(\lambda, A)x_1|| \cdot ||R(\lambda, A)x_2|| + (C^2 + 1)||R(\lambda, A)x_2||^2 \\
&\leq ||R(\lambda, A)x_1||^2 + 2C\Big(\frac{||R(\lambda, A)x_1||^2}{2} + \frac{||R(\lambda, A)x_2||^2}{2}\Big) + (C^2 + 1)||R(\lambda, A)x_2||^2
\end{aligned}$$

which results from Young's Inequality in the case $p = q = 2$. Finally, we have that the above may be bounded as

$$\begin{aligned}
&\leq ||R(\lambda, A)x_1||^2 + C||R(\lambda, A)x_1||^2 + C||R(\lambda, A)x_2||^2 + (C^2 + 1)||R(\lambda, A)x_2||^2 \\
&= (1 + C)||R(\lambda, A)x_1||^2 + (C^2 + C + 1)||R(\lambda, A)x_2||^2 \\
&\leq \max\{1 + C, C^2 + C + 1\} \cdot (||R(\lambda, A)x_1||^2 + ||R(\lambda, A)x_2||^2)
\end{aligned}$$

In particular, performing the same estimate for $||R(\lambda, \mathcal{A}^*)x||^2$, we have that for each $x = (x_1, x_2) \in \mathcal{H}$ and $\lambda \in \mathbb{C}_+$,

$$||R(\lambda, \mathcal{A})x||^2 \leq C_1(||R(\lambda, A)x_1||^2 + ||R(\lambda, A)x_2||^2) < \infty$$

and

$$||R(\lambda, \mathcal{A}^*)x||^2 \leq C_2(||R(\lambda, A^*)x_1||^2 + ||R(\lambda, A^*)x_2||^2) < \infty$$

by the first of the above lemmas, since $\{T(t)\}_{t\geq 0}$ is a bounded semigroup. Using this lemma again, we see that $\mathcal{T}$ must be a bounded semigroup as well. By definition, this means

$$\sup_{t\geq 0} ||tT(t)(-A)^{-\alpha}|| < \infty$$

and by the density of $D(\mathcal{A})$ in $\mathcal{H}$, along with the first theorem of this decay section, it follows that

$$||tT(t)(-A)^{-\alpha}|| = o(1) \qquad (t \to \infty \text{ and } x \in \mathscr{H})$$

which completes the proof. □

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