Trees and How to Count Them

Jeremy L. Martin
Department of Mathematics
University of Kansas

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A **graph** is a pair $G = (V, E)$, where

- $V$ is a set of **vertices**, and
- $E$ is a set of **edges**, each joining two vertices (its **endpoints**).

The **degree** of a vertex is the number of edges incident to it.
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Cycle graph $C_8$

Complete graph $K_6$

Cube graph $Q_3$

Complete bipartite graph $K_{5,3}$
Why study graphs?

- Real-world applications
  - Combinatorial optimization (routing, scheduling. . .)
  - Computer science (data structures, sorting, searching. . .)
  - Biology (evolutionary descent. . .)
  - Chemistry (molecular structure. . .)
  - Engineering (roads, rigidity. . .)
  - Network models (social networks, the Internet. . .)

- Pure mathematics
  - Combinatorics (ubiquitous!)
  - Discrete dynamical systems (chip-firing game. . .)
  - Algebra (quivers, Cayley graphs. . .)
  - Discrete geometry (polytopes, sphere packing. . .)
Spanning Trees

Definition  A **spanning tree** of a graph $G$ is a set of edges $T$ (or a subgraph $(V, T)$) such that:

1. $(V, T)$ is **connected**: every pair of vertices is joined by a path
2. $(V, T)$ is **acyclic**: there are no cycles
3. $|T| = |V| - 1$.

Any two of these conditions together imply the third.
Spanning Trees

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Spanning Trees

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Trees and How to Count Them
Counting Spanning Trees

\[ \mathcal{T}(G) = \text{set of spanning trees of } G \]
\[ \tau(G) = \text{number of spanning trees of } G \]

- \( \tau(\text{tree}) = 1 \)
- \( \tau(C_n) = n \)
- \( \tau(K_n) = n^{n-2} \) (Cayley's formula; highly nontrivial!)
- \( \tau(K_{m,n}) = n^{m-1} m^{n-1} \)
- Many other enumeration formulas for nice graphs
Deletion and Contraction

Let $e \in E(G)$. 

Theorem $\tau(G) = \tau(G - e) + \tau(G/e)$. 

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Trees and How to Count Them
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- **Deletion** $G - e$: Remove $e$
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This formula allows easy calculation of \( \tau(G) \) and some fun results:

\[ G \]

Unfortunately:

▶ "easy" does not mean "efficient": \( 2 |E| \) steps are required to calculate \( \tau(G) \) this way.

▶ Useful only for graph families with recursive deletion/contraction structure (not \( K_n \), \( K_m \), \( n \), \( Q_n \), etc.).
Theorem \( \tau(G) = \tau(G - e) + \tau(G/e) \).

This formula allows easy calculation of \( \tau(G) \) and some fun results:

\[
\begin{array}{cccccccc}
G & \bullet & \quad & \quad & \quad & \quad & \quad & \quad \\
\tau(G) & 1 & 1 & 2 & 3 & 5 & 8 & 13 \\
\end{array}
\]
**Theorem** \( \tau(G) = \tau(G - e) + \tau(G/e) \).

This formula allows easy calculation of \( \tau(G) \) and some fun results:

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Unfortunately:

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- Useful only for graph families with recursive deletion/contraction structure (not \( K_n, K_{m,n}, Q_n \), etc.).
The Matrix-Tree Theorem

**Definition** Let $G$ be a connected graph with vertices $1, \ldots, n$ and no loops. The **Laplacian** of $G$ is the $n \times n$ matrix $L = [\ell_{ij}]$:

$$
\ell_{ij} = \begin{cases} 
\deg_G(i) & \text{if } i = j, \\
-(\text{number of edges from } i \text{ to } j) & \text{if } i \neq j.
\end{cases}
$$

- $L$ is symmetric and positive semi-definite
  - $L = \partial\partial^T$, where $\partial = \text{signed vertex-edge incidence matrix}$
- $\text{rank } L = n - 1$
- $\ker L$ is spanned by the all-1’s vector
The Matrix-Tree Theorem

Example

\[ G = \begin{array}{ccc}
1 & & 2 \\
\bullet & \bullet & \bullet \\
3 & & 4
\end{array} \]

\[ L = \begin{bmatrix}
3 & -1 & -2 & 0 \\
-1 & 3 & -1 & -1 \\
-2 & -1 & 3 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix} \]
The Matrix-Tree Theorem (Kirchhoff, 1847)

(1) Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of $L$. Then the number of spanning trees of $G$ is

$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}.$$
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(2) Let $1 \leq i \leq n$. Form the reduced Laplacian $L_i$ by deleting the $i^{th}$ row and $i^{th}$ column of $L$. Then

$$\tau(G) = \det L_i.$$
The Matrix-Tree Theorem: Example

\[ \tau(G) = 5 \]

\[ L = \begin{bmatrix}
3 & -1 & -2 & 0 \\
-1 & 3 & -1 & -1 \\
-2 & -1 & 3 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix} \]

\[ L_1 = \begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & 0 \\
-1 & 0 & 1
\end{bmatrix} \]

Eigenvalues: 0, 1, 4, 5

\[ (1 \cdot 4 \cdot 5) / 4 = 5 \]

\[ \text{det } L_1 = 5 \]
The hypercube graph $Q_n$ has $2^n$ vertices, labeled by strings of $n$ bits (0’s and 1’s), with two vertices adjacent if they agree in all but one bit.

Theorem The eigenvalues of the Laplacian of $Q_n$ are $0, 2, 4, \ldots, 2n$, with $2k$ having multiplicity $\binom{n}{k}$. Therefore,

$$\tau(Q_n) = 2^{2^n-n-1} \prod_{k=2}^{n} k^{\binom{n}{k}}.$$
Threshold Graphs

A graph with vertex set \( \{1, 2, \ldots, n\} \) is a \textbf{threshold graph} if, whenever \( ab \) is an edge, so is \( a'b' \) for all \( a' \leq a \) and \( b' \leq b \).
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A graph with vertex set \{1, 2, \ldots, n\} is a threshold graph if, whenever \(ab\) is an edge, so is \(a'b'\) for all \(a' \leq a\) and \(b' \leq b\).
**Theorem** [Merris 1994] The eigenvalues of the Laplacian of a threshold graph $G$ on vertices $1, \ldots, n$ are the columns $\lambda'_j$ of the partition $\lambda = \lambda(G)$ whose rows are the vertex degrees.

**Corollary** $\tau(G) = \lambda'_2 \lambda'_3 \cdots \lambda'_{n-1}$. 
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![Diagram of a threshold graph](chart.png)
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Vertex degrees: 4, 4, 3, 3, 2
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**Corollary**  $\tau(G) = \lambda'_2 \lambda'_3 \cdots \lambda'_{n-1}$.

\[ \tau = 5 \times 4 \times 2 = 40 \quad \text{Laplacian eigenvalues: 5, 5, 4, 2, 0} \]
Theorem [Cayley–Prüfer]

\[ \sum_{T \in \mathcal{T}(K_n)} x_1^{\deg_T(1)} \cdots x_n^{\deg_T(n)} = x_1 \cdots x_n (x_1 + \cdots + x_n)^{n-2} \]

- Setting \( x_i = 1 \) for all \( i \) recovers \( \tau(K_n) = n^{n-2} \)
- Can be proved either bijectively (Prüfer code) or by a souped-up version of the Matrix-Tree Theorem
- Other weighted tree counting formulas:
  - *Via bijections*: Fiedler-Sedláček (complete bipartite graphs), Knuth, Kelmans, Remmel-Williamson, etc.
  - *Via MTT*: JLM–Reiner (threshold graphs, hypercubes)
Theorem [JLM–Reiner 2005] Let $G$ be a threshold graph on vertices $1, \ldots, n$ with degree sequence $\lambda$. Weight each edge $e = ij$ with $i < j$ by $x_i y_j$. Then the bidegree generating function is

$$
\sum_{T \in \mathcal{T}(G)} \prod_{e: i < j} x_i y_j = x_1 y_n \prod_{r=2}^{n-1} \left( \sum_{i=1}^{\lambda'_r} x_{\min(i,r)} y_{\max(i,r)} \right)
$$

and therefore (setting $y_i = x_i$) the degree generating function is

$$
\sum_{T \in \mathcal{T}(G)} \prod_{i=1}^{n} x_i^{\deg(i)} = x_1 \cdots x_n \prod_{r=2}^{n-1} \left( \sum_{i=1}^{\lambda'_r} x_i \right)
$$
Bidegree generating function:

\[ x_1 y_5 \left( x_1 y_2 + x_2 y_2 + x_2 y_3 + x_2 y_4 + x_2 y_5 \right) \]
\[ \times \left( x_1 y_3 + x_2 y_3 + x_3 y_3 + x_3 y_4 \right) \left( x_1 y_4 + x_2 y_4 \right) \]

Degree generating function:

\[ x_1 x_2 x_3 x_4 x_5 \left( x_1 + x_2 + x_3 + x_4 + x_5 \right) \left( x_1 + x_2 + x_3 + x_4 \right) \left( x_1 + x_2 \right) \]
A **d-simplex** is the convex hull of \( d + 1 \) general points in \( \mathbb{R}^{d+1} \).
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$d = 0$  \hspace{1cm}  $d = 1$  \hspace{1cm}  $d = 2$  \hspace{1cm}  $d = 3$

A **simplicial complex** is a space built (properly!) from simplices.
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$d = 0 \quad d = 1 \quad d = 2 \quad d = 3$

A **simplicial complex** is a space built (properly!) from simplices.
Combinatorially, a simplicial complex is a set family $\Delta \subseteq 2\{1,2,...,n\}$ such that if $\sigma \in \Delta$ and $\sigma' \subseteq \sigma$, then $\sigma' \in \Delta$.

$\Delta_1 = \langle 12, 14, 24, 24, 25, 35 \rangle$

$\Delta_2 = \langle 124, 245, 35 \rangle$

- **faces or simplices**: elements of $\Delta$
- **dimension**: $\dim \sigma = |\sigma| - 1$
- **facet**: a maximal face
- **pure complex**: all facets have equal dimension
Definition Let $\Delta$ be a simplicial complex of dimension $d$. A subcomplex $\Upsilon \subseteq \Delta$ is a **simplicial spanning tree** (SST) if:

1. $\Upsilon$ contains all simplices of $\Delta$ of dimension $< d$.
2. $\Upsilon$ is “acyclic” and “connected”.

   ▶ **Technically:** $\tilde{H}_d(\Upsilon; \mathbb{Q}) = \tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$.
   
   ▶ **Intuitively:** $\Upsilon$ has no “bubbles” whose boundary is an orientable $d$- or $(d-1)$-manifold.

As before, we’ll write $\mathcal{T}(\Delta)$ for the set of SSTs of $\Delta$. 
Examples of SSTs

- \( \text{dim} \Delta = 1: \mathcal{T}(\Delta) = \text{graph-theoretic spanning trees} \)
Examples of SSTs

- $\dim \Delta = 1$: $\mathcal{F}(\Delta) =$ graph-theoretic spanning trees
- $\dim \Delta = 0$: $\mathcal{F}(\Delta) =$ vertices of $\Delta$
Examples of SSTs

- $\dim \Delta = 1$: $\mathcal{T}(\Delta) =$ graph-theoretic spanning trees

- $\dim \Delta = 0$: $\mathcal{T}(\Delta) =$ vertices of $\Delta$

- If $\Delta$ is contractible: it has only one SST, namely itself.
  - Contractible complexes $\approx$ acyclic graphs
  - Some noncontractible complexes also qualify, notably $\mathbb{RP}^2$
Examples of SSTs

- dim $\Delta = 1$: $\mathcal{T}(\Delta) =$ graph-theoretic spanning trees

- dim $\Delta = 0$: $\mathcal{T}(\Delta) =$ vertices of $\Delta$

- If $\Delta$ is contractible: it has only one SST, namely itself.
  - Contractible complexes $\approx$ acyclic graphs
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- If $\Delta$ is a simplicial sphere: SSTs are $\Delta \setminus \{\sigma\}$, where $\sigma \in \Delta$ is any facet (maximal face)
  - Simplicial spheres are analogous to cycle graphs
Pop quiz: How many spanning trees does the equatorial bipyramid
\( \Delta = \langle 123, 124, 134, 234, 125, 135, 235 \rangle \) have?
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**Solution:** 15.
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Solution: 15.

- Either remove triangle 123 and any other triangle (6 SSTs)...

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- ... or one each “northern” and “southern” triangle (9 SSTs).
Examples of SSTs

**Pop quiz:** How many spanning trees does the equatorial bipyramid \( \Delta = \langle 123, 124, 134, 234, 125, 135, 235 \rangle \) have?

\[123 \quad 124 \quad 134 \quad 234 \quad 125 \quad 135 \quad 235\]

Solution: 15.

- Either remove triangle 123 and any other triangle (6 SSTs)...
- ... or one each “northern” and “southern” triangle (9 SSTs).
Every simplicial complex $\Delta$ of dimension $d$ has a **Laplacian matrix** $L = L(\Delta) = [\ell_{ij}]$ with

- rows and columns indexed by $(d - 1)$-faces (**ridges**) of $\Delta$
- $\ell_{ii} =$ number of facets containing $i$
- $\ell_{ij} = \pm 1$ if ridges $i, j$ lie in a common facet, 0 otherwise

\[
\begin{bmatrix}
2 & -1 & -1 & 1 & 1 & 0 \\
-1 & 2 & -1 & -1 & 0 & 1 \\
-1 & -1 & 2 & 0 & -1 & -1 \\
1 & -1 & 0 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Simplicial Matrix-Tree Theorem
(Bolker, Kalai, Adin, Duval–Klivans–JLM, . . .)

Let $\Delta$ be a simplicial complex of dimension $d$.

Form a reduced Laplacian $L_T(\Delta)$ from $L(\Delta)$ by deleting the rows and columns corresponding to a $(d - 1)$-dimensional SST $T \subseteq \Delta$.

Then the “number” of spanning trees of $\Delta$ is $\det L_T$, divided by a correction factor given by $T$. 
Counting SSTs: The Bad News

In dimension $d \geq 2$, spanning trees can have torsion.

- **Technically:** Torsion of $\Upsilon \in \mathcal{T}(\Delta) = |\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})|$
- **Intuitively:** Some piece of $\Upsilon$ is twisted in a funny way (e.g., a non-orientable $d$–manifold)

**Simplicial Matrix-Tree Theorem**

$$\tau(\Delta) \overset{\text{def}}{=} \sum_{\Upsilon \in \mathcal{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})|^2 = \text{(correction factor)} \times \det \hat{L}_T$$

- If $d = 1$ then all summands are 1!
- In many natural cases, the correction factor is 1 as well.
Kalai’s Theorem

Simplicial generalization of the complete graph:

\[ K_{n,d} = \{ F \subseteq \{1, \ldots, n\} \mid \dim F \leq d \} \]
Kalai’s Theorem

Simplicial generalization of the complete graph:

\[ K_{n,d} = \{ F \subseteq \{1, \ldots, n\} \mid \dim F \leq d \} \]

**Theorem**  [Kalai 1983]

\[ \tau(K_{n,d}) = n^{\binom{n-2}{d}}. \]

More generally,

\[
\sum_{\gamma \in \mathcal{T}(K)} \left| \bar{H}_{d-1}(\gamma; \mathbb{Z}) \right|^2 \prod_{i=1}^{n} x_i^{\deg \gamma(i)} = (x_1 \cdots x_n)^{\binom{n-2}{d-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{d}}.
\]
Kalai’s theorem reduces to $\tau(K_n) = n^{n-2}$ when $d = 1$, and the weighted version reduces to Cayley-Prüfer.

Bolker (1976): Observed that $n \binom{n-2}{d}$ is an exact count of trees for small $n, d$, but fails for $n = 6, d = 2$.

The problem is torsion — $\mathbb{RP}^2$ requires six vertices to triangulate.

Adin (1992): Analogous formula for complete colorful complexes, generalizing $\tau(K_{n,m}) = n^{m-1}m^{n-1}$.
A simplicial complex $\Delta$ with vertex set $\{1, 2, \ldots, n\}$ is \textit{shifted} if whenever $a_1 a_2 \cdots a_k \in \Delta$ and $b_i \leq a_i$ for all $i$, then $b_1 b_2 \cdots b_k \in \Delta$.

(So one-dimensional shifted complexes are just threshold graphs.)

**Theorem**  [Duval–Reiner 2002]

Let $\lambda_i =$ number of max-dim faces containing $i$.
Then eigenvalues of $L(\Delta) = \text{column lengths of } \lambda$.
(Generalization of Merris’ Theorem)

**Theorem**  [Duval–Klivans–JLM 2009]

Factorization of multidegree g.f. for spanning trees of a shifted complex. (Generalization of JLM–Reiner formula)
Further Directions

- Theory of SSTs and the Matrix-Tree Theorem generalize easily from simplicial complexes to cell complexes
  - Cellular MTT discovered independently in contexts of probability [Lyons 2009] and mathematical physics [Catanzaro–Chernyak–Klein 2015]

- Simplicial/cell complexes that have integer Laplacian eigenvalues “should” have factorizable weighted tree g.f.’s
  - Matroid complexes; others?

- Critical groups:
  - Complex $\Delta \Rightarrow$ abelian group $K(\Delta)$ of size $\tau(\Delta)$
  - Cuts, flows, sandpile theory, “algebraic geometry on graphs”
  - Group structure very mysterious