10.2: Hyperbolas

- Geometric definition: The set of points whose difference in distances from two points (called Foci) is constant.

- Using the geometric definition to find a formula

Note that in the picture the difference in distances between the vertex \((a, 0)\) and each of the foci is \(2a\) so the differences in the distances between any point on the ellipse and each of the foci should be equal to \(2a\).

\[
\sqrt{(x - c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} = 2a
\]

Manipulate the same way you would when solving equations with two radicals. Solving for \(x\) or \(y\) renders the same answer:

- Isolate one radical:
  \[
  \sqrt{(x - c)^2 + y^2} = 2a + \sqrt{(x + c)^2 + y^2}
  \]
- Raise to power 2:
  \[
  (x - c)^2 + y^2 = 4a^2 + 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2
  \]
- Use binomial expansion and simplify:
  \[
  0 = 4a^2 + 4a\sqrt{(x + c)^2 + y^2} + 4xc
  \]
- Factor 4 and isolate the radical:
  \[
  -a\sqrt{(x + c)^2 + y^2} = a^2 + cx
  \]
- Raise to power 2 again:
  \[
  a^2(x + c)^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2
  \]
- Simplify:
  \[
  a^2x^2 + a^2c^2 + a^2y^2 = a^4 + c^2x^2
  \]
- Isolate the terms with variables:
  \[
  (a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2
  \]
- Denote \(a^2 - c^2\) by \(-b^2\):
  \[
  -b^2x^2 + a^2y^2 = a^2b^2
  \]
- Divide by \(-a^2b^2\):
  \[
  \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
  \]
• Graphs of hyperbolas where axis are vertical or horizontal

Horizontal hyperbola
Standard Equation: \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \)

Vertical hyperbola
Standard Equation: \( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \)

Horizontal hyperbola and center \((h, k)\)
Standard Equation: \( \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \)

Vertical Hyperbola and center \((h, k)\)
Standard Equation: \( \frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1 \)

• How to find different parameters for a hyperbola using its equation:

1. If the equation is anything other than the above equations, reformat to one of the above.
2. In standard form if \(x\) term is positive, then the hyperbola is horizontal. Otherwise, the hyperbola is vertical.
3. If hyperbola is horizontal, to find the vertices, plug in \(y = 0\) or \(y = k\) and solve. If hyperbola is vertical, to find the vertices, plug in \(x = 0\) or \(x = h\) and solve.
4. Notice the asymptotes when drawing the parabolas.
5. Find \(c\) using the equation \(c^2 = a^2 + b^2\).
1. Write each of the following hyperbola equations in standard form and find $a$ and $b$. Then find $c$.

(a) $25x^2 - 4y^2 = 100$ 
(b) $4(y - 3)^2 - 36(x - 5)^2 = 36$ 
(c) $25y^2 - 4x^2 = 1$

2. Use completing the square to find the standard equation of the following hyperbola, then Find the center of the hyperbola. $4x^2 - 24x - 9y^2 + 18y - 9 = 0$
3. Which of the following is an equation for a hyperbola with foci (0, 5) and (0, −5) and asymptotes $y = \pm \frac{3x}{4}$?

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

(b) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

(c) $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(d) $\frac{y^2}{9} - \frac{x^2}{16} = 1$

4. Sketch the graph of the hyperbola $\frac{x^2}{64} - \frac{y^2}{16} = 1$. Label the vertices, foci and asymptotes.