Exam 2 Study Guide
Math 126 - Calculus II, Fall 2015

The following is a list of important concepts from each section that will be tested on exam 2. This is not a complete list of the material that you should know for the course, but it is a good indication of what will be emphasized on exam. A thorough understanding of all of the following concepts will help you perform well on exam 1. Some places to find problems on these topics are the following: in the book, in the homework, on quizzes, and online (for example the COW webpage).

Section 6.4 This section is on computing arclength of curves.

- You should be able to compute the arc length of a curve defined by \( y = f(x) \) for \( a \leq x \leq b \) using the equation
  \[
  L = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx.
  \]

- You should be able to compute the arc length of a curve defined by \( x = g(y) \) for \( c \leq y \leq d \) using the equation
  \[
  L = \int_c^d \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy.
  \]

- You should be able to compute the arc length of parametric equation \( x = f(t), y = g(t) \) for \( a \leq t \leq b \) using the equation
  \[
  L = \int_a^b \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt.
  \]

Section 6.5 This section is on the average value of a function over an interval.

- **Important note:** The *average value* of a function and the *mean* of a probability distribution are different! See section 6.8 for information on the mean of a probability density function (aka pdf or probability distribution).

- You should know the definition of the average value of \( f \) on the interval \( f_{[a,b]} \)
  \[
  f_{[a,b]} = \frac{1}{b-a} \int_a^b f(x) \, dx.
  \]

- Be able to compute average values of functions given the function definition (e.g. \( f(x) = x^2 \)) or given the graph of the function (see quiz 4, problem 2 for an example).

- You should know the interpretation of average value in terms of area. The equation
  \[
  f_{[a,b]} \cdot (b-a) = \int_a^b f(x) \, dx,
  \]
  interpreted geometrically means that the area of a rectangle of width \( b-a \) and height \( f_{[a,b]} \) is the same as the area under the curve \( y = f(x) \) on \([a,b]\).
Section 6.6 This section is on applications of integration to physics.

- You should know the formulas \( W = F \cdot d \) (work = force \times distance) and \( F = m \cdot a \) (force = mass \times acceleration).

- You should know how to compute the work done in three types of problems: work done raising a rope/cable (see quiz 4, problem 3 for example), work done to empty a water tank (see section 6.6, problems 19-22 for example), and work done to stretch a spring using Hooke’s law (see section 6.6, problems 5-7 for example).

Section 6.8 This section is on applications of integration to probability.

- You should know the definition of a probability density function (pdf). A function \( f(x) \) is a pdf if \( f(x) \geq 0 \) for all \( x \) and \( \int_{-\infty}^{\infty} f(x)dx = 1 \).

- You should be able to do problems of the form “find the constant \( k \) that makes \( f \) a pdf” (see section 6.8, problems 5, 6 for example).

- You should know how to compute probabilities using a pdf. The probability that \( x \) is between \( a \) and \( b \) is computed by
  \[
P(a \leq x \leq b) = \int_{a}^{b} f(x)dx.
\]

Similarly,
\[
P(x \geq a) = \int_{a}^{\infty} f(x)dx \quad \text{and} \quad P(x \leq b) = \int_{-\infty}^{b} f(x)dx.
\]

- You should know the definition of the mean of a pdf. If \( f(x) \) is a probability density function, then the mean, usually denoted \( \mu \), is
  \[
  \mu = \int_{-\infty}^{\infty} x \cdot f(x)dx.
  \]

Don’t confuse this with the median. We talked briefly about the median and you had a couple homework problems dealing with the median, but you will not be tested on finding medians.

- Important note: The average value of a function and the mean of a probability distribution are different! See section 6.5 for information on the mean of a probability density function (aka pdf or probability distribution).

A general remark for chapter 6: Many of the topics above involve integration. The new material from the sections is only how to set up integrals to represent the appropriate quantity (arc length, work, average value, probabilities, etc.). Once you set up the integral, you will have to compute it, which uses skills we developed in sections 5.5-5.10. It is helpful to keep this in mind while you are studying. Make sure that you understand the new material to set up the appropriate integral, and then use what you learned from earlier sections to evaluate it. This also means that you will need to be comfortable with integration techniques from those sections (primarily \( u \)-substitution and integration by parts), and you will have to be comfortable with improper Riemann integrals (particularly those of the form \( \int_{a}^{\infty} f(x)dx \), \( \int_{-\infty}^{b} f(x)dx \), and \( \int_{-\infty}^{\infty} f(x)dx \)). Make sure that you review these techniques and that you are comfortable with them.

Section 7.1 This section is on differential equations.
• You should be able to check if a proposed function is a solution to a given differential equation. This can be done by “computing the left and right hand side” of the equation, and verifying that they are the same. For example, determine if \( y = \frac{1}{x+C} \) is a solution to \( y' = -y^2 \) for a fixed constant \( C \).

\[
\begin{align*}
\text{LHS} & \quad y = \frac{1}{x+C} \\
\text{RHS} & \quad y = \frac{1}{x+C} \\
y' & = -\frac{1}{(x+C)^2} \\
-y^2 & = -\left(\frac{1}{x+C}\right)^2 = -\frac{1}{(x+C)^2} \\
\end{align*}
\]

The left and right hand side are equal here, and so \( y = \frac{1}{x+C} \) is a solution to the differential equation.

• Given a general solution to a differential equation and an associate initial value problem, you should be able to find the particular solution. For example, given that \( y = \frac{1}{x+C} \) is a solution to \( y' = -y^2 \) for any fixed \( C \), find a solution to the initial value problem

\[
\begin{align*}
\begin{cases}
y' &= -y^2 \\
y(1) &= 7.
\end{cases}
\end{align*}
\]

Since \( y = \frac{1}{x+C} \) is a solution to the equation for any \( C \), we just need to make sure that \( y(1) = 7 \). So we should find a \( C \) so this equation holds, which you can do by solving the equation \( 7 = y(1) = \frac{1}{1+C} \) for \( C \).

In this case, you get \( C = -\frac{6}{7} \). So the solution to the initial value problem is \( y = \frac{1}{x-\frac{6}{7}} \).

Section 7.2 This section is on direction fields for differential equations.

• You should be able to draw the direction field for a given differential equations. You should also be able to match a direction field to a differential equation. Remember one way to interpret this is drawing “little pieces of tangent lines” for solutions to the differential equations.

• You should be able to use a direction field to sketch a qualitative solution to an initial value problem.

Sequences and Series

We worked with sequences and series in a number of different capacities. In order to compute limits of sequences and determine the convergence behavior of series, often the problems can be reduced to what terms tend to infinity “faster” or “slower.” For example, think of computing the following limits

\[
\lim_{n \to \infty} \frac{2^n}{n^2 + 1} \quad \text{or} \quad \lim_{n \to \infty} \frac{n!}{n^n}
\]

Computing these limits reduces to determining whether the numerator or denominator tend to infinity “faster”. In the above examples, \( 2^n \) grows much faster (exponentially) than \( n^2 + 1 \). So the first limit above tends to infinity. On the other hand \( n^n = n \cdot n \cdot n \cdot \ldots \cdot n \cdot n \) is much larger than \( n! = n(n-1)(n-2)\cdots3\cdot2 \) for large \( n \). So the second limit tends to zero as \( n \to \infty \). These are not actual proofs that these limits are \( \infty \) and 0, but the ability to qualitatively evaluate these limits is crucial for the problems on exam 3 relating to sequences and series. For this reason, it is very helpful to know which common expressions tend to infinity “faster” or “slower.” The table below summarizes the growth rate of some common expressions.
### Growth rate

<table>
<thead>
<tr>
<th>Growth rate</th>
<th>( a_n )</th>
<th>( n = 10 )</th>
<th>( n = 50 )</th>
<th>( n = 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>tends to infinity slower</td>
<td>( \ln(n) )</td>
<td>( \ln(10) \approx 2.3 )</td>
<td>( \ln(50) \approx 3.9 )</td>
<td>( \ln(100) \approx 4.6 )</td>
</tr>
<tr>
<td>( \uparrow )</td>
<td>( n^p ) for ( p &gt; 0 )</td>
<td>( 10^2 = 100 )</td>
<td>( 50^2 = 2,500 )</td>
<td>( 100^2 = 10,000 )</td>
</tr>
<tr>
<td>e.g. ( n^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \downarrow )</td>
<td>( a^n ) for ( a &gt; 1 )</td>
<td>( 2^{10} = 1024 )</td>
<td>( 2^{50} \approx 10^{15} )</td>
<td>( 2^{100} \approx 10^{30} )</td>
</tr>
<tr>
<td>e.g. ( 2^n )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tends to infinity faster</td>
<td>( \frac{n!}{n^n} )</td>
<td>( 10^{10} = 10,000,000,000 )</td>
<td>( 50^{50} \approx 8 \cdot 10^{84} )</td>
<td>( 100^{100} = 10^{200} )</td>
</tr>
<tr>
<td></td>
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</tbody>
</table>

See also the worksheet on growth rates of sequences, which is available on the homework page on my website. If you develop a good intuition about which of these common terms tend to infinity “faster” it will be easier to determine limits of sequences and convergence behavior of series. For example, using the table above as an intuitive guide it should be easier to compute the following limits

\[
\lim_{n \to \infty} \frac{\ln(n)}{n} = 0, \quad \lim_{n \to \infty} (n^3 + n^2 + n + 1)2^{-n} = 0, \quad \lim_{n \to \infty} \frac{n^n}{n!} = \infty, \quad \lim_{n \to \infty} \frac{100^n}{n!} = 0,
\]

\[
\lim_{n \to \infty} \frac{50^n}{n^n} = 0, \quad \lim_{n \to \infty} \frac{2^n}{100^n} = \infty, \quad \lim_{n \to \infty} \frac{n^{50}}{n!}10^n = 0, \quad \lim_{n \to \infty} \frac{\ln(n)}{n^{100}} = \infty.
\]

Notice that it is not difficult to construct sequences that tend to infinity “faster” or “slower” than anything on the list above. For example, both \( n^{n^2} \) and \( n^n \) tend to infinity “faster” than \( n^n \) and both \( \ln(\ln(n)) \) and \( \ln(\ln(\ln(n))) \) tend to infinity “slower” than \( \ln(n) \).

### Section 8.1

This section is on limits of sequences.

- Understand the intuitive definition of a sequential limit. What does it mean for \( \lim_{n \to \infty} a_n = L \). The limit of a sequence behaves in one of three ways:

  1. The limit exists: \( \lim_{n \to \infty} a_n = L \). The sequence eventually “settles in” on a number \( L \) and stays close to it for large \( n \).

  2. The limit tends to \( \pm \infty \): \( \lim_{n \to \infty} a_n = \pm \infty \). In the case where \( a_n \to \infty \), the values of the sequence become large and remain large for large \( n \).

  3. The limit does not exist: \( \lim_{n \to \infty} a_n \) DNE. The values of \( a_n \) have to “bounce around” or oscillate in some sense. Think of \((-1)^n\) as a typical example.

- Be able to compute limits using ideas from the table above and techniques from section 2.5 and 4.5 on limits at infinity (section 4.5 is L'Hopital's rule, which you will be expected to be able to use correctly).

- You should know the definition of increasing, decreasing, monotonic, and bounded sequences.

- You should know the monotone convergence theorem, and intuitively why it is true. The monotone convergence theorem states: If a sequences \( a_n \) is monotonic and bounded, then \( \lim_{n \to \infty} a_n \) exists.

- Be able to compute limits of recursively defined sequences using the monotone convergence theorem. For example, find the limit of the sequence defined by \( a_0 = 1 \) and \( a_{n+1} = \sqrt{3}a_n \). First check that \( a_n \)
is increasing and bounded. Then by the monotone convergence theorem, we know that \(a_n \rightarrow a\) as \(n \rightarrow \infty\). Then

\[
a = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \sqrt{3a_n} = \sqrt{3a}.
\]

Then \(a\) satisfies \(a = \sqrt{3a}\). This equation simplifies to \(a(a - 3) = 0\). The solutions to this equation are 0 and 3. Since \(a_n \geq 1\) for all \(n\), the limit must be \(a = 3\). Therefore \(a_n \rightarrow 3\) as \(n \rightarrow \infty\).

**Section 8.2** This section is on the definition of convergence/divergence of series and computing some series.

- Know the definition of convergence versus divergence of series. In particular, the convergence of series is defined in terms of the limit of partial sums. For a sequence \(a_n\), the sequence of partial sum \(S_N\) is defined

\[
S_N = \sum_{n=1}^{N} a_n,
\]

which is itself a sequence.

- Know the divergence test for series and how to use it: if \(\lim_{n \rightarrow \infty} a_n \neq 0\), then \(\sum_{n=1}^{\infty} a_n\) diverges.

- Be able to recognize a geometric series, determine whether it converges or diverges, and compute the value of convergent geometric series.

\[
\sum_{n=0}^{\infty} x^n = \begin{cases} 
\frac{1}{1-x} & |x| < 1 \\
\text{diverges} & |x| \geq 1
\end{cases} \quad \text{and} \quad \sum_{n=1}^{\infty} x^n = \begin{cases} 
\frac{x}{1-x} & |x| < 1 \\
\text{diverges} & |x| \geq 1
\end{cases}
\]

You should also be able to use this to write a repeating decimal as a fraction. For example,

\[
23.232\overline{3} = 23 + .23 + .0023 + .000023 + \cdots \\
= 23 \cdot \left(\frac{1}{100}\right)^0 + 23 \cdot \left(\frac{1}{100}\right)^1 + 23 \cdot \left(\frac{1}{100}\right)^2 + 23 \cdot \left(\frac{1}{100}\right)^3 + \cdots \\
= \sum_{n=0}^{\infty} 23 \cdot \left(\frac{1}{100}\right)^n = 23 \cdot \frac{1}{\frac{1}{100} - 1} = \frac{2,300}{99}
\]

- Be able to recognize and compute telescoping sums. For example,

\[
\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right) = \lim_{N \rightarrow \infty} \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots \\
\cdots + \left(\frac{1}{N-2} - \frac{1}{N}\right) + \left(\frac{1}{N-1} - \frac{1}{N+1}\right) + \left(\frac{1}{N} - \frac{1}{N+2}\right) \\
= \lim_{N \rightarrow \infty} 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} = \frac{3}{2}
\]

Note that this may involve using a partial fraction decomposition.