4.1: Exponential Functions

An exponential function is of the form $f(x) = a \cdot b^x$ over the domain of all real numbers, where

- $a$ is a non-zero real number called the **initial** value and $b$ is any positive real number such that $b \neq 1$.
- The domain of $f$ is all real numbers.
- The range of $f$ is all positive real numbers if $a > 0$.
- The range of $f$ is all negative real numbers if $a < 0$.
- The **y-intercept** is $(0, a)$, and the **horizontal asymptote** is $y = 0$. The graph has NO **x-intercept**.
- If $b > 1$, then the function is an exponential growth. That is, if $a > 0$, as $x \to \infty$ $y \to \infty$.
- If $0 < b < 1$, then the function is an exponential decay. That is, as $x \to \infty$, $y \to 0$.
- A few graphs:
Compound Interest

- \( A(t) = P \left(1 + \frac{r}{n}\right)^{nt} \)

  \( A(t) \) = Amount after \( t \) years.  
  \( P \) = Principal 
  \( r \) = Annual Percentage Rate (APR) 
  \( n \) = Number of compounding per year 

  \( t \) = Number of years 
  \( \frac{r}{n} \) = Interest rate per period 
  \( nt \) = Number of compound in calculation

Annual Percentage Yield

- The annual percentage yield (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year.

  \[
  \text{APY} = \left(1 + \frac{r}{n}\right)^n - 1
  \]

Euler Number \( e \)

- The letter \( e \) represents the irrational number \( \left(1 + \frac{1}{n}\right)^n \rightarrow e \) as \( n \) increases without bound.

  \( e \approx 2.718282 \) and is the natural base for many real-world exponential models.

Continuous Growth

- We use the natural base \( e \) for continuous growth. \( A(t) = ae^{rt} \).

  \( A(t) \) = Amount after time \( t \). 
  \( a \) = Initial value 
  \( r \) = Rate of continuous growth 
  \( t \) = Time elapsed

Continuous Compounding

- \( A(t) = Pe^{rt} \)

  \( A(t) \) = Amount after \( t \) years.  
  \( P \) = Principal 
  \( r \) = Annual Percentage Rate (APR) 
  \( t \) = Number of years
1. Identify the exponential functions.

(a) \( f(x) = x^{100} + 5x^{50} \) \( \times \)  

(b) \( g(x) = 3(5^{-x}) \) \( \checkmark \) \( a = 3 \) and \( b = 5^{-1} \)  

(c) \( h(y) = 3e^{y-2} \) \( \checkmark \) \( a = 3e^{-2} \) and \( b = 1 \)  

(d) \( i(t) = 0.5(2^{2t-1}) \) \( \checkmark \) \( a = 0.5(2^{-1}) = 0.25 \) and \( b = 2^2 = 4 \)  

(e) \( j(x) = 0.2^{2x} \) \( \checkmark \) \( a = 1 \) and \( b = 0.2^2 = 0.04 \)  

(f) \( k(t) = t(t - 1) \) \( \times \)  

2. Evaluate each function at the value given.

(a) \( f(x) = e^{-x} \) at \( x = 2 \)  
\[ f(2) = e^{-2} = \frac{1}{e^2} \]  

(b) \( g(t) = 2^t \) at \( t = \pi \)  
\[ g(\pi) = 2^\pi \]  

(c) \( h(y) = 5(0.5)^y \) at \( y = -1 \)  
\[ h(-1) = 5(0.5)^{-1} = 5(2) = 10 \]  

(d) \( i(x) = 9^{-x} \) at \( x = 0.5 \)  
\[ i(0.5) = 9^{-0.5} = \frac{1}{9^{0.5}} = \frac{1}{3} \]  

3. If $3000 is invested at the rate of 6% per year. Find the amount in the account after 5 years if interest is compounded (a) annually, (b) semi-annually and (c) daily. (Round to nearest dollar.)

**Solution:** \( P = 3000, r = 0.06 \) and \( t = 5. \)

(a) \( P = 3000(1 + 0.06)^5 \approx 4015 \)  
(b) \( P = 3000(1 + 0.06/2)^{10} \approx 4032 \)  
(c) \( P = 3000(1 + 0.06/365)^{365 \times 5} \approx 4049 \)  

4. If $2000 is invested at an interest rate of 3.5% per year, compounded continuously, find the future value of the investment after the given number of years:

(a) 2 years.  
(b) 4 years

**Solution:**  
Use formula \( A(t) = Pe^{rt} \) where \( P = 2000 \) and \( r = 0.035. \)

(a) \( A(2) \approx 2145 \) and (b) \( A(4) \approx 2300.55 \)
5. A radioactive substance decays in such a way that the amount of mass remaining after \( t \) days is given by the function \( m(t) = 13e^{-0.015t} \) Where \( m(t) \) is measured in kilograms.

(a) Find the mass at time \( t = 0 \).
(b) How much of the mass remains after 20 days.

**Solution:**

(a) \( m(0) = 13 \) and (b) \( m(20) \approx 9.63 \)

6. Chloe invested a total of $5000, part at 3% simple interest and part at 4% simple interest. At the end of 1 year, the investments had earned $176 interest. How much was invested at each rate?

**Solution:**

Let \( x \) be the amount invested at 3\% rate, then \( 5000 - x \) is invested at 4\% rate.

Now total interest is \( 0.03 \times x + 0.04(5000 - x) = 200 - 0.01 \times x \)

Set that equal to 176 to get \( 176 = 200 - 0.01x \implies x = \frac{24}{0.01} = 2400 \) at 3\% and \( 500 - 2400 = 2600 \) at 4\% rate.