9.3: Systems of Nonlinear Equations and Inequalities

- **A system of nonlinear equations** is a system of two or more equations where at least one equation is nonlinear.

- **How to solve when one of the two equations is linear**: Solve for one variable in the linear equation and replace in the other equation. Then solve for the other variable.

- **Other methods to solve**: Eliminate one or more variables. Solve for remaining variables. Substitute and solve for the variables that were eliminated.

- **Nonlinear systems of inequalities**: Solve for intersections using the above methods. Graph the equations dividing the entire plane into two or more regions. Choose which regions fit the inequality.

- **Applications**: Aside from direct applications in solving models with inequalities, in calculus we often use the techniques to find area and volume of different regions.
1. Solve each of the following systems of equations, then match each system with the corresponding graph. Mark the solutions on the corresponding graph.

(a) \[
\begin{align*}
y &= x^2 - 1 \\
y &= x + 1
\end{align*}
\]
(b) \[
\begin{align*}
y &= x^2 - 1 \\
y &= 2x - 2
\end{align*}
\]
(c) \[
\begin{align*}
y &= x^2 \\
y &= x - 1
\end{align*}
\]

![Graphs](image)

No solution. One solution. Two solutions.

**Solution:**

(a) \[
\begin{align*}
y &= x^2 - 1 \\
y &= x + 1
\end{align*}
\]
Eliminate \( y \) between the two equations.

\[
x^2 - 1 = x + 1
\]
\[
\Rightarrow x^2 - x - 2 = 0
\]
Quadratic formula

\[
x = -1 \text{ or } 2
\]

Plug \( x \) back in the second equation to find \( y \):

\[
\begin{align*}
x &= -1 & \Rightarrow & y &= 0 \\
x &= 2 & \Rightarrow & y &= 3
\end{align*}
\]

The solutions are \((-1, 0)\) and \((2, 3)\).

That is Graph iii.

(b) \[
\begin{align*}
y &= x^2 - 1 \\
y &= 2x - 2
\end{align*}
\]
Eliminate \( y \) between the two equations.

\[
x^2 - 1 = 2x - 2
\]
\[
\Rightarrow x^2 - 2x + 1 = 0
\]
Quadratic formula

\[
x = 1
\]

Plug \( x \) back in the second equation to find \( y \):

\[
x = 1 \Rightarrow y = 0
\]

The solution is \((1, 0)\).

That is Graph ii.

(c) \[
\begin{align*}
y &= x^2 \\
y &= x - 1
\end{align*}
\]
Eliminate \( y \) between the two equations.

\[
x^2 = x - 1
\]
\[
\Rightarrow x^2 - x + 1 = 0
\]
Quadratic formula

No solutions.

That is Graph i.
2. Solve the following system of equations and mark the solutions on the graph of two functions.

\[
\begin{cases}
y = 2x^2 \\
y = 2\sqrt{x}
\end{cases}
\]

**Solution:**

Eliminate \( y \) between the two equations:

\[
\begin{align*}
&\begin{cases}
y = 2x^2 \\
y = 2\sqrt{x}
\end{cases} \\
\iff & 2x^2 - 2\sqrt{x} = 0 \\
\iff & x^2 = \sqrt{x} \\
\iff & x^4 = x \\
\iff & x^4 - x = 0 \\
\iff & x(x^3 - 1) = 0 \\
\iff & x = 0 \text{ or } x = 1.
\end{align*}
\]

Real solutions are \( x = 0 \) or \( x = 1 \).

Plug in to find \( y \) values:

\[
\begin{align*}
&\begin{cases}
x = 0 \\
x = 1
\end{cases} \\
\iff & \begin{cases}
y = 2(0)^2 = 0 \\
y = 2(1)^2 = 2
\end{cases} \\
\iff & (0, 0) \text{ and } (1, 2)
\end{align*}
\]

3. (a) Solve the following system.

\[
\begin{align*}
x^2 + y^2 &= 16 \\
\frac{x}{3} + \frac{y}{2} &= 1
\end{align*}
\]

(b) Which of the following figures best describes your answer?

(i) No solution.  (ii) One solution.  (iii) Two solutions.

**Solution:**

(a) Solve for \( x \) in the linear equation:

\[
\begin{align*}
x &= 3 - \frac{3y}{2}
\end{align*}
\]

Replace \( x \) in the first equation:

\[
\begin{align*}
\left(3 - \frac{3y}{2}\right)^2 + y^2 &= 16
\end{align*}
\]

Solve:

\[
\begin{align*}
9 - 9y + \frac{9y^2}{4} + y^2 &= 16 \\
13y^2 - 9y - 7 &= 0 \\
y &\approx -0.63 \text{ or } y \approx 3.4
\end{align*}
\]

Use the second equation to solve for \( x \) in each case:

\[
(3.95, -0.63) \quad (-2.1, 3.4)
\]

(b) Figure iii.
4. (a) Solve the following system.

\[
\begin{align*}
  x^2 + y^2 &= 16 \\
  \frac{x^2}{9} + \frac{y^2}{25} &= 1
\end{align*}
\]

(b) Which of the following figures best describes your answer?

(i) No solution.  (ii) One solution.  (iii) Two solutions.  (iv) Three solutions.  (v) Four solutions.

**Solution:**

(a) Multiply the second row by $-9$:

\[
\begin{align*}
  x^2 + y^2 &= 16 \\
  -x^2 - \frac{9y^2}{25} &= -9
\end{align*}
\]

Add the two equations together to eliminate $x$:

\[
\frac{16y^2}{25} = 7
\]

Plug $y$ back in the first equation:

\[
\begin{align*}
  x^2 + \frac{175}{16} &= 16 \\
  x^2 &= \frac{81}{16} \\
  x &= \pm \frac{9}{4}
\end{align*}
\]

So the answers are:

\[
\begin{align*}
  \left(\frac{9}{4}, \frac{5\sqrt{7}}{4}\right) & \quad \left(\frac{9}{4}, -\frac{5\sqrt{7}}{4}\right) \\
  \left(\frac{9}{4}, -\frac{5\sqrt{7}}{4}\right) & \quad \left(-\frac{9}{4}, -\frac{5\sqrt{7}}{4}\right)
\end{align*}
\]

(b) Figure v.
5. (a) Solve the following system.

\[
\begin{align*}
    x^2 + y^2 &= 4 \\
    \frac{x^2}{4} + \frac{y^2}{25} &= 1
\end{align*}
\]

(b) Which of the following figures best describes your answer?

(i) No solution.
(ii) One solution.
(iii) Two solutions.
(iv) Three solutions.
(v) Four solutions.

Solution:

(a)\[\implies\begin{align*}
    x^2 + y^2 &= 4 \\
    -x^2 - \frac{4y^2}{25} &= -4
\end{align*}\[
\]Multiply the second row by \(-4\)

\[
\implies\begin{align*}
    y^2 &= 0 \\
    x^2 + 0 &= 4
\end{align*}\[
\]Add the two equations together to eliminate \(x\);

\[
\implies\begin{align*}
    \frac{21y^2}{25} &= 0
\end{align*}\[
\]So the solutions are \((\pm 2, 0)\)

(b) Figure iii.
6. (a) Solve the following system.

\[
\begin{cases}
  x^2 + y^2 = 16 \\
  \frac{(x - 7)^2}{9} + \frac{y^2}{25} = 1
\end{cases}
\]

(b) Which of the following figures best describes your answer?

(i) No solution.  (ii) One solution.  (iii) Two solutions.  (iv) Three solutions.  (v) Four solutions.

Solution:

\[
\begin{align*}
\text{(a) } & \implies x^2 + y^2 = 16 \\
& \quad \implies x^2 - \frac{25(x - 7)^2}{9} - y^2 = -25 \\
& \quad \implies \frac{x^2}{9} - \frac{25(x - 7)^2}{9} = -9 \\
& \quad \implies 9x^2 - 25(x - 7)^2 = -9(9) \\
& \quad \implies 9x^2 - 25x^2 + 9(7)^2 = -9(9) \\
& \quad \implies -16x^2 + 350 - 1144 = 0 \\
& \quad \implies x = 4 \\
\text{Plug } x = 4 \text{ back in the first equation:} \\
& \quad (4)^2 + y^2 = 16 \\
& \quad y^2 = 0 \\
& \quad y = 0 \\
\end{align*}
\]

So the answer is \((4, 0)\).

(b) Figure ii.
7. To solve the system of inequalities \( \begin{align*}
&\begin{cases}
y &\geq x^2 - 2 \\
y &< x
\end{cases}
\end{align*} \), we graphed two functions \( y = x^2 - 2 \) and \( y = x \). Shade the region of solutions to the system of inequalities noting dashed lines versus solid lines and points of intersections of the two graphs.

![Graph of the system of inequalities](image)

**Solution:**

Solve the system: \( \begin{align*}
&\begin{cases}
y &= x^2 - 2 \\
y &= x
\end{cases}
\end{align*} \) \( \implies \) \( x^2 - x - 2 = 0 \) \( \implies \) using quadratic formula: \( x = -1 \) or \( x = 2 \)

Plug \( x = -1 \) back in the 2nd equation: \( y = -1 \). Plug \( x = 2 \) back in the 2nd equation: \( y = 2 \).

Points of intersections are: \( (-1, -1) \) or \( (2, 2) \).

Next, find the correct graph and choose and shade the correct region. Note < versus \( \leq \), use dashed line versus solid lines. For example, point \((0,-1)\) satisfies the both inequalities but \((2,0)\) only satisfies the second inequality and \((0,2)\) only satisfies the first inequality. So the region between two graph is the answer.

8. To solve the system of inequalities \( \begin{align*}
&\begin{cases}
y &> 2x^2 \\
y &\leq 2\sqrt{x}
\end{cases}
\end{align*} \), we graphed two functions \( y = 2x^2 \) and \( y = 2\sqrt{x} \). Shade the region of solutions to the system of inequalities noting dashed lines versus solid lines and points of intersections of the two graphs.

![Graph of the system of inequalities](image)

**Solution:**

Solve the system:

\( \begin{align*}
&\begin{cases}
y &= 2x^2 \\
y &= 2\sqrt{x}
\end{cases}
\end{align*} \) \( \implies \) \( 2x^2 - 2\sqrt{x} = 0 \) \( \implies \) \( x^4 = x \) \( \implies \) \( x^4 - x = 0 \) \( \implies \) \( x(x^3 - 1) = 0 \)

\( \implies \) \( x = 0 \) or \( x = 1 \).

Real solutions are \( (0,0) \) and \( (1,2) \).

Plug in to find \( y \) values: \( (0,0) \) and \( (1,2) \).

Find the correct graph and choose the correct region.
9. A car braked with a constant deceleration of 16 ft/s², producing skid marks measuring 200 feet before coming to a stop. The velocity function for the car is \( v(t) = -16t + v_0 \) and the position function is \( s(t) = -8t^2 + v_0t \), where \( v_0 \) is the initial velocity when the brakes were applied. How fast was the car traveling when the brakes were first applied? (That is, find \( v_0 \).)

Hints: You need to solve a non-linear system of equations in two variables to solve for \( v_0 \) and \( T_f \), where \( T_f \) is the time that it takes for the car to stop.

Solution:

Let \( t_f \) denote the time that it takes to stop and \( v_0 \) be the initial velocity. These two are the unknown. We form a system of two equations and two variables.

Two pieces of information is given. At the stopping time, the velocity is zero (this one is tricky) and that the car travels 200 ft when \( t_f \) seconds passed. That is, \( s(T_f) - s(0) = 200 \) ft.

Apply the skid mark information \( s(T_f) - s(0) = 200 \):

\[
-8T_f^2 + v_0T_f = 200
\]

where \( T_f \) is the time to take to stop.

Apply the fact that the car stopped at \( t_f \):

\[
v(T_f) = 0 \implies -16(T_f) + v_0 = 0
\]

Solve the system of two equations and two variables:

\[
\begin{aligned}
-8T_f^2 + v_0T_f &= 200 \\
-16(T_f) + v_0 &= 0
\end{aligned}
\]

Replace \( v_0 \) in the first equation by \( 16T_f \):

\[
-8T_f^2 + 16T_f^2 = 200 \implies T_f^2 = 25 \implies T_f = 5 \text{ seconds.}
\]

Notice that we did not accept \( T_f = -5 \).

Solve for \( v_0 \):

\[
-16(5) + v_0 = 0 \implies v_0 = 80 \text{ ft/s}^2
\]
Related Videos:

1. **Example 1:** https://mediahub.ku.edu/media/MATH+-+System+of+Non-linear+Equations+1/1.99vwube5

2. **Example 2:** https://mediahub.ku.edu/media/MATH+-+System+of+Non-linear+Equations+2.m4v/1.y6gicatg