In the next three sections, we discuss conical sections, starting with ellipses.

Conical Sections

- Ellipse
- Hyperbola
- Parabola
Ellipses

- **Geometric definition:** The set of points whose sum of distances from two points (each called a focus and together called **Foci**) is constant.

- **Using the geometric definition to find a formula**

Note that in the picture the sum of distances between the vertex \((a, 0)\) from each of the two foci is \(2a\) so the sum of the distances between any point on the ellipse and each of the foci should be equal to \(2a\).

\[
\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a
\]

Manipulate the same way you would when solving equations with two radicals. Solving for \(x\) or \(y\) renders the same answer:

\[
\Rightarrow \quad \sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}
\]

Isolate one radical:

\[
(x - c)^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2
\]

Raise to Power 2:

\[
-4xc = 4a^2 - 4a\sqrt{(x + c)^2 + y^2}
\]

Use binomial expansion and simplify:

\[
a\sqrt{(x + c)^2 + y^2} = a^2 + cx
\]

Factor 4 and isolate the radical:

\[
a^2(x + c)^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2
\]

Raise to power 2 again:

\[
a^2x^2 + a^2c^2 + a^2y^2 = a^4 + c^2x^2
\]

Simplify:

\[
(a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2
\]

Isolate the terms with variables:

\[
b^2x^2 + a^2y^2 = a^2b^2
\]

Denote \(a^2 - c^2\) by \(b^2\):

\[
\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
• Graphs of ellipses where axis are vertical or horizontal

Case: \(a > b\)
Standard Equation: \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\)

Case: \(b > a\)
Standard Equation: \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\)

Case: \(a > b\) and center \((h, k)\)
Standard Equation: \(\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1\)

Case: \(b > a\) and center \((h, k)\)
Standard Equation: \(\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1\)

• How to find different parameters for an ellipse using its equation:

1. If the equation is anything other than the above equations, reformat to one of the above.
2. Use the equation to find the horizontal and vertical line containing each axis. If center is \((0, 0)\), then the x and y axis contain the ellipse’s axis. If the center is \((h, k)\), then lines \(x = h\) and \(y = k\) contain the axis of ellipse.
3. To find end points of the horizontal axis, plug in \(y = 0\) or \(y = k\) and solve for \(x\).
4. To find the end points of vertical axis, plug in \(x = 0\) or \(x = h\) and solve for \(y\).
5. Find \(c\) using the equation \(c^2 = |a^2 - b^2|\).

Equation of a Circle

By setting \(a = b\) in any of standard equations of ellipse, equation of a circle is obtained.
1. Write each of the following ellipse equations in standard form and find $a$ and $b$. Then find $c$.

   (a) $25x^2 + 4y^2 = 100$  
   (b) $4(x - 3)^2 + 36(y - 5)^2 = 36$  
   (c) $25x^2 + 4y^2 = 1$

2. Use completing the square to find the standard equation of each of the following ellipses, then find the center of each ellipse.

   (a) $4x^2 - 24x + 9y^2 + 18y + 9 = 0$  
   (b) $3x^2 - 12x + 4y^2 - 8y + 8 = 0$
3. Which of the following is an equation for an ellipse with foci (0, 3) and (0, −3) and major axis of length 10?

(a) \( \frac{x^2}{16} + \frac{y^2}{100} = 1 \)  
(b) \( \frac{x^2}{25} + \frac{y^2}{16} = 1 \)  
(c) \( \frac{x^2}{9} + \frac{y^2}{25} = 1 \)  
(d) \( \frac{x^2}{16} + \frac{y^2}{25} = 1 \)

4. Sketch the graph of the ellipse \( \frac{x^2}{36} + \frac{y^2}{16} = 1 \). Label the vertices and foci.
5. Find the foci and vertices of the following ellipses.

(a) \( \frac{x^2}{9} + \frac{y^2}{16} = 1 \)

(b) \( \frac{(x - 1)^2}{9} + \frac{(y + 3)^2}{16} = 1 \)

(c) \( 28x^2 + 4y^2 = 7 \)
6. Find the standard equation of an ellipse where $c = \sqrt{7}/2$, foci are on y-axis and the length of major axis is 4.

7. Throughout this problem, let $P$ be the point $(2, -4)$ and $Q$ be the point $(-8, 6)$.

   (a) Find the midpoint of the line segment connecting $P$ and $Q$. (Note that x-value of the midpoint is the average of x-values. Y-value of the midpoint is the average of y-values.)

   (b) Find the equation of a circle with $P$ and $Q$ as endpoints of a diameter.

   (c) Find an equation for the line passing through $P$ and $Q$.

   (d) Find an equation for the line perpendicular to the line from part (c) and passing through $(0, 4)$. 
8. Find the distance, d, between a point on the graph of \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) and point \((0, 0)\) as a function of \(x\).

9. Find the area of a rectangle circumscribed in \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) as a function of \(x\).

Watch how the area is changing here: [https://ggbm.at/dbdd5az7](https://ggbm.at/dbdd5az7)

10. Use parameterization to find the maximum area of a rectangle circumscribed in \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \). Find the dimensions of such rectangle.
11. Fill in the blank to match each nonlinear system with one of the figures. Do NOT solve the systems.

**Figure:** ________ \[\begin{align*}
&\begin{cases}
  x^2 + y^2 = 9 \\
  \frac{x^2 + y^2}{16} = 1
\end{cases} \\
&\begin{cases}
  x^2 + y^2 = 9 \\
  \frac{(x - 0.5)^2}{6.25} + \frac{y^2}{4} = 1
\end{cases}
\]

**Figure:** ________ \[\begin{align*}
&\begin{cases}
  x^2 + y^2 = 9 \\
  \frac{(x - 1)^2}{16} + \frac{y^2}{4} = 1
\end{cases} \\
&\begin{cases}
  x^2 + y^2 = 9 \\
  \frac{x^2}{4} + \frac{y^2}{9} = 1
\end{cases}
\]

(A) (One solution.) (B) (Two solutions.) (C) (Three solutions.) (D) (Four solutions.)