3.4: Graphs of Polynomials

A Few Definitions

- **Global maximum**: highest point on a graph; \(f(a)\) where \(f(a) \geq f(x)\) for all \(x\). **Global minimum**: lowest point on a graph; \(f(a)\) where \(f(a) \leq f(x)\) for all \(x\).

- **The Intermediate Value Theorem**:

  Let \(f\) be continuous on \([a, b]\) and \(M\) a value between \(f(a)\) and \(f(b)\), then there exist number \(r\) in \([a, b]\) such that \(f(r) = M\).

- **Intermediate Value Theorem, in Other words**: for two numbers \(a\) and \(b\) in the domain of \(f\), if \(a < b\) and \(f(a) \neq f(b)\), then the function \(f\) takes on every value between \(f(a)\) and \(f(b)\); specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis.

- **Existence of a Zero of a Continuous Function**.

  Let \(f\) be continuous on \([a, b]\) and \(f(a)\) and \(f(b)\) have different signs, then \(f\) has a zero in \([a, b]\).

- **Multiplicity**: the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form \((x - h)^p\), \(x = h\) is a zero of multiplicity \(p\).

- **There are no breaks or holes** in the graph of a polynomial.

**Given a polynomial function, sketch the graph.**

- **Find the intercepts.** \textbf{x-intercepts} are obtained by setting the function equal to zero. The \textbf{y-intercept} is obtained by plugging in zero in the function.

- **Check for symmetry.** If the function is an even function, its graph is symmetrical about the y-axis, that is, \(f(-x) = f(x)\). If a function is an odd function, its graph is symmetrical about the origin, that is, \(f(-x) = -f(x)\).

- **Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.**
• Determine the end behavior by examining the leading term.

- Use the end behavior and the behavior at the intercepts to sketch a graph.
- Use test points.
- Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
- Optionally, use technology to check the graph.

**An Interesting Result of Intermediate Value Theorem:** In solving inequalities involving polynomials only, we have been told to find the zeros of the polynomial and divide the number line into pieces and find the sign of each piece using test points. How we know that the sign doesn’t change on each piece is using the Intermediate Value Theorem.
1. Which of the following describes the end behavior of \( f(x) = -5x^4 + 3x^3 - 11x + 1 \)?

(a) \( y \to -\infty \) as \( x \to \infty \) and \( y \to \infty \) as \( x \to -\infty \)
(b) \( y \to -\infty \) as \( x \to \infty \) and \( y \to -\infty \) as \( x \to -\infty \)
(c) \( y \to \infty \) as \( x \to \infty \) and \( y \to -\infty \) as \( x \to -\infty \)
(d) \( y \to \infty \) as \( x \to \infty \) and \( y \to \infty \) as \( x \to -\infty \)

**Solution:**

(b) is correct.

As \( x \to \infty \), \( x^4 \to \infty \) so \(-5x^4 \to -\infty\)

As \( x \to -\infty \), \( x^4 \to \infty \) so \(-5x^4 \to -\infty\)

2. Use the given information about the graph of the polynomial to write the equation.

(a) Degree = 3. Zeros at \( x = -5, x = -2, \) and \( x = 1 \). y-intercept at \((0,6)\).

(b) Degree = 4. Root of multiplicity 2 at \( x = 4 \), and a root of multiplicity 1 at \( x = 1 \) and \( x = -2 \). y-intercept at \((0,-3)\).

(c) Degree = 3. With zeros \(-2, 0 \) and \( 5 \).

**Solution:**

(a) First find the factors: \((x + 5)(x + 2)(x - 1)\) since degree is 3, there is no other factor. Then multiply by a scalar: \( P(x) = a_3(x + 5)(x + 2)(x - 1) \).

Use the test point to find the scale: \( P(0) = 6 \implies a_3(5)(2)(-1) = 6 \implies a_3 = -10 \)

So \( P(x) = -10(x + 5)(x + 2)(x - 1) \).

(b) Find the factors \((x - 4)^2(x - 1)(x + 2)\) and since the polynomial is of degree 4, there is no other factor.

Multiply by a scalar: \( P(x) = a_4(x - 4)^2(x - 1)(x + 2) \). Use the test point to find the scalar:

\( P(0) = -3 \implies a_4(-4)(-1)(2) = -3 \implies a_4 = \frac{-3}{8} \)

So \( P(x) = \frac{-3}{8}(x - 4)^2(x - 1)(x + 2) \).

(c) There is not enough information to have a unique polynomial in this case so just find fine polynomial with the information \( P(x) = (x + 2)(x)(x - 5) \). The answer can vary by a scalar.
3. Find a polynomial with integer coefficients of degree 5 with zeros $1/3$, $-2$ and $5$ of multiplicities 1, 1, and 3 respectively.

**Solution.**

\[ f(x) = (x - 5)^3(x + 2)(x - \frac{1}{3})(3) = (x - 5)^3(x + 2)(3x - 1) \]

Fun project: For the following polynomial, use a graphing calculator or this applet: [https://ggbm.at/uxvrnkzn](https://ggbm.at/uxvrnkzn) to approximate local minima and maxima or the global minimum and maximum.

(a) \( f(x) = x^3 - x - 1 \)

(b) \( f(x) = x^4 - x^3 + 1 \)

4. A right circular cone has a radius of \(3x + 6\) units and a height 3 units less.

   (a) Express the volume of the cone as a polynomial function. The volume of a cone is 
   \[ V = \frac{1}{3}\pi r^2 h \]
   for radius \( r \) and height \( h \).

   (b) What is the leading coefficient of this polynomial?

**Solution.**

(a) \( r = 3x + 6 \) and \( h = (3x + 6) - 3 = 3x + 3 \).

\[
V(x) = \frac{1}{3}\pi(3x + 6)^2(3x + 3)
\]

(b) By foiling, you can see that the leading coefficient is \( 9\pi \).
5. Graph \( P(x) = (x - 1)(x - 3)(x + 1) \).

**Solution.**

**General Guidelines:**

- Find x-intercepts \( x = -1, 1, 3 \) from setting \( (x - 1)(x - 3)(x + 1) = 0 \) and y-intercept \( P(0) = 3 \).
- Find test points.
- End behavior. (After factoring, we get the coefficient of \( x^3 \) is 1 so the end behavior is odd degree with positive coefficient.)
- Start at one end behavior, graph, and be sure that the number of turning points are not bigger than the one less than degree of the polynomial (3). Also make sure that you have gone through each root, y-intercept and the other end behavior is satisfied.

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<td>5.625</td>
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</tbody>
</table>
6. Graph \( P(x) = -(x - 1)(x - 3)(x + 1) + 2 \)

**Solution:**

Normally, we graph \( y = -(x - 1)(x - 3)(x + 1) \) and then shift upward 2 units but since we graphed \( y = (x - 1)(x - 3)(x + 1) \) in the last question, we can reflect that graph over the x-axis and then shift.
7. Graph \( p(x) = (x - 2)^3(x + 1)^2 \).

**Solution:**

**General Guidelines:**

- Find \( x \)-intercepts \((x = -1 \text{ of degree } 2, x = 2 \text{ of degree } 3)\) from setting \((x + 1)^2(x - 2)^3 = 0\) and \( y \)-intercept \((P(0) = (-2)^3(1)^2 = -8)\).
- Find test points.
- End behavior. (After foiling, we get the coefficient of \( x^5 \) is one so the end behavior is odd degree with positive coefficient.)
- Graph and be sure that the number of turning points are not bigger than the degree of the polynomial (5).

\[ \begin{array}{c|c}
 x & P(x) \\
-2 & -64 \\
-1 & 0 \\
-0.5 & -3.9 \\
0 & -8 \\
0.25 & -8.37 \\
0.5 & -7.59 \\
1 & -4 \\
2 & 0 \\
3 & 16
\end{array} \]
8. We are building a rectangular box with **no lid** has height \(h\) units and a square base with dimension \(x\) units. The box is made out of cardboard with total area of 96 unit\(^2\)s.

(a) Express \(h\) as a function of \(x\).

(b) Express the volume as a function of \(x\).

Fun Project: Use the applet provided on this week’s folder to graph the polynomial and find the maximum volume that can be attained.

**Solution:**

(a) Remember the surface area with no lid is the total area of the 4 walls plus the area of the base. (Refer to the picture below.) That is, \(V = x^2(h)\). That is, \(S(x) = x^2 + 4hx\). The surface area is a constraint and has to be equal to 96.

This way, you can solve for \(h\):

\[
96 = x^2 + 4xh \implies h = \frac{96 - x^2}{4x} \implies h(x) = \frac{24}{x} - \frac{x}{4} \quad \leftarrow \text{Separate the two fractions}
\]

The area of the base \((B)\):

\[
B = x^2
\]

Area of each side: \(A = h \times x\)

Total area of all four sides

\[
h \times 4A = 4 \times h \times x = 4hx
\]

(b) Remember the volume of rectangular box is the area of the base times the height. \(V = x^2(h)\).

Replace \(h\) by its rule as a function of \(x\): \(V(x) = x^2\left(\frac{24}{x} - \frac{x}{4}\right)\)

\[
\implies V(x) = 24x - \frac{1}{4}x^3 \quad \leftarrow \text{Distribute } x^2
\]