Section 17.2
Stokes’ Theorem
Suppose $S$ is an oriented surface with unit normal vector $\vec{n}$. Recall that the **boundary** $\partial S$ consists of a set of closed curves.

We can orient $\partial S$ as follows: *If you walk along $\partial S$ with your head in the direction of $\vec{n}$, then $S$ should be on your left.*

This is sometimes called a **right-hand-rule orientation**. If your right thumb points up (toward $\vec{n}$), then you should be able to curl your fingers from the direction of travel along $\partial S$ inward toward $S$. 
Stokes’ Theorem

Let $S$ be an oriented surface with smooth, simple closed boundary curves. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives. Then

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_{S} \text{curl}(\vec{F}) \cdot d\vec{S}$$

where the components of $\partial S$ are oriented using a right-hand-rule orientation.

Green’s Theorem is a special case of Stokes’ Theorem. If $D$ is a region in the plane and $\partial D$ is given a right-hand-rule orientation (with $\vec{n} = \vec{k}$), then Stokes’ Theorem becomes

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_{D} \text{curl}(\vec{F}) \cdot \vec{k} \, dA$$

which is exactly Green’s Theorem.
Stokes’ Theorem: Examples

**Example 1:** Verify Stokes’ Theorem for the vector field
\( \vec{F}(x, y, z) = \langle -y^2, x, z \rangle \) and the surface \( S \) obtained by intersecting the plane \( y + z = 2 \) and the solid cylinder \( x^2 + y^2 \leq 1 \).

**Solution:** First, \( \text{curl}(\vec{F}) = \langle 0, 0, 1 + 2y \rangle \).

The surface \( S \) can be parametrized over the unit disk \( D \) by \( G(x, y) = (x, y, 2 - y) \). The normal is
\[
G_x \times G_y = \langle 1, 0, 0 \rangle \times \langle 0, 1, -1 \rangle = \langle 0, 1, 1 \rangle
\]
which points upwards.

The double-integral side of Stokes’ Theorem is
\[
\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_D 1 + 2y \, dA = \int_0^{2\pi} \int_0^1 (1 + 2r \sin(\theta)) \, r \, dr \, d\theta = \pi.
\]
Example 1 (continued): For reference, \( \vec{F}(x, y, z) = \langle -y^2, x, z \rangle \).

The boundary \( \partial S \) is the ellipse with parametrization

\[
\vec{r}(t) = \langle \cos(t), \sin(t), 2 - \sin(t) \rangle
\]

\[
\vec{r}'(t) = \langle -\sin(t), \cos(t), -\cos(t) \rangle
\]

so

\[
\vec{F}(\vec{r}(t)) = \langle -\sin^2(t), \cos(t), 2 - \sin(t) \rangle
\]

Therefore, the contour-integral side of Stokes' Theorem is

\[
\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]

\[
= \int_0^{2\pi} \sin^3(t) + \cos^2(t) - 2 \cos(t) + \sin(t) \cos(t) \, dt
\]

\[
= \int_0^{2\pi} \cos^2(t) \, dt = \pi.
\]
Stokes’ Theorem: Examples

Example 2: Compute \( \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} \) where \( \vec{F}(x, y, z) = \langle xz, yz, xy \rangle \) and \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 25 \) inside the cylinder \( x^2 + y^2 = 16 \) above the \( xy \)-plane.

Solution: \( C = \partial S \) can be parametrized as:

\[
\vec{r}(t) = \langle 4 \cos(t), 4 \sin(t), 3 \rangle \\
\vec{r}'(t) = \langle -4 \sin(t), 4 \cos(t), 0 \rangle \\
\vec{F}(\vec{r}(t)) = \langle 12 \cos(t), 12 \sin(t), 16 \cos(t) \sin(t) \rangle
\]

Using Stokes’ Theorem,

\[
\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_0^{2\pi} 0 \, dt = 0.
\]

Varying the radius of the cylinder would not change the answer of 0.
Curl and Circulation in Green and Stokes

Green’s and Stokes’ Theorems both say that if a vector field pushes stuff (counter)clockwise around the boundary of a surface $\mathbb{R}^2$, then it rotates stuff (counter)clockwise in the surface itself.

The circulation per unit area is $\text{curl}(\vec{F})_z$ (Green) or $\text{curl}(\vec{F}) \cdot \vec{n}$ (Stokes).
The more complicated the math, the dumber you sound explaining it.

Stokes' Theorem? Yeah, that's how if you draw a loop around something, you can tell how much swirly is in it.
Vector Potentials and Surface-Independence

A vector potential for a vector field $\mathbf{F}$ is a vector field $\mathbf{A}$ such that

$$\mathbf{F} = \text{curl}(\mathbf{A}).$$

If $\mathbf{F}$ has a vector potential, then its integral over a surface $S$ depends only on $\partial S$, because Stokes’ Theorem says that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{r}.$$ 

For example, if $\mathbf{F} = \nabla \times \mathbf{A}$ then $\mathbf{F}$ has the same flux through the two surfaces $S_1, S_2$ shown to the right, because $\partial S_1 = \partial S_2$.

This is very useful: we can often compute flux through a complicated surface by replacing it with a simpler surface with the same boundary.
If $S_1$ and $S_2$ have the same boundaries (including orientation(s)), then

$$\int\int_{S_1} \nabla \times \vec{F} \cdot d\vec{S} = \int\int_{S_2} \nabla \times \vec{F} \cdot d\vec{S}.$$ 

**Example 3:** Consider the surface $S$ in $\mathbb{R}^3$ defined by $x^2 + y^2 = 2 + z - e^{2z}$ for $z \geq 0$, oriented by a right-hand rule, and let $\vec{F}(x, y, z) = \langle x^2 y, -xy^2, xyz \rangle$. Evaluate

$$\int\int_S \nabla \times \vec{F} \cdot d\vec{S}.$$ 

**Solution:** Parametrizing $S$ and doing the integral directly is very hard.

However, since $\partial S$ is just the unit circle $C$ ($x^2 + y^2 = 1$, $z = 0$), we can use Stokes’ Theorem to replace $S$ with the unit disk $D$ (oriented upwards), because $\partial S = \partial D$. 
Example 3 continued: Since $D$ is a region in the plane, the surface integral over $D$ is just a double integral, which we can calculate using polar coordinates:

$$
\iiint_S \nabla \times \vec{F} \cdot d\vec{S} = \iiint_D \nabla \times \vec{F} \cdot d\vec{S}
$$

$$
= \iiint_D \langle xz, -yz, -x^2 - y^2 \rangle \cdot \mathbf{k} \, dA
$$

$$
= -\iiint_D x^2 + y^2 \, dA
$$

$$
= -\int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta
$$

$$
= -\pi/2.
$$
Example 4: Let \( \vec{F} = \text{curl}(\vec{A}) \), where

\[
\vec{A}(x, y, z) = \langle y + z, \sin(xy), e^{xyz} \rangle.
\]

Find the flux of \( \vec{F} \) outward through each of the surfaces \( S_1 \) and \( S_2 \) whose common boundary \( C \) is the unit circle in the \( xz \)-plane.

Solution: Parametrize \( C \) as \( \vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle \).

The orientation of \( C \) has the surface \( S_1 \) on the left.

\[
\iint_{S_1} \vec{F} \cdot d\vec{S} = \oint_{C} \vec{A} \cdot d\vec{r} = \int_{0}^{2\pi} -\sin^2(t) + \cos(t) \, dt = -\pi.
\]

The orientation of \( C \) has the surface \( S_2 \) on the right.

\[
\iint_{S_2} \vec{F} \cdot d\vec{S} = \oint_{-C} \vec{A} \cdot d\vec{r} = -\oint_{C} \vec{A} \cdot d\vec{r} = \pi.
\]

Note that we did not need to know what \( S_1 \) and \( S_2 \) were!