### 3.6: Zeros of Polynomial Functions

#### A few Theorems and Rules for finding Roots

- **Descartes Rule of Signs**: a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $P(x)$ and $P(-x)$.

  Let $P$ be a polynomial with real coefficients.

  1. The number of **positive real** zeros of $P(x)$ is either equal to the number of variation in sign in $P(x)$ or less than that by an even whole number.

  2. The number of **negative real** zeros of $P(x)$ is equal to the number of variation in sign of $P(-x)$ or less than that by an even whole number.

- **Factor Theorem**: $x - k$ is a factor of a polynomial $P$ if and only if $P(k) = 0$.

- **Fundamental Theorem of Algebra**: a polynomial function with degree greater than 0 has at least one complex zero. (Note that real numbers are subsets of complex numbers.)

- **Linear Factorization Theorem**: Allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x - c)$, where $c$ is a complex number.

- **Linear and quadratic factors Theorem**: Every polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors with real coefficient.

- **Conjugate zeros theorem**: If a polynomial $P$ has real coefficients and if the complex number $z$ is a zero of $P$, then the complex conjugate $\bar{z}$ is also a zero of $P$.

- **Rational Zero Theorem**: If the polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ has integer coefficient, then every possible rational zero of $P$ is of the form: $\frac{p}{q}$ where:

  - $p$ is a factor of the constant coefficient $a_0$ and
  - $q$ is a factor of the leading coefficient, $a_n$.

- **Remainder Theorem**: If polynomial $P(x)$ is divided by $x - k$, then the remainder is the value $P(k)$. 

1. List all possible rational zeros of \( P(x) = 5x^5 - 18x^4 - 6x^3 + 91x^2 - 60x + 9 \) given by the Rational Zeros Theorem.

2. By Descartes Rule, how many real zeros does \( p(x) = 5x^5 - 18x^4 - 6x^3 + 91x^2 - 60x + 9 \) have?

3. Find the remainder of dividing \( x^{5005} + x^{304} + 5 \) by \( x + 1 \).
4. Use all theorems to find all rational roots of \( P(x) = x^6 - 5x^5 + 8x^4 - 3x^3 - 7x + 6 \).

5. Find a polynomial of degree 4, with integer valued coefficients, whose roots include 2, \(-2\), \(1 - i\).
6. Let \( p(x) = x^4 + 2x^3 + x^2 + 12x + 20 \)

(a) List all of the rational numbers that could be zeros of \( p(x) \) according to the rational zeros theorem.

(b) Use long division/synthetic division and the quadratic formula to find all of the zeros of \( p(x) \). List the zeros along with their multiplicities.

(c) Write \( p(x) \) in its fully factored form. (That is, all factors should be linear complex factors.)