7.4: Sum to Product and Product to Sum Formulas

- Derivation of Formulas:

\[
\begin{align*}
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y) \\
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \sin(x+y) + \sin(x-y) &= 2\sin(x)\cos(y) \\
\Rightarrow \frac{\sin(x)\cos(y)}{2} &= \frac{1}{2} (\sin(x+y) + \sin(x-y)) \\
\end{align*}
\]

Or let \( x = \frac{u+v}{2} \) and \( y = \frac{u-v}{2} \)

\[
\begin{align*}
\sin(u) + \sin(v) &= 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\
\end{align*}
\]

(1)

\[
\begin{align*}
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y) \\
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \sin(x+y) - \sin(x-y) &= 2\cos(x)\sin(y) \\
\Rightarrow \frac{\sin(x)\cos(y)}{2} &= \frac{1}{2} (\sin(x+y) + \sin(x-y)) \\
\end{align*}
\]

Or let \( x = \frac{u+v}{2} \) and \( y = \frac{u-v}{2} \)

\[
\begin{align*}
\sin(u) - \sin(v) &= 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) \\
\end{align*}
\]

(2)

\[
\begin{align*}
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\
\cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y) \\
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \cos(x+y) + \cos(x-y) &= 2\cos(x)\cos(y) \\
\Rightarrow \frac{\cos(x)\cos(y)}{2} &= \frac{1}{2} (\cos(x+y) + \cos(x-y)) \\
\end{align*}
\]

Or let \( x = \frac{u+v}{2} \) and \( y = \frac{u-v}{2} \)

\[
\begin{align*}
\cos(u) + \cos(v) &= 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\
\end{align*}
\]

(3)

\[
\begin{align*}
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\
\cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y) \\
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \cos(x+y) - \cos(x-y) &= -2\sin(x)\sin(y) \\
\Rightarrow \frac{\sin(x)\sin(y)}{2} &= \frac{-1}{2} (\cos(x+y) - \cos(x-y)) \\
\end{align*}
\]

Or let \( x = \frac{u+v}{2} \) and \( y = \frac{u-v}{2} \)

\[
\begin{align*}
\sin(u) - \sin(v) &= -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) \\
\end{align*}
\]

(4)
Notice that subtraction the first set does not render any new information for product to sum formula. (It renders another formula for product of mix sine and cosine.)

**A List of Formulas to Use in Problems**

- **Product to Sum or Difference Identities:**
  
  \[
  \sin(u) \cos(v) = \frac{1}{2} \left( \sin(u + v) + \sin(u - v) \right)
  \]
  
  \[
  \cos(u) \cos(v) = \frac{1}{2} \left( \cos(u + v) + \cos(u - v) \right)
  \]
  
  \[
  \sin(u) \sin(v) = \frac{1}{2} \left( \cos(u - v) - \cos(u + v) \right)
  \]

- **Sum to Product Identities:**
  
  \[
  \sin(x) + \sin(y) = 2 \sin \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right)
  \]
  
  \[
  \sin(x) - \sin(y) = 2 \cos \left( \frac{x + y}{2} \right) \sin \left( \frac{x - y}{2} \right)
  \]
  
  \[
  \cos(x) + \cos(y) = 2 \cos \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right)
  \]
  
  \[
  \cos(x) - \cos(y) = -2 \sin \left( \frac{x + y}{2} \right) \sin \left( \frac{x - y}{2} \right)
  \]
1. Write the sum \( \sin(3t) - \sin(5t) \) as a product.

2. Write the sum \( \cos(7t) - \cos(5t) \) as a product.

3. Write the product \( \cos(7x) \cos(3x) \) as a sum.
4. Write the product $\sin(7x) \cos(5x)$ as a sum.

5. Write the product $\sin(7x) \sin(5x)$ as a sum.
6. Simplify \( \frac{\sin(10x) - \sin(2x)}{\cos(10x) + \cos(2x)}. \)

7. One of the reasons we use sum to product rule is to find out about the amplitude modulation of two signals that are adding. When the product is found, one factor of signal is asymptotically present in amplitude and the frequency of oscillation follows the frequency of the signal with higher frequency (Whichever has a bigger \( B. \))

In this phenomena the amplitude changes periodically. This is used in am radio (amplitude modulation)\(^1\) It is only interesting when \( B_1 \) and \( B_2 \) are “close” in value.

In here, the blue graph is \( u(t) = \cos(10t) - \cos(9t). \) The green graphs are the enveloping graphs which are only asymptotes. Find the period of either of the green graphs and period of the blue graph.

\(^1\)That is, the signal transfers using amplitude modulation.