### 9.5: Matrices and Matrix Operations

- **Definitions and Dimensions of Matrices:**
  - A rectangular array of numbers, functions or parameters is called a matrix.
  - A matrix with **dimension** $m \times n$ has $m$ rows and $n$ columns. When we refer to $ij$-entry, we mean the entry on the $i$th row and $j$th column.
  - We denote matrices with capital letters. The following is the general representation of an $m \times n$ matrix $A$:
    \[
    A = [a_{ij}] = \\
    \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
    \end{bmatrix}
    \]

- **Matrix addition and subtraction:**
  Let $A$ and $B$ be a $m \times n$ matrices:
    \[
    A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
    \end{bmatrix}
    \quad \text{and} \quad
    B = \begin{bmatrix}
    b_{11} & b_{12} & \cdots & b_{1n} \\
    b_{21} & b_{22} & \cdots & b_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{m1} & b_{m2} & \cdots & b_{mn}
    \end{bmatrix}
    \]
  Then $A \pm B = [a_{ij} \pm b_{ij}] = 
\[
\begin{bmatrix}
    a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\
    a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn}
    \end{bmatrix}
\]
  Note that matrix addition/subtraction has to be done on matrices with **same dimensions** and it produces a matrix of the same dimensions with entries that are addition/subtraction of the same corresponding entries of $A$ and $B$.

- **Scalar multiplication:**
  Let $A$ be a $n \times m$ matrix $A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
    \end{bmatrix}$ and $k$ be a number. Then the **Scalar multiplication** of the matrix $A$ by $k$ is
  \[
  kA = k[a_{ij}] = \\
  \begin{bmatrix}
    ka_{11} & ka_{12} & \cdots & ka_{1n} \\
    ka_{21} & ka_{22} & \cdots & ka_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    ka_{m1} & ka_{m2} & \cdots & ka_{mn}
    \end{bmatrix} = [ka_{ij}]
  \]
Some properties of matrix addition/subtraction:

(1) Commutative Property Of Addition: \( A + B = B + A \)

(2) Associative Property Of Addition \( A + (B + C) = (A + B) + C \)

The identity of matrix addition The \( m \times n \) matrix \( O_{m \times n} \) is the following matrix:

\[
O_{m \times n} = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}
\]

Note that \( O_{m \times n} + A_{m \times n} = A_{m \times n} \).

Matrix Multiplication: The matrix multiplication is only defined if the number of columns of the first matrix is equal to number of rows of the second matrix.

Let \( A \) of be an \( m \times n \) and \( B \) be an \( n \times k \) matrices, then the dimension of matrix \( AB \) is \( m \times k \).

\((A_{m \times n}B_{n \times k} = AB_{m \times k})\) The \( m \)th row, \( k \)th entry of \( AB \) is the product of \( m \)th row of \( A \) and \( k \)th column of \( B \) as described below:

Example: Here is the multiplication of \( A_{2 \times 3} \) by \( B_{3 \times 3} \):

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23}
\end{bmatrix}, \quad B = \begin{bmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{bmatrix}
\]

\[
AB = \begin{bmatrix}
a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\
a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}
\end{bmatrix}
\]

Note that the matrix \( BA \) is not defined.

The identity of matrix multiplication

The identity matrix \( I_n \) or \( I_{n \times n} \) is a square matrix of size \( n \) where all entries on the major diagonal are one and all the other entries are zero.

\[
I_n = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}
\]

Important properties of the identity matrix

Given any \( m \times n \) matrix \( A \) the following is true.

\[
I_mA = A \quad I_nA = A
\]

Equal matrices: Let \( A \) and \( B \) be two \( m \times n \) matrices:

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1n} \\
b_{21} & b_{22} & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
b_{m1} & b_{m2} & \cdots & b_{mn}
\end{bmatrix}
\]

Then \( A = B \) means that \( a_{ij} = b_{ij} \) for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).
1. What is a general representation for a $2 \times 3$ matrix?

**Solution:**

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23}
\end{bmatrix}
\]

2. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 0 \end{bmatrix}$.

(a) What is the dimension of $A$? 
(b) What is $a_{23}$ entry of matrix $A$?

**Solution:**

(a) $2 \times 3$ 
(b) $a_{23} = 0$

3. Determine which of the following operations is defined?

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 3 & 11 \\ 5 & 2 \end{bmatrix}$ 
(b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 11 & 7 \\ 5 & 2 & 1 \end{bmatrix}$ 
(c) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 
(d) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 0 & 5 \end{bmatrix}$

**Solution:**

(a) Not defined. The inner dimension of the product $(2 \times 3) \times (2 \times 2)$ does not match.
(b) Defined and equal to $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
(c) Defined and equal to $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 0 \end{bmatrix}$.
(d) The dimensions are not the same so it is not defined.
4. Perform the following matrix operations.

(a) \[
\begin{bmatrix}
1 & -2 & 3 \\
2 & 5 & 0
\end{bmatrix}
+ \begin{bmatrix}
3 & 11 & 1 \\
5 & 2 & 1
\end{bmatrix}
\]

(b) \[
3 \begin{bmatrix}
3 & 11 & 7 \\
5 & 2 & 1
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & -2 & 3 \\
2 & 5 & 0
\end{bmatrix}
- \begin{bmatrix}
3 & 11 & 1 \\
5 & 2 & 1
\end{bmatrix}
\]

(d) \[
\begin{bmatrix}
1 & 1 \\
2 & 0 \\
5 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 3 \\
2 & 0
\end{bmatrix}
\]

Solution:

(a) \[
\begin{bmatrix}
1 & -2 & 3 \\
2 & 5 & 0
\end{bmatrix}
+ \begin{bmatrix}
3 & 11 & 1 \\
5 & 2 & 1
\end{bmatrix}
= \begin{bmatrix}
4 & 9 & 4 \\
7 & 7 & 1
\end{bmatrix}
\]

(b) \[
3 \begin{bmatrix}
3 & 11 & 7 \\
5 & 2 & 1
\end{bmatrix}
= \begin{bmatrix}
9 & 33 & 21 \\
15 & 6 & 3
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & -2 & 3 \\
2 & 5 & 0
\end{bmatrix}
- \begin{bmatrix}
3 & 11 & 1 \\
5 & 2 & 1
\end{bmatrix}
= \begin{bmatrix}
-2 & -13 & 2 \\
-3 & 3 & -1
\end{bmatrix}
\]

(d) \[
\begin{bmatrix}
1 & 1 \\
2 & 0 \\
5 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 3 \\
2 & 0
\end{bmatrix}
= \begin{bmatrix}
(1 \times 1 + 1 \times 2) & (1 \times 3 + 1 \times 0) \\
(2 \times 1 + 0 \times 2) & (2 \times 3 + 0 \times 0) \\
(5 \times 1 + 1 \times 2) & (5 \times 3 + 1 \times 0)
\end{bmatrix}
= \begin{bmatrix}
3 & 3 \\
2 & 6 \\
7 & 15
\end{bmatrix}
\]

Note that in Part (d), we highlighted a row of the first and a column of the second matrix. This explains that to find the 3rd row 2nd column entry, multiply the third row of the first matrix and the second column of the second matrix.
5. Find $AB$ and $BA$ for the following choices of matrix $A$ and matrix $B$, or say that they are not defined. In case that they are both defined, determine if $AB = BA$ or not.

(a) $A = \begin{bmatrix} 1 & 3 \\ 5 & 0.6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

\textbf{Solution}:

(a) $AB = \begin{bmatrix} 5 & 1 \\ 10.6 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 7 & 6.6 \\ 1 & 3 \end{bmatrix}$ They are both defined but $AB \neq BA$

(b) $AB = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$. $AB$ and $BA$ are both defined and $AB = BA$

(c) $AB$ is not defined because $A$ is $2 \times 2$ and $B$ is $3 \times 2$ and the inner dimension does not match.

$BA = \begin{bmatrix} 6 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$

6. Let $A = \begin{bmatrix} 1 & 3 \\ 5 & 6 \\ 0.5 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Find any of the following multiplication if it is defined.

(a) $I_3A$

(b) $AI_3$

(c) $I_2A$

(d) $AI_2$

\textbf{Solution}:

(a) $I_3A = \begin{bmatrix} 1 & 3 \\ 5 & 6 \\ 0.5 & 1 \end{bmatrix}$

(b) Not defined.

(c) Not defined.

(d) $AI_2 = \begin{bmatrix} 1 & 3 \\ 5 & 6 \\ 0.5 & 1 \end{bmatrix}$
7. (a) Calculate \[
\begin{bmatrix}
1 & 3 & 2 \\
5 & 6 & 1 \\
0.5 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}.
\]

(b) Express the matrix equality \[
\begin{bmatrix}
1 & 3 & 2 \\
5 & 6 & 1 \\
0.5 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
1 \\
3 \\
2
\end{bmatrix}
\]
in terms of three equations.

**Solution:**

(a) By matrix multiplication:
\[
\begin{bmatrix}
1 & 3 & 2 \\
5 & 6 & 1 \\
0.5 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
x + 3y + 2z \\
5x + 6y + z \\
0.5x + y + 2z
\end{bmatrix} \quad \text{(This is a 3 × 1 matrix.)}
\]

(b) \[
\begin{bmatrix}
x + 3y + 2z \\
5x + 6y + z \\
0.5x + y + 2z
\end{bmatrix} = \begin{bmatrix}
1 \\
3 \\
2
\end{bmatrix} \quad \Rightarrow \quad \begin{cases}
x + 3y + 2z = 1 \\
5x + 6y + z = 3 \\
0.5x + y + 2z = 2
\end{cases}
\]

Each pair of corresponding entries are equal.
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1. Example 1: https://mediahub.ku.edu/media/MATH++-Matrices.m4v/1_pn42m74z
2. Example 2: https://mediahub.ku.edu/media/MATH++Matrix+Multiplication+1.m4v/1_b0z677nu
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