A BILINEAR T(B) THEOREM FOR SINGULAR INTEGRAL OPERATORS

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ABSTRACT. In this work, we present a bilinear Tb theorem for singular integral operators of Calderón-Zygmund type. We prove some new accretive type Littlewood-Paley results and construct a bilinear paraproduct for a para-accretive function setting. As an application of our bilinear Tb theorem, we prove product Lebesgue space bounds for bilinear Riesz transforms defined on Lipschitz curves.

1. INTRODUCTION

In the development of Calderón-Zygmund singular integral operator theory, measuring cancellation of operators via testing conditions has become a central theme through T1 and Tb theorems. In the 1980’s, David-Journé [10] proved the original T1 theorem, which gave a characterization of $L^2$ boundedness for Calderón-Zygmund operators. Driven by the Cauchy integral operator, in the late 1980’s David-Journé-Semmes [11] and McIntosh-Meyer [30] proved Tb theorems, which are also characterizations of $L^2$ bounds for Calderón-Zygmund operators based on perturbed testing conditions. We state the version from [11] to compare to the bilinear version we present in this work.

**Tb Theorem.** Let $b_0, b_1$ be para-accretive functions. Assume that $T$ is a singular integral operator of Calderón-Zygmund type associated to $b_0, b_1$. Then $T$ can be extended to a bounded operator on $L^2$ if and only if $M_{b_0}T M_{b_1}$ satisfies the weak boundedness property and $M_{b_0}T(b_1), M_{b_1}T^*(b_0) \in BMO$.

From the late 1980’s to the early 2000’s, multilinear Calderón-Zygmund theory was developed and multilinear Tb theorems and boundedness results were obtained by Christ-Journé [8], Kenig-Stein [26], and Grafakos-Torres [17]. Many analogs of the linear theory have been found in the multilinear setting, but to date there has been no multilinear Tb theorem. In this work we prove a bilinear Tb theorem (which can be naturally extended also to a higher degree of multilinearity). The proof presented in this work does not rely on the linear Tb theorem of David-Journé-Semmes [11] or McIntosh-Meyer [30]. Furthermore a new proof of the linear Tb theorem can be easily extracted from the work in this paper. We state our main result.

**Theorem 1.1.** Let $b_0, b_1, b_2$ be para-accretive functions. Assume that $T$ is a bilinear singular integral operator of Calderón-Zygmund type associated to $b_0, b_1, b_2$. Then $T$ can be extended to a bounded operator from $L^{p_1} \times L^{p_2}$ into $L^p$ for all $1 < p_1, p_2 < \infty$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ if and only if $M_{b_0}T(M_{b_1} \cdot, M_{b_2} \cdot)$ satisfies the weak boundedness property and $M_{b_0}T(b_1, b_2), M_{b_1}T^{*1}(b_0, b_2), M_{b_2}T^{*2}(b_1, b_0) \in BMO$.

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The meaning of $M_b T(b_1, b_2) \in BMO$ is not a priori clear here, but we define this notations in Section 2. We also define other terminology use throughout the work, such as para-accretive, singular integral operators of Calderón-Zygmund type, weak boundedness property, etc.

Calderón [5] proved some convergence results for a reproducing formula of the form

$$\int_0^\infty \phi_t * \phi_t * f \frac{dt}{t} = f,$$

for appropriate functions $\phi_t$, which came to be known as Calderón’s reproducing formula. The convergence of Calderón’s reproducing formula holds in many function space topologies; see for example Calderón [5, 6], Janson-Taibleson [24], Frazier-Jawerth-Weiss [14], and the references therein. This formula has since been generalized and reformulated in many ways. In this work, we consider discrete versions of Calderón’s formula where we replace convolution with $\phi_t$ with certain non-convolution integral operators indexed by a discrete parameter $k \in \mathbb{Z}$ instead of the continuous parameter $t > 0$. We prove criterion for extending the convergence of perturbed discrete Calderón reproducing formulas from $L^p$ spaces to $H^1$. More precisely, we will prove:

**Theorem 1.2.** Let $b \in L^{\infty}$ be para-accretive functions and $\theta_k$ be a collection of Littlewood-Paley square function kernels such that $\Theta_k b = \Theta_k^* b = 0$ for all $k \in \mathbb{Z}$. Also assume that

$$\sum_{k \in \mathbb{Z}} M_b \Theta_k M_b f = b f$$

for any $f \in C^\delta_0$ such that $bf$ has mean zero, where the convergence holds in $L^p$ for some $1 < p < \infty$. If $\phi \in C^\delta_0$ for some $1 < \delta \leq 1$ such that $b\phi$ has mean zero, then $b\phi \in H^1$ and

$$\sum_{k \in \mathbb{Z}} M_b \Theta_k M_b \phi = b\phi,$$

where the convergence holds in $H^1$.

Here, we take the typical definition of $H^1$ with norm $||f||_{H^1} = ||f||_{L^1} + \sum_{\ell=1}^{n} ||R_\ell f||_{L^1}$, where $R_\ell$ is the $\ell^{th}$ Reisz transform in $\mathbb{R}^n$ for $\ell = 1, \ldots, n$, $R_\ell f = c_n \left( p.v. \frac{\gamma^\ell}{|y|^{n+\ell}} * f \right)$ and $c_n$ is a dimensional constant. Theorem 1.2 tells use that anytime we have convergence of Calderón’s reproducing formula in $L^p$ for some $p$, then it also converges in $H^1$, for appropriate operators and functions.

This article is organized the following way: In Section 2, we set notation and give a few pertinent definitions. In Section 3, we prove a few almost orthogonality estimates for bilinear Littlewood-Paley square function kernels and operators. In Section 4, we prove a number of convergence results in various spaces, including the $H^1$ convergence stated in Theorem 1.2. In Section 5, we prove an estimate closely related to bilinear Littlewood-Paley square function theory, which will serve as an estimate for truncated Calderón-Zygmund operators. In Section 6, we complete the proof of Theorem 1.1 by proving a reduced $Tb$ theorem and constructing a bilinear paraproduct for para-accretive perturbed setting. In Section 7, we apply our bilinear $Tb$ theorem 1.1 to bilinear Riesz transforms defined by principle value operators along Lipschitz curves.

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2. Definitions and Preliminaries

We first define para-accretive functions as one of several equivalent definitions provided by David-Journé-Semmes [11].

**Definition 2.1.** A function $b \in L^\infty$ is para-accretive if $b^{-1} \in L^\infty$ and there is a $c_0 > 0$ such that for every cube $Q$, there exists a sub-cube $R \subset Q$ such that

$$\frac{1}{|Q|} \left| \int_R b(x) dx \right| \geq c_0.$$

Many results involving para-accretive functions were proved by David-Journé-Semmes [11], McIntosh-Meyer [30], and by Han in [19]. We will use a number of the results from those works.

2.1. Bilinear Singular Integrals Associated to Para-Accretive Functions. Next we introduce the Hölder continuous spaces and para-accretive perturbed Hölder spaces. These are the functions spaces that we use to form our initial weak continuity assumption for $T$ in Theorem 1.1, similar to the linear $Tb$ theorem in [11].

**Definition 2.2.** Define for $0 < \delta \leq 1$ and $f : \mathbb{R}^n \to \mathbb{C}$

$$||f||_\delta = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\delta};$$

and the space $C^\delta = C^\delta(\mathbb{R}^n)$ to be the collection of all functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $||f||_\delta < \infty$. Also define $C^\delta_0 = C^\delta_0(\mathbb{R}^n)$ to be the subspace of all compactly supported functions in $C^\delta$. It follows that $|| \cdot ||_\delta$ is a norm on $C^\delta_0$. Despite conventional notation, we will take $C^1$ and $C^1_0$ to be the spaces of Lipschitz continuous functions to keep our notation consistent.

Let $b$ be a para-accretive function and define $bC^\delta_0$ to be the collection of functions $bf$ such that $f \in C^\delta_0$ with norm $||bf||_{b\delta} = ||f||_\delta$. Also let $(bC^\delta_0)'$ be the collection of all sequentially continuous linear functionals on $bC^\delta_0$, i.e. a linear functional $W : bC^\delta_0 \to \mathbb{C}$ is in $(bC^\delta_0)'$ if and only if

$$\lim_{k \to \infty} ||f_k - f||_\delta = 0 \quad \text{where} \quad f_k, f \in C^\delta_0 \quad \implies \quad \lim_{k \to \infty} \langle W, bf_k \rangle = \langle W, bf \rangle,$$

where these are both limits of complex numbers. Given a topological space $X$, we say that an operator $T : X \to (bC^\delta_0)'$ is continuous if

$$\lim_{k \to \infty} x_k = x \in X \quad \implies \quad \lim_{k \to \infty} \langle T(x_k), bf \rangle = \langle T(x), bf \rangle \quad \text{for all} \quad f \in C^\delta_0.$$

Given a bilinear operator $T : b_1 C^\delta_0 \times b_2 C^\delta_0 \to (b_0 C^\delta_0)'$ for some $\delta > 0$, define the transposes of $T$ for $f_1, f_2, 3 \in C^\delta_0$

$$\langle T^{1*}(b_0 f_0, b_2 f_2), b_1 f_1 \rangle = \langle T^{2*}(b_1 f_1, b_0 f_0), b_1 f_1 \rangle = \langle T(b_1 f_1, b_2 f_2), b_0 f_0 \rangle.$$

Then the transposes of $T$ are bilinear operators acting on the following spaces: $T^{1*} : b_0 C^\delta_0 \times b_2 C^\delta_0 \to (b_1 C^\delta_0)'$ and $T^{2*} : b_1 C^\delta_0 \times b_0 C^\delta_0 \to (b_2 C^\delta_0)'$. One could more generally define the transpose $T^{1*}$ on $(b_1 C^\delta_0)'' \times b_1 C^\delta_0$, but this is not necessary for this work. So we restrict the first spot of $T^{1*}$ to $b_1 C^\delta_0$ instead of $(b_1 C^\delta_0)'$. Likewise for $T^{2*}$.
Definition 2.3. A function $K : \mathbb{R}^3 \setminus \{(x,x,x) : x \in \mathbb{R}^n\} \to \mathbb{C}$ is a standard bilinear Calderón-Zygmund kernel if

$$|K(x,y_1,y_2)| \lesssim \frac{1}{(|x-y_1| + |x-y_2|)^{2n}} \text{ when } |x-y_1| + |x-y_2| \neq 0$$

$$|K(x,y_1,y_2) - K(x',y_1,y_2)| \lesssim \frac{|x-x'|}{(|x-y_1| + |x-y_2|)^{2n+2}}$$

when $|x-x'| < \max(|x-y_1|, |x-y_2|)/2$

$$|K(x,y_1,y_2)| \lesssim \frac{|y_1 - y_1'|}{(|x-y_1| + |x-y_2|)^{2n}}$$

when $|y_1 - y_1'| < \max(|x-y_1|, |x-y_2|)/2$

$$|K(x,y_1,y_2) - K(x,y_1',y_2')| \lesssim \frac{|y_2 - y_2'|}{(|x-y_1| + |x-y_2|)^{2n}}$$

when $|y_2 - y_2'| < \max(|x-y_1|, |x-y_2|)/2$.

Let $b_0, b_1, b_2 \in L^\infty(\mathbb{R}^n)$ be para-accretive functions. We say a bilinear operator $T : b_1 C_0^\delta \times b_2 C_0^\delta \to (b_0 C_0^\delta)'$ is a bilinear singular integral operator of Calderón-Zygmund type associated to $b_0, b_1, b_2$, or for short a bilinear C-Z operator associated to $b_0, b_1, b_2$, if $T$ is continuous from $b_1 C_0^\delta \times b_2 C_0^\delta$ into $(b_0 C_0^\delta)'$ for some $\delta > 0$ and there exists a standard Calderón-Zygmund kernel $K$ such that for all $f_1, f_2, f_3 \in C_0^\delta$ with disjoint support

$$\langle T(M_{b_1} f_1, M_{b_2} f_2), M_{b_0} f_0 \rangle = \int_{\mathbb{R}^n} K(x,y_1,y_2) \prod_{i=0}^2 f_i(y_i) b_i(y_i) dy_i.$$

Note that this continuity assumption for $T$ from $b_1 C_0^\delta \times b_2 C_0^\delta$ into $(b_0 C_0^\delta)'$ is equivalent to the following: For any $f_0, f_1, f_2, g, k \in C_0^\delta$ such that $g_k \to g$ in $C_0^\delta$, we have

$$\lim_{k \to \infty} \langle T(M_{b_1} g_k, M_{b_2} f_2), M_{b_0} f_0 \rangle = \langle T(M_{b_1} g, M_{b_2} f_2), M_{b_0} f_0 \rangle,$$

$$\lim_{k \to \infty} \langle T(M_{b_1} f_1, M_{b_2} g_k), M_{b_0} f_0 \rangle = \langle T(M_{b_1} f_1, M_{b_2} g), M_{b_0} f_0 \rangle,$$

$$\lim_{k \to \infty} \langle T(M_{b_1} f_1, M_{b_2} f_2), M_{b_0} g_k \rangle = \langle T(M_{b_1} f_1, M_{b_2} f_2), M_{b_0} g \rangle.$$

It follows that the continuity assumptions for a bilinear singular integral operator $T$ associated to para-accretive functions $b_0, b_1, b_2$ is symmetric under transposes. That is, $T$ is a bilinear C-Z operator associated to $b_0, b_1, b_2$ if and only if $T^\ast$ is a bilinear C-Z operator associated to $b_1, b_0, b_2$ if and only if $T^{2\ast}$ is a bilinear C-Z operator associated to $b_2, b_1, b_0$.

Definition 2.4. A function $\phi \in C_0^\infty$ is a normalized bump of order $m \in \mathbb{N}$ if $\text{supp}(\phi) \subset B(0,1)$ and

$$\sup_{|\alpha| \leq m} ||\partial^\alpha \phi||_{L^\infty} \leq 1.$$

Let $b_0, b_1, b_2 \in L^\infty$ be para-accretive functions, and $T$ be an bilinear C-Z operator associated to $b_0, b_1, b_2$. We say that $M_{b_0} T(M_{b_1} \cdot, M_{b_2} \cdot)$ satisfies the weak boundedness property (written $M_{b_0} T(M_{b_1} \cdot, M_{b_2} \cdot)$ satisfies the WBP) if there exists an $m \in \mathbb{N}$ such that for all normalized bumps $\phi_0, \phi_1, \phi_2 \in C_0^\infty$ of order $m, x \in \mathbb{R}^n$, and $R > 0$

$$\langle T(M_{b_1} \phi_1^{+R}, M_{b_2} \phi_2^{+R}), M_{b_0} \phi_0^{+R} \rangle \lesssim R^m$$

where $\phi^{+R}(u) = \phi(\frac{u-R}{R})$. 

It follows by the symmetry of this definition that $M_{b_0}T(M_{b_1} \cdot, M_{b_2} \cdot) \in WBP$ if and only if $M_{b_1}T^{1+}(M_{b_0} \cdot, M_{b_2} \cdot) \in WBP$ if and only if $M_{b_2}T^{2+}(M_{b_0} \cdot, M_{b_1} \cdot) \in WBP$. Next we define $T$ on $(b_1C^0 \cap L^1) \times (b_2C^0 \cap L^1)$, so that we can make sense of the testing condition $M_{b_0}T(b_1, b_2) \in BMO$ as well as the transpose conditions. The definition we give is essentially the same as the one given by Torres [39] in the linear setting and by Grafakos-Torres [17] in the multilinear setting. Here we use the definition from [17] with the necessary modifications for the accretive functions $b_0, b_1, b_2$. A benefit of this definition versus the ones (see e.g. [10] or [11]) is that we define $T(b_1, b_2)$ paired with any element of $b_0C^0_0$, not just the ones with mean zero. Although one must still take care to note that the definition of $T$ agrees with the given definition of $T$ when paired with elements of $b_0C^0_0$ with mean zero. This is all made precise in the next definition and the remarks that follow it.

**Definition 2.5.** Let $b_0, b_1, b_2$ be para-accretive function, $T$ be a bilinear singular integral operator associated to $b_0, b_1, b_2$, and $f_1, f_2 \in C^0 \cap L^\infty$. Also fix functions $\eta_R^i \in C^\infty_0$ for $R > 0$, $i = 1, 2$ such that $\eta_R^i \equiv 1$ on $B(0, R)$ and $\text{supp}(\eta_R^i) \subset B(0, 2R)$. Then we define

$$
\langle T(b_1f_1, b_2f_2), b_0f_0 \rangle = \lim_{R \to \infty} \langle T(\eta_R^1b_1f_1, \eta_R^2b_2f_2), b_0f_0 \rangle
$$

(2.1) 

$$
= \int_{\mathbb{R}^3} K(0, y_1, y_2)b_0(x)f_0(x)\prod_{i=1}^2f_i(y_i)(\eta_R^1(y_i) - \eta_R^0(y_i))b_i(y_i)dx dy_1 dy_2,
$$

where $f_0 \in C^0_0$ and $R_0 > 0$ is minimal such that $\text{supp}(f_0) \subset B(0, R_0/2)$. When $R > 2R_0$, we have

$$
\langle T(\eta_R^1b_1f_1, \eta_R^2b_2f_2), b_0f_0 \rangle = \langle T(\eta_R^1b_1f_1, \eta_R^2b_2f_2), b_0f_0 \rangle + \langle T(\eta_R^1b_1f_1, \eta_R^2 - \eta_R^0)b_2f_2), b_0f_0 \rangle
$$

$$
+ \langle T((\eta_R^1 - \eta_R^0)b_1f_1, \eta_R^2b_2f_2), b_0f_0 \rangle + \langle T((\eta_R^1 - \eta_R^0)b_1f_1, \eta_R^2 - \eta_R^2)b_2f_2), b_0f_0 \rangle
$$

$$
= \langle T(\eta_R^1b_1f_1, \eta_R^2b_2f_2), b_0f_0 \rangle
$$

$$
+ \int_{\mathbb{R}^3} K(0, y_1, y_2)\eta_R^1(y_1)(\eta_R^1(y_2) - \eta_R^0(y_2))\prod_{i=0}^2b_i(y_i)f_i(y_i)dy_0 dy_1 dy_2
$$

$$
+ \int_{\mathbb{R}^3} K(0, y_1, y_2)(\eta_R^1(y_1) - \eta_R^0(y_1))\eta_R^2(y_2)\prod_{i=0}^2b_i(y_i)f_i(y_i)dy_0 dy_1 dy_2
$$

$$
+ \langle T((\eta_R^1 - \eta_R^0)b_1f_1, \eta_R^2 - \eta_R^2)b_2f_2), b_0f_0 \rangle
$$

$$
= I + II + III + IV.
$$

The first term $I$ is well defined since $\eta_R^0b_1f_1 \in b_2C^0_0$ for a fixed $R_0$ (depending on $f_0$). We check that the first integral term $II$ is absolutely convergent: The integrand of $II$ is bounded by $||b_0||_{L^\infty} \prod_{i=1}^2||b_i||_{L^\infty} ||f_i||_{L^\infty}$ times

$$
|K(y_0, y_1, y_2)\eta_R^1(y_1)(\eta_R^1(y_2) - \eta_R^0(y_2))f_0(y_0)| \lesssim \frac{|\eta_R^1(y_1)(\eta_R^1(y_2) - \eta_R^0(y_2))f_0(y_0)|}{(|y_0 - y_1| + |y_0 - y_2|)^{2\alpha}}
$$

$$
\lesssim \frac{|\eta_R^1(y_1)f_0(y_0)|}{(|y_0 - y_1| + |y_0 - y_2|/2 + (R_0 - R_0/2)/2)^{2\alpha}}
$$

$$
\lesssim \frac{|\eta_R^1(y_1)f_0(y_0)|}{(R_0 + |y_0 - y_2|)^{2\alpha}}.
$$
This is an \( L^1(\mathbb{R}^{3n}) \) function that is independent of \( R \) (as long as \( R > 4R_0 \)),
\[
\int_{\mathbb{R}^{3n}} \frac{|\eta^1_{R_0}(y_1)f_0(y_0)|}{(R_0 + |y_0 - y_2|)^{2n}} dy_0 dy_1 dy_2 \lesssim \int_{\mathbb{R}^{2n}} \frac{|\eta^1_{R_0}(y_1)f_0(y_0)|}{R_0^n} dy_0 dy_1 \lesssim \|f_0\|_{L^0}.
\]
Since \( \eta_R \to 1 \) pointwise, by dominated convergence the following limit exists:
\[
\lim_{R \to \infty} \int_{\mathbb{R}^{3n}} K(y_0, y_1, y_2) \eta^1_{R_0}(y_1)(\eta^2_R(y_2) - \eta^2_{R_0}(y_2)) \prod_{i=0}^2 b_i(y_i) f_i(y_i) dy_0 dy_1 dy_2
\]
\[
= \int_{\mathbb{R}^{3n}} K(y_0, y_1, y_2) \eta^1_{R_0}(y_1)(1 - \eta^2_{R_0}(y_2)) \prod_{i=0}^2 b_i(y_i) f_i(y_i) dy_0 dy_1 dy_2.
\]
So \( \lim_{R \to \infty} II \) exists. A symmetric argument holds for \( \lim_{R \to \infty} III \). Finally, we consider IV minus the integral term from (2.1)
\[
IV - \int_{\mathbb{R}^3} K(0, y_1, y_2)b_0(y_0)f_0(y_0) \prod_{i=1}^2 f_i(y_i) \eta^1_R(y_i)b_i(y_i) dy_0 dy_1 dy_2
\]
\[
= \int_{\mathbb{R}^{3n}} (K(y_0, y_1, y_2) - K(0, y_1, y_2))b_0(y_0)f_0(y_0)
\]
\[
\times \prod_{i=1}^2 (\eta^1_R(y_i) - \eta^1_{R_0}(y_i)) f_i(y_i)b_i(y_i) dy_0 dy_1 dy_2.
\]
Again we bound the integrand by \( \|b_0\|_{L^2} \prod_{i=1}^2 \|b_i\|_{L^2} \|f_i\|_{L^2} \)-times
\[
|K(y_0, y_1, y_2) - K(0, y_1, y_2)||f_0(y_0)|(\eta^1_R(y_1) - \eta^1_{R_0}(y_1)) \lesssim \frac{|y_0|^\gamma|\eta^1_R(y_1) - \eta^1_{R_0}(y_1)|}{(|y_0 - y_1| + |y_0 - y_2|)^{2n+\gamma}}|f_0(y_0)|
\]
\[
\lesssim \frac{|y_0|^\gamma|\eta^1_R(y_1) - \eta^1_{R_0}(y_1)|}{(4R_0 + |y_0 - y_1| + |y_0 - y_2|)^{2n+\gamma}}|f_0(y_0)|
\]
\[
\lesssim \frac{R_0^n|f_0(y_0)|}{(R_0 + |y_0 - y_1| + |y_0 - y_2|)^{2n+\gamma}},
\]
which is an \( L^1(\mathbb{R}^{3n}) \) function:
\[
\int_{\mathbb{R}^{3n}} \frac{|f_0(y_0)|}{(R_0 + |y_0 - y_1| + |y_0 - y_2|)^{2n+\gamma}} dy_0 dy_1 dy_2 \lesssim \int_{\mathbb{R}^{2n}} \frac{|f_0(y_0)|}{(R_0 + |y_0 - y_1|)^{\gamma+\gamma}} dy_0 dy_1
\]
\[
\lesssim \int_{\mathbb{R}^{n}} \|f_0(y_0)\|dy_0 \lesssim \|f_0\|_{L^0}.
\]
Then it follows again by dominated convergence that
\[
\lim_{R \to \infty} \langle T((\eta^1_R - \eta^1_{R_0})b_1 f_1, (\eta^2_R - \eta^2_{R_0})b_2 f_2, b_0 f_0) \rangle
\]
\[
- \int_{\mathbb{R}^3} K(0, y_1, y_2)b_0(y_0)f_0(y_0) \prod_{i=1}^2 f_i(y_i) (\eta^1_R(y_i) - \eta^1_{R_0}(y_i)) b_i(y_i) dy_0 dy_1 dy_2
\]
\[
= \int_{\mathbb{R}^{3n}} (K(x, y_1, y_2) - K(0, y_1, y_2))b_0(x)f_0(x) \prod_{i=1}^2 (1 - \eta^1_{R_0}(y_i)) f_i(y_i)b_i(y_i) dy_1 dy_2 dx,
\]
which is an absolutely convergent integral. Therefore \( T(b_1 f_1, b_2 f_2) \) is well defined as an element of \( (b_0C^0_\gamma)' \) for \( f_1, f_2 \in C^0 \cap L^\alpha \). Furthermore if \( f_0, f_1, f_2 \in C^0 \) and \( b_0 f_0 \) has mean
zero, then this definition of $T$ is consistent with the a priori definition of $T$ since
\[
\lim_{R \to \infty} \int_{R^3} K(0,y_0,y_2)b_0(y_0)f_0(y_0)\sum_{i=1}^{n} f_i(y_i)(\eta^i_R(y_i) - \eta^i_{R_0}(y_i))b_i(y_i)dy_0 dy_1 dy_2
\]

\[
= \left( \int_{R^3} K(0,y_0,y_2)\prod_{i=1}^{n} b_i(y_i) f_i(y_i)(1 - \eta^i_{R_0}(y_i))dy_1 dy_2 \right) \left( \int_{R^0} b_0(y_0)f_0(y_0)dy_0 \right) = 0,
\]
since both of these integrals are absolutely convergent. Also, when $b_0 f_0$ has mean zero in this way, the definition of $\langle T(b_1,b_2), b_0 f_0 \rangle$ is independent of the choice of $\eta^i_R$ and $\eta^i_{R_0}$. We will also use the notation $M_{b_0} T(b_1,b_2) \in BMO$ or $M_{b_0} T(b_1,b_2) = \beta$ for $\beta \in BMO$ to mean that for all $f_0 \in C^0_0$ such that $b_0 f_0$ has mean zero, the following holds:
\[
\langle T(b_1,b_2), b_0 f_0 \rangle = \langle \beta, b_0 f_0 \rangle.
\]

Here the left hand side makes sense since $T(b_1,b_2)$ is defined in $(b_0C^0_0)'$. The right hand side also makes sense since $b_0 f_0 \in H^1$ for $f_0 \in C^0_0$ where $b_0 f_0$ has mean zero. The condition $M_{b_0} T(b_1,b_2) \in BMO$ defined here is weaker than (possibly equivalent to) $T(b_1,b_2) \in BMO$ when we can make sense of $T(b_1,b_2)$ as a locally integrable function. This is because our definition of $M_{b_0} T(b_1,b_2) \in BMO$ only requires this equality to hold when paired with a subset of the predual space of $BMO$, namely we require this to hold for $\{ b_0 f : f \in C^0_0 \}$ and $b_0 f$ has mean zero $\} \subseteq H^1$. It is possible that this is equivalent through some sort of density argument, but that is not of consequence here. So we do not pursue it any further, and use the definition of $M_{b_0} T(b_1,b_2) \in BMO$ that we have provided. Furthermore, if $T$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^p$ for some $1 \leq p_1, p_2, p < \infty$, then $T$ can be defined on $L^\infty \times L^\infty$ and is bounded from $L^\infty \times L^\infty$ into $BMO$. This is result is due to Peetre [31], Spanne [34], and Stein [35] in the linear setting and Grafakos-Torres [17] in the bilinear setting. Hence, if $T$ is bounded, then $M_{b_0} T(b_1,b_2), M_{b_1} T^* (b_0,b_2), M_{b_2} T^*(b_1,b_0) \in BMO$.

2.2. Function, Operator, and General Notations. Define for $N > 0, k \in \mathbb{Z}$, and $x \in \mathbb{R}^n$
\[
\Phi^N_k(x) = \frac{2^{kn}}{(1 + 2^k |x|)^N}.
\]

For $f : \mathbb{R}^n \to \mathbb{C}$, we use the notation $f_k(x) = 2^{kn} f(2^k x)$. We will say indices $0 < p, p_1, p_2 < \infty$ satisfy a Hölder relationship if
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.
\]

**Definition 2.6.** Let $\theta_k$ be a function from $\mathbb{R}^{2n}$ into $\mathbb{C}$ for each $k \in \mathbb{Z}$. We call $\{ \theta_k \}_{k \in \mathbb{Z}}$ a collection of Littleswood-Paley square function kernels of type $LPK(A,N,\gamma)$ for $A > 0$, $N > n$, and $0 < \gamma \leq 1$ if for all $x,y,y' \in \mathbb{R}^n$ and $k \in \mathbb{Z}$
\[
|\theta_k(x,y)| \leq A \Phi^{N+\gamma}_k(x-y)
\]
(2.3)
\[
|\theta_k(x,y) - \theta_k(x,y')| \leq A (2^k |y-y'|)^\gamma \left( \Phi^{N+\gamma}_k(x-y) + \Phi^{N+\gamma}_k(x-y') \right).
\]
(2.4)

We say that $\{ \theta_k \}_{k \in \mathbb{Z}}$ is a collection of smooth Littleswood-Paley square function kernels of type $SLPK(A,N,\gamma)$ for $A > 0$, $N > n$, and $0 < \gamma \leq 1$ if it satisfies (2.3), (2.4), and for all $x,x',y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$
\[
|\theta_k(x,y) - \theta_k(x',y)| \leq A (2^k |x-x'|)^\gamma \prod_{j=1}^{\gamma} \left( \Phi^{N+\gamma}_k(x'-y) - \Phi^{N+\gamma}_k(x'-y) \right).
\]
(2.5)
If \( \{ \theta_k \} \) is a collection of Littlewood-Paley square function kernels of type \( LPK(A,N,\gamma) \) (respectively \( SLPK(A,N,\gamma) \)) for some \( A > 0, N > n \), and \( 0 < \gamma \leq 1 \), then write \( \{ \theta_k \} \in LPK \) (respectively \( \{ \theta_k \} \in SLPK \)). We also define for \( k \in \mathbb{Z}, x \in \mathbb{R}^n, \) and \( f \in L^1 + L^\infty \)
\[
\Theta_k f(x) = \int_{\mathbb{R}^n} \theta_k(x,y) f(y) dy.
\]

**Definition 2.7.** Let \( \theta_k \) be a functions from \( \mathbb{R}^{3n} \) into \( \mathbb{C} \) for each \( k \in \mathbb{Z} \). We call \( \{ \theta_k \}_{k \in \mathbb{Z}} \) a collection of bilinear Littlewood-Paley square function kernels of type \( BLPK(A,N,\gamma) \) for \( A > 0, N > n \), and \( 0 < \gamma \leq 1 \) if it satisfies (2.3)-(2.5) and for all \( x,x',y_1,y_2 \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \)
\[
|\theta_k(x,y_1,y_2) - \theta_k(x,y_1',y_2)| \leq A(2^k|y_1 - y_1'|+\gamma|y_1-y_1'|)\Phi_k^N(x),
\]
\[
|\theta_k(x,y_1,y_2) - \theta_k(x,y_2,y_2')| \leq A(2^k|y_2 - y_2'|+\gamma|y_2-y_2'|)\Phi_k^N(x).
\]

We say that \( \{ \theta_k \}_{k \in \mathbb{Z}} \) is a collection of smooth Littlewood-Paley square function kernels of type \( SBLPK(A,N,\gamma) \) for \( A > 0, N > n \), and \( 0 < \gamma \leq 1 \) if it satisfies (2.3)-(2.5) and for all \( x,x',y_1,y_2 \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \)
\[
|\theta_k(x,y_1,y_2) - \theta_k(x',y_1,y_2)| \leq A(2^k|y_2 - y_2'|+\gamma|y_2-y_2'|)\Phi_k^N(x).
\]

If \( \{ \theta_k \} \) is a collection of bilinear Littlewood-Paley square function kernels of type \( BLPK(A,N,\gamma) \) (respectively of type \( SBLPK(A,N,\gamma) \)) for some \( A > 0, N > n \), and \( 0 < \gamma \leq 1 \), then we write \( \{ \theta_k \} \in BLPK \) (respectively \( \{ \theta_k \} \in SBLPK \)). We also define for \( k \in \mathbb{Z}, x \in \mathbb{R}^n, \) and \( f_1, f_2 \in L^1 + L^\infty \)
\[
\Theta_k(f_1,f_2)(x) = \int_{\mathbb{R}^{2n}} \theta_k(x,y_1,y_2) f_1(y_1) f_2(y_2) dy_1 dy_2.
\]

**Remark 2.8.** Let \( \theta_k \) be a function from \( \mathbb{R}^{3n} \) to \( \mathbb{C} \) for each \( k \in \mathbb{Z} \). There exists \( A_1 > 0, N_1 > n \), and \( 0 < \gamma \leq 1 \) such that \( \{ \theta_k \} \) is a collection of Littlewood-Paley square function kernels of type \( SBLPK(A_1,N_1,\gamma) \) if and only if there exist \( A_2 > 0, N_2 > n \), and \( 0 < \gamma \leq 1 \) such that for all \( x,y_1,y_2,y_1',y_2 \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \)
\[
|\theta_k(x,y_1,y_2)| \leq A_2 \Phi_k^N(x-y_1)\Phi_k^N(x-y_2)
\]
\[
|\theta_k(x,y_1,y_2) - \theta_k(x,y_1',y_2)| \leq A_2 2^{2nk}(2^k|y_1-y_1'|+\gamma|y_1-y_1'|)
\]
\[
|\theta_k(x,y_1,y_2) - \theta_k(x,y_2,y_2')| \leq A_2 2^{2nk}(2^k|y_2-y_2'|+\gamma|y_2-y_2'|)
\]
\[
|\theta_k(x,y_1,y_2) - \theta_k(x',y_1,y_2)| \leq A_2 2^{2nk}(2^k|x-x'|+\gamma|x-x'|).
\]

A similar equivalence holds for smooth square function kernels of type \( BLPK(A,N,\gamma) \), \( LPK(A,N,\gamma) \), and \( SLPK(A,N,\gamma) \) with the appropriate modifications.
Proof. Assume that \(\{\theta_k\} \in \text{SBLPK}(A_1, N_1, \gamma)\), and define \(A_2 = 2A_1, N_2 = N_1 + \gamma\), and \(\gamma = \gamma\). It follows easily that (2.10) holds. Also
\[
|\theta_k(x,y_1,y_2) - \theta_k(x,y_1',y_2)| \leq A_1(2^k |y_1 - y_1'|)^\eta \Phi_k^{N_1+\gamma}(x-y_2) \\
\times \left( \Phi_k^{N_1+\gamma}(x-y_1) + \Phi_k^{N_1+\gamma}(x-y_1') \right) \\
\leq 2A_1 2^{nk} (2^k |y_1 - y_1'|)^\eta.
\]
A similar argument holds for regularity in the \(y_2\) and \(x\) spots. Then \(\theta_k\) satisfies (2.10)-(2.13).

Conversely we assume that (2.10)-(2.13) hold. Define \(\eta = \frac{N_1-n}{2(N_2+n)}\), \(A_1 = A_2, N_1 = N_2(1-\eta) - \eta \gamma\), and \(\gamma = \eta \gamma\). Estimate (2.6) easily follows since \(N_1 + \gamma < N_2\). Estimate (2.7) also follows since
\[
|\theta_k(x,y_1,y_2) - \theta_k(x,y_1',y_2)| \leq A_2(2^k |y_1 - y_1'|)^\eta \Phi_k^{N_2(1-\eta)}(x-y_2) \\
\times \left( \Phi_k^{N_2(1-\eta)}(x-y_1) + \Phi_k^{N_2(1-\eta)}(x-y_1') \right) \\
\leq A_1(2^k |y_1 - y_1'|)^\eta \Phi_k^{N_1+\gamma}(x-y_2) \\
\times \left( \Phi_k^{N_1+\gamma}(x-y_1) + \Phi_k^{N_1+\gamma}(x-y_1') \right).
\]

Note that this selection satisfies
\[
N_1 = N_2 - \eta(N_2 + \gamma) = \frac{N_2 + n}{2} > n.
\]
Then (2.7) holds for this choice of \(A_1, N_1\), and \(\gamma\) as well. Estimates (2.8) and (2.9) follow with a similar argument, and hence \(\{\theta_k\}\) is a collection of Littlewood-Paley square function kernel of type \(\text{BLPK}(A_1, N_1, \gamma)\). The proofs of the other equivalences are contained in the proof of this one. □

Remark 2.9. If \(\{\lambda_k^i\} \in \text{LPK}\) (respectively \(\{\lambda_k^i\} \in \text{SLPK}\)) for \(i = 1, 2\), then \(\{\theta_k\} \in \text{BLPK}\) (respectively \(\{\theta_k\} \in \text{SBLPK}\)) where \(\theta_k\) is defined, \(\theta_k(x,y_1,y_2) = \lambda_k^1(x,y_1)\lambda_k^2(x,y_2)\).

3. Almost Orthogonality Estimates

In this section, we prove some almost orthogonality estimates for kernel functions and for operators. These type of estimates have been well-developed over the years. The linear version of these results were implicit in many classical works by Besov [2, 3], Taibleson [36, 37, 38], Peetre [31, 32, 33], Triebel [40, 41], and Lizorkin [27], and they appear explicitly in the work of Frazier-Jawerth [13]. The bilinear versions appear in the work of Grafakos-Torres [18].

3.1. Kernel Almost Orthogonality. We first mention a well known almost orthogonality estimate for non-negative functions: If \(M, N > n\), then for all \(j, k \in \mathbb{Z}\)
\[
\int_{\mathbb{R}^n} \Phi^M_j(x-u)\Phi^N_k(u-y) du \lesssim \Phi^M_j(x-y) + \Phi^N_k(x-y).
\]
Then next result is also a result for integrals with non-negative integrands, but this one involves regularity estimates on the functions.
Proposition 3.1. If \( \{ \theta_k \}_{k \in \mathbb{Z}} \in \text{BLPK} \), then for all \( j, k \in \mathbb{Z}, x, y, x_1, y_1 \in \mathbb{R}^n \)

\[
\int_{\mathbb{R}^n} |\theta_j(x, y_1, y_2) - \theta_j(x, u, y_2)| \Phi_k^{N + \gamma} (u - y_1) du \lesssim 2^{\gamma(j-k)} (\Phi_j^N (x - y_1) + \Phi_k^N (x - y_1)) \Phi_j^N (x - y_2),
\]

\[
\int_{\mathbb{R}^n} |\theta_j(x, y_1, y_2) - \theta_j(x, y, u_2)| \Phi_k^{N + \gamma} (u - y_2) du \lesssim 2^{\gamma(j-k)} \Phi_j^N (x - y_1) (\Phi_j^N (x - y_2) + \Phi_k^N (x - y_2)),
\]

and

\[
\int_{\mathbb{R}^{2n}} |\theta_j(x, y_1, y_2) - \theta_j(x, u_1, u_2)| \Phi_k^{N + \gamma} (u_1 - y_1) \Phi_k^{N + \gamma} (u_2 - y_2) du_1 du_2 \lesssim 2^{\gamma(j-k)} \prod_{i=1}^2 (\Phi_j^N (x - y_i) + \Phi_k^N (x - y_i)).
\]

Proof. Since \( \{ \theta_k \}_{k \in \mathbb{Z}} \) is of type \( \text{BLPK}(A, N, \gamma) \), it follows that

\[
\int_{\mathbb{R}^n} |\theta_j(x, y_1, y_2) - \theta_j(x, u, y_2)| \Phi_k^{N + \gamma} (u - y_1) du \lesssim \Phi_j^N (x - y_2) \int_{\mathbb{R}^n} (2^{|u - y_1|})^{\gamma} (\Phi_j^{N + \gamma} (x - y_1) + \Phi_k^{N + \gamma} (x - u)) \Phi_k^{N + \gamma} (u - y_1) du \
\]

\[
\lesssim 2^{\gamma(j-k)} \Phi_j^N (x - y_2) \int_{\mathbb{R}^n} (\Phi_j^{N + \gamma} (x - y_1) + \Phi_k^{N + \gamma} (x - u)) \Phi_k^N (u - y_1) du \
\]

\[
\lesssim 2^{\gamma(j-k)} (\Phi_j^N (x - y_1) + \Phi_k^N (y_1 - x_1)) \Phi_j^N (x - y_2).
\]

By symmetry the second estimate holds as well. For the third estimate, we make a similar argument,

\[
\int_{\mathbb{R}^{2n}} |\theta_j(x, y_1, y_2) - \theta_j(x, u_1, u_2)| \Phi_k^{N + \gamma} (u_1 - y_1) \Phi_k^{N + \gamma} (u_2 - y_2) du_1 du_2 \lesssim 2^{\gamma(j-k)} \int_{\mathbb{R}^{2n}} (\Phi_j^N (x - y_1) + \Phi_j^N (x - y_2)) \Phi_k^{N + \gamma} (u_1 - y_1) \Phi_k^N (u_2 - y_2) du_1 du_2 \
\]

\[
+ 2^{\gamma(j-k)} \int_{\mathbb{R}^{2n}} (\Phi_j^N (x - y_1) + \Phi_j^N (x - u_1)) \Phi_k^{N + \gamma} (u_1 - y_1) \Phi_k^N (u_2 - y_2) du_1 du_2 \
\]

\[
\lesssim 2^{\gamma(j-k)} (\Phi_j^N (x - y_1) + \Phi_j^N (x - y_2)) (\Phi_j^N (x - y_1) + \Phi_k^N (x - y_2)).
\]

This completes the proof of the proposition. 

\( \square \)

3.2. Operator Almost Orthogonality Estimates. It is well-known that if \( N > n \) and \( f \in L^1 + L^n \), then \( \Phi_k * |f|(x) \lesssim Mf(x) \) for all \( k \in \mathbb{Z} \), where \( M \) is the Hardy-Littlewood maximal function

\[
Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy,
\]

and here the supremum is taken over all balls \( B \) containing \( x \). Next we use the kernel function almost orthogonality estimates to prove pointwise estimates for some operators.
Proposition 3.2. If \( \{\lambda_k\}, \{\theta_k\} \in L^p K \) and there exists a para-accretive function \( b \) such that \( \Lambda_k(b) = \Theta_k(b) = 0 \) for all \( k \in \mathbb{Z} \), then for all \( f \in L^1 + L^\infty \) and \( j, k \in \mathbb{Z} \)

\[
(3.1) \quad |\Theta_j M_b \Lambda_k^* f(x)| \lesssim 2^{-j(k-j)} |M f(x)|.
\]

If \( \{\lambda_k\} \in L^p K, \{\theta_k\} \in S^p L^p K \) and there exists a para-accretive functions \( b \) such that \( \Lambda_k(b) = 0 \) and

\[
\int_{\mathbb{R}^n} \theta_k(x, y_1, y_2) b(x) dx = 0
\]

for all \( k \in \mathbb{Z} \) and \( y_1, y_2 \in \mathbb{R}^n \), then for all \( f_1, f_2 \in L^1 + L^\infty \) and \( j, k \in \mathbb{Z} \)

\[
(3.2) \quad |\Lambda_k M_b \Theta_j (f_1, f_2)(x)| \lesssim 2^{-j(k-j)} |M f_1 M f_2(x)|
\]

Finally, if \( \{\lambda_k^1\}, \{\lambda_k^2\} \in L^p K, \{\theta_k\} \in B L^p K \) and there exist para-accretive functions \( b_1, b_2 \) and \( i \in \{1, 2\} \) such that \( \Lambda_k^1(b_1) \cdot \Lambda_k^2(b_2) = \Theta_k(b_1, b_2) = 0 \) for all \( k \in \mathbb{Z} \), then for all \( f_1, f_2 \in L^1 + L^\infty \) and \( j, k \in \mathbb{Z} \)

\[
(3.3) \quad |\Theta_j (M_{b_1} \Lambda_k^1 f_1, M_{b_2} \Lambda_k^2 f_2)(x)| \lesssim 2^{-j(k-j)} |M f_1 M f_2(x)|.
\]

Here we use capital \( \Lambda_k \) to be the operator defined by integration against the kernel lower case \( \lambda_k \), just like \( \Theta_k \) and \( \theta_k \).

Proof. We first prove (3.1). Using that \( \Lambda_k^*(b) = 0 \) and Proposition 3.1

\[
|\Theta_j M_b \Lambda_k^* f(x)| \leq \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (\theta_j(x, u) - \theta_j(x, y)) b(u) \lambda_k(y, u) du \right| |f(y)| dy
\]

\[
\lesssim \int_{\mathbb{R}^n} |\theta_j(x, u) - \theta_j(x, y)| |\Phi_k^{N+\gamma}(y-u)| |f(y)| dy du
\]

\[
\lesssim 2^{\gamma(j-k)} (\Phi_j^N * f)(x) + \Phi_k^N * f(x)
\]

\[
\lesssim 2^{\gamma(j-k)} |M f(x)|.
\]

With a symmetric argument, the same estimate holds replacing \( 2^{\gamma(j-k)} \) with \( 2^{\gamma(k-j)} \). Therefore (3.1) holds. Now we prove (3.2). We first use that \( \Lambda_k(b) = 0 \) to estimate

\[
|\Lambda_k M_b \Theta_j (f_1, f_2)(x)|
\]

\[
\leq \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \lambda_k(x, u) b(u) (\theta_j(u, y_1, y_2) - \theta_j(x, y_1, y_2)) du \right| |f_1(y_1), f_2(y_2)| dy_1 dy_2
\]

\[
\leq \int_{\mathbb{R}^n} \Phi_k^{N+\gamma}(x-u) (2^{j|x-u|} |\Phi_j^{N+\gamma}(u-y_1)| \Phi_j^{N+\gamma}(u-y_2) + \Phi_j^{N+\gamma}(x-y_1) \Phi_j^{N+\gamma}(x-y_2))
\]

\[
\times |f_1(y_1), f_2(y_2)| dy_1 dy_2
\]

\[
\lesssim 2^{\gamma(j-k)} \int_{\mathbb{R}^n} \Phi_k^N (x-u) (\Phi_j^{N+\gamma}(u-y_1) \Phi_j^{N+\gamma}(u-y_2) + \Phi_j^{N+\gamma}(x-y_1) \Phi_j^{N+\gamma}(x-y_2))
\]

\[
\times |f_1(y_1), f_2(y_2)| dy_1 dy_2
\]

\[
= 2^{\gamma(j-k)} \int_{\mathbb{R}^n} \Phi_k^N (x-u) \prod_{i=1}^2 (\Phi_i^N * f_i(u) + \Phi_i^N * |f_i(x)|) du
\]

\[
\lesssim 2^{\gamma(j-k)} |M (M f_1, M f_2)(x)|.
\]
We also have 

\[ |\Lambda_k M_x \Theta_j(f_1, f_2)(x)| \]

\[ \leq \int_{\mathbb{R}^n} \left| \sum_{k=1}^{2^n} (\lambda_k(x,u) - \lambda_k(x,y_1)) b(u) \Theta_j(u,y_1,y_2) du \right| f_1(y_1) f_2(y_2) dy_1 dy_2 \]

\[ \lesssim 2^{(k-j)^2} \int_{\mathbb{R}^n} (\Phi^N_k(x-u) + \Phi^N_k(x-y_1)) \prod_{i=1}^2 \Phi^N_j(u-y_i) |f_i(y_i)| dy_i du \]

\[ \leq 2^{(k-j)^2} \int_{\mathbb{R}^n} \Phi^N_k(x-u) \prod_{i=1}^2 \Phi^N_j(u-y_i) |f_i(y_i)| dy_i du \]

\[ + 2^{(k-j)^2} \int_{|x-y_1| \leq |x-u|/2} \Phi^N_k(x-y_1) \prod_{i=1}^2 \Phi^N_j(u-y_i) |f_i(y_i)| dy_i du \]

\[ + 2^{(k-j)^2} \int_{|x-y_1| \leq |x-u|/2} \Phi^N_k(x-y_1) \prod_{i=1}^2 \Phi^N_j(u-y_i) |f_i(y_i)| dy_i du \]

\[ = 2^{(k-j)^2}(I + II + III). \]

Note that \(2^{(k-j)^2} I \lesssim 1\), which is on the right hand side of (3.2). In II, replace \(\Phi^N_k(x-y_1)\) with \(\Phi^N_k((x-u)/2)\) and it follows that \(II \lesssim I\). So II is bounded appropriately as well. The final term, III is bounded by

\[ \int_{|x-y_1| \leq |x-u|/2} \Phi^N_k(x-y_1) \frac{2^{j_n} |f_1(y_1)|}{(1 + 2/|x-u| - |x-y_1|)^N} \Phi^N_j * |f_2|(u) dy_1 du \]

\[ \lesssim \int_{|x-y_1| \leq |x-u|/2} \Phi^N_k(x-y_1) \Phi^N_j(x-u) |f_1(y_1)| \Phi^N_j * |f_2|(u) dy_1 du \]

\[ \lesssim \left( \int_{\mathbb{R}^n} \Phi^N_k(x-y_1) |f_1(y_1)| dy_1 \right) \left( \int_{\mathbb{R}^n} \Phi^N_j(x-u) \Phi^N_j * |f_2|(u) du \right) \]

\[ \lesssim \Phi^N_k * |f_1|(x) \Phi^N_j * |f_2|(x) \]

\[ \leq M_\mathcal{M} (M f_1 \cdot M f_2)(x). \]

This verifies that (3.2) holds. For estimate (3.3) when \(j \leq k\), we use that \(\Lambda_x^k (b_1) \cdot \Lambda_x^k (b_2) = 0\) and Proposition 3.1

\[ |\Theta_j(M_{b_1} \Lambda_x^{1+} f_1, M_{b_2} \Lambda_x^{2+} f_2)(x)| \]

\[ \leq \int_{\mathbb{R}^n} |\Theta_j(x, u_1, u_2) - \Theta_j(x, y_1, y_2)| \prod_{i=1}^2 |b_i(u) \lambda^1_i(y_i, u_i) f_i(y_i)| dy_i du_i \]

\[ \lesssim 2^{k-j} \int_{\mathbb{R}^n} \prod_{i=1}^2 (\Phi^N_k(x-u_i) + \Phi^N_k(x-y_i)) \Phi^N_j(u_i-y_i) |f_i(y_i)| du_i dy_i \]

\[ \lesssim 2^{k-j} \prod_{i=1}^2 \int_{\mathbb{R}^n} (\Phi^N_k(x-y_i) + \Phi^N_k(x-y_i)) |f_i(y_i)| dy_i \]

\[ \lesssim 2^{k-j} M f_1(x) M f_2(x). \]
Finally using that $\Theta_j(b_1, b_2) = 0$, it follows that

$$|\Theta_j(M_{b_1}\Lambda_1^* f_1, M_{b_2}\Lambda_2^* f_2)(x)|$$

$$\leq \int_{\mathbb{R}^{2n}} |\theta_j(x, u_1, u_2)| \left( \prod_{i=1}^2 \lambda_i^j(y_i, u_i) - \prod_{i=1}^2 \lambda_i^j(y_i, x) \right) \left( \prod_{i=1}^2 \prod_{j=1}^m |b_i(u_i)f_i(y_i)|dy_i du_i \right)$$

$$\lesssim \int_{\mathbb{R}^{2n}} \left( \prod_{i=1}^2 \lambda_i^j(y_i, u_i) - \prod_{i=1}^2 \lambda_i^j(y_i, x) \right) \left( \prod_{i=1}^2 \prod_{j=1}^m \Phi^j_{\alpha_i} \gamma(y_i - y_i)du_i \right) \prod_{i=1}^m |f_i(y_i)|dy_i$$

$$\lesssim 2^{n(k-j)} (\Phi^j_k * |f_1|)(x) + \Phi^j_k * |f_1|(x) \left( \Phi^j_k * |f_2|(x) + \Phi^j_k * |f_2|(x) \right)$$

$$\lesssim 2^{n(k-j)} M f_1(x) M f_2(x).$$

Note that we use Remark 2.9 to see that $\lambda_1^j(x, y_1)\lambda_2^j(x, y_2)$ form a collection of kernels of type BLPK. Then (3.3) holds as well. □

### 4. CONVERGENCE RESULTS

In this section, we prove convergence results for various function spaces. Most of these results are well known, see e.g. the work of Davide-Journé-Semmes [11] or Han [19], but for convenience we include them here. We also introduce a criterion for extending the convergence of some reproducing formulas in $L^p$ for to convergence in $H^1$.

#### 4.1. Approximation to Identities.

**Proposition 4.1.** Suppose $p_k : \mathbb{R}^n \to \mathbb{C}$ for $k \in \mathbb{Z}$ satisfy $|p_k(x, y)| \lesssim \Phi^N_k(x - y)$ and $N > n$, and define $P_k$

$$P_k f(x) = \int_{\mathbb{R}^n} p_k(x, y) f(y) dy$$

for $f \in L^1 + L^\infty$. If

$$\int_{\mathbb{R}^n} p_k(x, y) dy = 1$$

for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, then $P_k f \to f$ in $L^p$ as $k \to \infty$ for all $f \in L^p$ when $1 \leq p < \infty$ and $P_k f \to 0$ in $L^q$ as $k \to \infty$ for all $f \in L^p \cap L^q$ for $1 \leq q < \infty$.

**Proof.** For $f \in L^p$ with $1 \leq p < \infty$

$$||P_k f - f||_{L^p} = \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} p_k(x, y) f(y) dy - \int_{\mathbb{R}^n} p_k(x, y) f(x) dy \right) dx \right)^{1/p}$$

$$= \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} p_k(x, x - 2^{-k}y) f(x - 2^{-k}y) - f(x) 2^{-kr}dy \right) dx \right)^{1/p}$$

$$\lesssim \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \Phi^N_k(y) |f(x - 2^{-k}y) - f(x)|^p dy \right)^{1/p} dx$$

$$\lesssim \int_{\mathbb{R}^n} \Phi^N_k(y) ||f(\cdot - 2^{-k}y) - f||_{L^p} dy.$$  

Note that $\Phi^N_k(y) ||f(\cdot - 2^{-k}y) - f||_{L^p} \lesssim 2 ||f||_{L^p} \Phi^N_k(y)$ which is an $L^1(\mathbb{R}^n)$ function independent of $k$. So by Lebesgue dominated convergence and the continuity of translation in $|| \cdot ||_{L^p}$,

$$\lim_{k \to \infty} ||P_k f - f||_{L^p} \leq \int_{\mathbb{R}^n} \Phi^N_k(y) \lim_{k \to \infty} ||f(\cdot - 2^{-k}y) - f||_{L^p} dy = 0.$$
Next we compute
\[
|P_k f(x)| \leq \|\Phi^N_k\|_{L^p} \|f\|_{L^q} = 2^{kn/q} \|\Phi^N_k\|_{L^p} \|f\|_{L^q}.
\]
So \(P_k f \to 0\) almost everywhere as \(k \to -\infty\). We also have
\[
|P_k f(x)| \leq \Phi^N_k * |f|(x) \lesssim \mathcal{M} f(x),
\]
and since \(f \in L^p(\mathbb{R}^n)\), it follows that \(\mathcal{M} f \in L^p(\mathbb{R}^n)\) as well when \(1 < p < \infty\). So by dominated convergence
\[
\lim_{k \to -\infty} \|P_k f\|_{L^p}^p = \int_{\mathbb{R}^n} \lim_{k \to -\infty} |P_k f(x)|^p dx = 0.
\]
This proves the proposition. \(\square\)

**Corollary 4.2.** Let \(b\) be a para-accretive function. Suppose \(s_k : \mathbb{R}^{2n} \to \mathbb{C}\) for \(k \in \mathbb{Z}\) satisfy
\[
|s_k(x, y)| \lesssim \Phi^N_k(x - y) \text{ for some } N > n, \text{ and define } S_k
\]
\[
S_k f(x) = \int_{\mathbb{R}^n} s_k(x, y) f(y) dy
\]
for \(f \in L^1 + L^\infty\). If
\[
\int_{\mathbb{R}^n} s_k(x, y) b(y) dy = 1
\]
for all \(k \in \mathbb{Z}\) and \(x \in \mathbb{R}^n\), then \(S_k M_b f \to f\) and \(M_b S_k f \to f\) in \(L^p\) as \(k \to \infty\) for all \(f \in L^p(\mathbb{R}^n)\)
when \(1 \leq p < \infty\). Also \(S_k M_{b_0} f \to 0\) and \(M_{b_0} S_k f \to 0\) in \(L^p\) as \(k \to -\infty\) for all \(f \in L^p \cap L^q\) for \(1 \leq q < p < \infty\).

**Proof.** Define \(P_k f = S_k M_b f\) with kernel \(p_k\). It is obvious that \(|p_k(x, y)| \lesssim \Phi^N_k(x - y)\), and \(P_k(1) = S_k(b) = 1\). So by Proposition 4.1, since \(f \in L^p\) it follows that \(S_k M_b f = P_k f \to f\) in \(L^p\) when \(f \in L^p\) and \(1 \leq p < \infty\). Also when \(f \in L^p \cap L^q\) for \(1 \leq q < p < \infty\), it follows that \(S_k M_{b_0} f = P_k f \to 0\) as \(k \to -\infty\). Also \(M_{b_0} S_k f = M_{b_0} P_k(b^{-1} f)\), so the same convergence properties hold for \(M_{b_0} S_k\). \(\square\)

These approximations to identities perturbed by para-accretive functions are important to this work. They have been studied in depth by David-Journé-Semmes [11] and Han [19], among others.

**Definition 4.3.** Let \(b \in L^\infty\) be a para-accretive function. A collection of operators \(\{S_k\}_{k \in \mathbb{Z}}\) defined by
\[
S_k f(x) = \int_{\mathbb{R}^n} s_k(x, y) f(y) dy
\]
for kernel functions \(s_k : \mathbb{R}^{2n} \to \mathbb{C}\) is an approximation to identity with respect to \(b\) if \(\{s_k\} \in SLPK\), and
\[
|s_k(x, y) - s_k(x', y) - s_k(x, y') + s_k(x', y')| \leq A 2^{kn} (2^k |x - x'|)^\gamma (2^k |y - y'|)^\gamma
\]
\[
\times \left( \Phi_k^{N+\gamma}(x - y) + \Phi_k^{N+\gamma}(x' - y) + \Phi_k^{N+\gamma}(x - y') + \Phi_k^{N+\gamma}(x' - y') \right)
\]
\[
\int_{\mathbb{R}^n} s_k(x, y) b(y) dy = \int_{\mathbb{R}^n} s_k(x, y) b(x) dx = 1.
\]
We say that an approximation to identity with respect to \(b\) has compactly supported kernel if \(s_k(x, y) = 0\) whenever \(|x - y| > 2^{-k}\).
Remark 4.4. Given a para-accretive function $b$, we define a particular approximation to the identity with respect to $b$. Let $\varphi \in C_0^\infty$ be radial with integral 1 and $\text{supp}(\varphi) \subset B(0,1/8)$. Define $S_k^b = P_k M_{(P_k b)^{-1}} P_k$. It follows that $S_k^b$ is an approximation to identity with respect to $b$. Furthermore, $S_k^b$ is self-transpose and has compactly supported kernel. It is not trivial to see that $M_{(P_k b)^{-1}}$ is well a defined operator, but it was proved in [11] that whenever $b$ is a para-accretive function there exists $\varepsilon > 0$ such that $|P_k b| \geq \varepsilon > 0$ uniformly in $k$. With this fact, the proof of this remark easily follows.

Proposition 4.5. Let $b$ be a para-accretive function, $\{S_k\}$ be the approximation to identity with respect to $b$ that has compactly supported kernel, and $\delta_0 > 0$. Then $M_k S_k M_k f \to b f$ and $M_k S_k M_k f \to 0$ in $b C_0^\infty$ as $N \to \infty$ for all $f \in C_0^\infty$ and $0 < \delta < \min(\delta_0, \gamma)$, where $\gamma$ is the smoothness parameter for $\{S_k\} \in SLPK$. In particular these convergence results hold for the operators defined in Remark 4.4.

Proof. Let $f \in C_0^\infty$ and $0 < \delta < \delta_0$. Without loss of generality assume that $\gamma = \delta$, where $\gamma$ is the smoothness parameter of $s_k$. We must check that $\|S_k M_k f - f\|_0 \to 0$ as $N \to \infty$. So we start by estimating

$$|(S_k M_k f(x) - f(x)) - (S_k M_k f(y) - f(y))| = \left| \int_{\mathbb{R}^n} (s_k(x,u)(f(u) - f(x)))(y,u)(f(u) - f(y))b(u)du \right| \leq \|f\|_1 \int_{\mathbb{R}^n} |F_n^y(u) - F_n^x(u)| du$$

where $F_n^y(u) = s_k(x,u)(f(u) - f(x))$. Consider $u \in B(y,2^{-N})$, and it follows that

$$|F_n^y(u) - F_n^x(u)| = |s_k(x,u)(f(u) - f(x)) - s_k(y,u)(f(u) - f(y))| \leq |s_k(x,u)| |f(y) - f(x)| + |s_k(x,u) - s_k(y,u)| |f(u) - f(y)| \leq \|f\|_{\delta_0} 2^{nN}|x - y|^{\delta_0} + \|f\|_{\delta_0} 2^{nN} (2^N|x - y|)\delta_0|y - u|^{\delta_0} \leq \|f\|_{\delta_0} 2^{nN}|x - y|^{\delta_0}$$

With a similar argument, it follows that for $u \in B(x,2^{-N})$, $|F_n^y(u) - F_n^x(u)| \leq \|f\|_{\delta_0} 2^{nN}|x - y|^{\delta_0}$. Now we may also estimate $|F_n^y(u)|$ in the following way for $u \in B(x,2^{-N})$,

$$|F_n^y(u)| \leq 2^{nN}|f(u) - f(x)| \leq \|f\|_{\delta_0} 2^{nN}|u - x|^{\delta_0} \leq \|f\|_{\delta_0} 2^{nN} - \delta_0 N.$$

Using the support properties of $s_k$, we have that $\text{supp}(F_n^y - F_n^x) \subset B(x,2^{-N}) \cup B(y,2^{-N})$. Then it follows from (4.1), (4.2), and $\delta_{\delta_0} \in (0,1)$ that

$$|F_n^y(u) - F_n^x(u)| \leq \left(\|f\|_{\delta_0} 2^{nN}|x - y|^{\delta_0} \right)^{\delta} \left(\|f\|_{\delta_0} 2^{nN} - \delta_0 N\right)^{1 - \frac{\delta}{2}} \leq \|f\|_{\delta_0} 2^{nN}|x - y|^{\delta 2 - (\delta_0 - \delta) N}.$$ 

Therefore $S_k M_k f \to f$ in $\|\cdot\|_0$ since

$$\frac{|(S_k M_k f(x) - f(x)) - (S_k M_k f(y) - f(y))|}{|x - y|^{\delta}} \leq \frac{1}{|x - y|^{\delta}} \int_{\mathbb{R}^n} |F_n^y(u) - F_n^x(u)| du \leq \|f\|_{\delta_0} 2^{nN} \int_{B(x,2^{-N}) \cup B(y,2^{-N})} 2^{nN} du \leq \|f\|_{\delta_0} 2^{nN} - \delta_0 N.$$
This proves that $S_N M_b f \to f$ in $C_0^\delta$ as $N \to \infty$. Now we consider $S_{-N} M_b f$ as $N \to \infty$. We also have
\[
\frac{|S_N M_b f(x) - S_N M_b f(y)|}{|x-y|^{\delta}} \leq \frac{1}{|x-y|^{\delta}} \int_{\mathbb{R}^n} |s_{-N}(x,u) - s_{-N}(y,u)| |b(u)| f(u) |du
\leq \frac{||f||_{L^\infty}}{|x-y|^{\delta}} \left( \int_{|y-u|<2^N} + \int_{|y-u|>2^N} \right) 2^{-nN} (2^{-N}|x-y|)^{\delta} |du
\leq ||f||_{L^\infty} 2^{-nN} 2^{-\delta N}.
\]
Note that $||f||_{L^\infty} < \infty$ since $f$ is continuous and compactly supported. Therefore $S_N M_b f \to f$ and $S_{-N} M_b f \to 0$ as $N \to \infty$ in the topology of $C_0^\delta$.

4.2. Reproducing Formulas. We state a Calderón type reproducing formula for the para-accretive setting, which was constructed by Han in [19].

**Theorem 4.6.** Let $b \in L^\infty$ be a para-accretive function and $S^b_k$ for $k \in \mathbb{Z}$ be approximations to the identity operators with respect to $b$. Define $D^b_k = S^b_{k+1} - S^b_k$. There exist operators $\tilde{D}^b_k$ such that

\begin{equation}
\sum_{k \in \mathbb{Z}} \tilde{D}^b_k M_b D^b_k f = bf
\end{equation}

in $L^p$ for all $1 < p < \infty$ and $f \in C_0^\delta$ such that $bf$ has mean zero. Furthermore, $\tilde{D}^b_k(b) = \tilde{D}^b_k$ is defined by

\[
\tilde{D}^b_k f(x) = \int_{\mathbb{R}^n} d^b_k(x,y) f(y) dy
\]

where $\{d^b_k(y)\} \in LPK$, where $d^b_k(x,y) = \tilde{d}^b_k(x,y)$ are the kernels associated with $\tilde{D}^b_k$.

We will use this formula extensively, and in fact, we need this formula in $H^1$ as well to construct the accretive type para-product in Section 6. We will prove that this reproducing formula holds in $H^1$ in Theorem 1.2 and its Corollary 4.8. First we prove a lemma.

**Lemma 4.7.** If $f : \mathbb{R}^n \to \mathbb{C}$ has mean zero and

\[
|f(x)| \lesssim \Phi_N^j(x) + \Phi_N^k(x)
\]

for some $N > n$ and $j, k \in \mathbb{Z}$, then $f \in H^1$ and $||f||_{H^1} \lesssim 1 + |j-k|$, where the suppressed constant is independent of $j$ and $k$.

This is an extension of a result of Uchiyama [42], which is Lemma 4.7 when $j = k$. Initially in [23], we obtained a quadratic bound, $|j-k|^2$, for Lemma 4.7 using an argument involving atomic decompositions in $H^1$. Such a result suffices for our purposes, but thanks to suggestions from Atanas Stefanov we are able to obtain the linear bound stated here. We present Stefanov’s proof, which appears more natural.

**Proof.** The conclusion of Lemma 4.7 is well known for $j = k$, see e.g. the work of Uchiyama [42] or Wilson [43]. So without loss of generality we take $j \neq k$, and furthermore we suppose that $j < k$. It is easy to see that

\[
||f||_{L^1} \lesssim ||\Phi_N^j||_{L^1} + ||\Phi_N^k||_{L^1} \lesssim 1,
\]

so we may reduce the problem to proving that $||R_{\ell} f||_{L^1} \lesssim k-j$ for $\ell = 1, \ldots, n$. The strategy here is to split the norm $||R_{\ell} f||_{L^1}$ into two sets, where $|x| \leq 2^{-j}$ and where $|x| > 2^{-j}$. We
will control the first by \( k - j \) and the second by 1. Define \( p = 1 + \frac{1}{k - j} > 1 \), and use that 
\[ \|Rf\|_{L^p \to L^p} \lesssim p' \] 
to estimate 
\[ \|\chi_{[x] \leq 2^{-j} Rf}\|_{L^1} \leq \|\chi_{[x] \leq 2^{-j}}\|_{L^p} \|Rf\|_{L^p} \lesssim 2^{-n/j} p' \|f\|_{L^p} \]
\[ \lesssim (k - j)2^{-n/j} p' \left( 2^{n/j} p' + 2^{nk/p'} \right) \lesssim k - j. \] 

Note that here we use that \( p' = k - j + 1 \) and hence \( 2^{n(k-j)/p'} \leq 2^n \). Now it remains to control 
\[ \|\chi_{[x] > 2^{-j} Rf}\|_{L^1} \leq \sum_{m=-j}^{0} \|\chi_{2^{m-1} < [x] \leq 2^{m+1}} Rf\|_{L^1} \]
\[ \leq \sum_{m=-j}^{0} \|\chi_{2^{m-1} < [x] \leq 2^{m+1}} Rf(\chi_{[y] < 2^{m-1}})\|_{L^1} \]
\[ + \sum_{m=-j}^{0} \|\chi_{2^{m-1} < [x] \leq 2^{m+1}} Rf(\chi_{[y] > 2^{m-1}})\|_{L^1} = I + I'. \]

In order to estimate \( I \) from (4.5), we bound the terms of the sum by first breaking them into two pieces using the mean zero hypothesis on \( f \):
\[ \|\chi_{2^{m-1} < [x] \leq 2^{m+1}} Rf(\chi_{[y] < 2^{m-1}})\|_{L^1} = \int_{2^{m-1} < [x] \leq 2^{m+1}} \left| Rf(\chi_{[y] < 2^{m-1}})(x) - \int_{R} \frac{x_{y'}}{|x'|^{n+1}} f(y)dy \right| dx \]
\[ \leq \int_{[x] > 2^{m}} \int_{[y] < 2^{m-1}} \left| \frac{x_{y} - x_{y'}}{|x|^{n+1} - |x'|^{n+1}} \right| |f(y)|dydx \]
\[ + \int_{[x] > 2^{m}} \int_{[y] > 2^{m-1}} |f(y)|dydx = I_a + I_b. \]

Let \( \delta = \min(1, (N-n)/2) \) and \( N' = N - \delta > n \). Then the first term of (4.6) is bounded by 
\[ I_a \leq \int_{[x] > 2^{m}} \int_{[y] < 2^{m-1}} \frac{|y|}{|x'|^{n+1}} |f(y)|dydx \leq \int_{[x] > 2^{m}} \int_{[y] < 2^{m-1}} \frac{|y|\delta}{|x'|^{n+\delta}} |f(y)|dydx \]
\[ \lesssim 2^{-m}\delta \int_{R} \left( |\Phi_N^N(y) + \Phi_N^N(y)| \right) dy \]
\[ \lesssim 2^{-m}\delta \int_{R} \left( 2^{-j\delta} \Phi_N^N(y) + 2^{-k\delta} \Phi_N^N(y) \right) dy \]
\[ \lesssim 2^{-(j+m)\delta}. \]

Note that we absorb the \( 2^{-k\delta} \) term into the \( 2^{-j\delta} \) term since \( k > j \). The second term of (4.6) is bounded by 
\[ I_b \leq \int_{2^{m-1} < [x] \leq 2^{m+1}} \int_{[y] > 2^{m-1}} \frac{1}{|x|^{n}} |f(y)|dydx \leq 2^{-mn} \int_{2^{m-1} < [x] \leq 2^{m+1}} \int_{[y] > 2^{m-1}} |f(y)|dydx \]
\[ \leq \int_{[y] > 2^{m-1}} \left( 2^{-j(N-n)} + \frac{2^{-k(N-n)}}{|y|^{N-n}} \right) dy \]
\[ \lesssim 2^{-(j+m)(N-n)} + 2^{-(k+m)(N-n)} \]
\[ \lesssim 2^{-(j+m)(N-n)}. \]
Again we use that $2^{-k(N-n)} \leq 2^{-j(N-n)}$ since $k > j$ and $N > n$. Now in order to estimate $II$ from (4.5), we bound the terms of the sum using an $L^2$ bound for $R_\ell$

\[
|||_2 \leq \left( \sum_{\ell=1}^\infty |R_\ell f(x)|^2 \right)^{1/2}
\]

\[
\leq \left( \sum_{\ell=1}^\infty \left( \int_{|y|>2^{m+1}} \left( \Phi_\ell^N(y) + \Phi_\ell^N(y) \right)^2 \right) \right)^{1/2}
\]

\[
\leq \left( \sum_{\ell=1}^\infty \left( \int_{|y|>2^{m+1}} \left( \frac{2\ell N}{|y|^{2N}} + \frac{2\ell k(N-n)}{|y|^{2N}} \right) \right) \right)^{1/2}
\]

\[
\leq \left( \sum_{\ell=1}^\infty \left( 2^{-j(N-n)} + 2^{-k(N-n)} \right) \left( \int_{|y|>2^{m+1}} \frac{1}{|y|^{2N}} \right) \right)^{1/2}
\]

\[
\leq 2^{-(j+m)(N-n)}.
\]

Using these estimates, it follows that (4.5) is bounded in the following way:

\[
I + II \lesssim \sum_{m=-j}^{\infty} 2^{-j(N-n)} + \sum_{m=-j}^{\infty} 2^{-j+m}(N-n) \lesssim 1.
\]

Therefore using (4.4) and (4.5), it follows that $||R_\ell f||_L^2 \lesssim k - j$ for $\ell = 1, ..., n$ and hence $||f||_{H^1} \lesssim k - j$. □

Now we prove Theorem 1.2.

Proof. Define for $k \in \mathbb{Z}$, $f_k(x) = M_{\ell} \Theta_k M_{\phi} \phi$. It easily follows that

\[
\int_{\mathbb{R}^n} f_k(x) dx = \int_{\mathbb{R}^n} M_{\ell} \Theta_k M_{\phi}(x) \phi(x) dx = 0.
\]

Let $R$ be large enough so that supp($\phi$) $\subset B(0, R)$. We estimate

\[
|f_k(x)| \leq ||b||_L \left| \int_{\mathbb{R}^n} (\Theta_k(x,y) - \Theta_k(x,0)) b(y) \phi(y) dy \right|
\]

\[
\lesssim \int_{\mathbb{R}^n} (2^k |y|^\gamma (\Phi_k^N(x-y) + \Phi_k^N(x)) |\phi(y)| dy
\]

\[
\lesssim 2^k R^{\gamma} (\Phi_k^N(x) + \Phi_k^N(x))
\]

\[
\lesssim 2^k (\Phi_k^N(x) + \Phi_k^N(x)).
\]

We also estimate

\[
|f_k(x)| \leq ||b||_L \left| \int_{\mathbb{R}^n} \Theta_k(x,y) b(y) (\phi(y) - \phi(x)) dy \right|
\]

\[
\lesssim \int_{\mathbb{R}^n} \Phi_k^{N+\gamma}(x-y) |x-y|^\gamma (\Phi_k^N(y) + \Phi_k^N(x)) dy
\]

\[
\lesssim 2^{-k} \int_{\mathbb{R}^n} \Phi_k^{N+\gamma}(x-y) (\Phi_k^N(y) + \Phi_k^N(x)) dy
\]

\[
\lesssim 2^{-k} (\Phi_k^N(x) + \Phi_k^N(x)).
\]

So we have proved that $|f(x)| \lesssim 2^{-|k|} (\Phi_k^N(x) + \Phi_k^N(x))$. It follows from Lemma 4.7 applied with $j = 0$ that

\[
||f_k||_{H^1} \lesssim (1 + |k|) 2^{-|k|\gamma}.
\]
Therefore
\[ \left| \sum_{|k|<M} f_k \right|_{H^1} \leq \sum_{|k|<M} \| f_k \|_{H^1} \lesssim \sum_{k \in \mathbb{Z}} (1+|k|)2^{-|k|\gamma} < \infty. \]

Hence \( \sum_{|k|<M} f_k \) is a Cauchy sequence in \( H^1 \), and there exists \( \tilde{\phi} \in H^1 \) such that
\[ \tilde{\phi} = \sum_{k \in \mathbb{Z}} f_k = \sum_{k \in \mathbb{Z}} M_b \Theta_k M_b \phi. \]

But since the reproducing formula holds for \( b \phi \) in \( L^p \) for some \( 1 < p < \infty \), it follows that \( \tilde{\phi} = b \phi \) and the reproducing formula holds for \( b \phi \) in \( H^1 \), which completes the proof. \( \square \)

**Corollary 4.8.** Let \( b \in L^\infty \) be a para-accretive function, \( S^b_k \), \( D^b_k \), and \( \tilde{D}^b_k \) be approximation to identity and reproducing formula operator with respect to \( b \) as in Theorem 4.6. Then for all \( \delta > 0 \) and \( \phi \in C^\delta_b \) such that \( b \phi \) has mean zero,
\[ \sum_{k \in \mathbb{Z}} M_b \tilde{D}^b_k M_b D_k \phi = \sum_{k \in \mathbb{Z}} M_b D_k M_b \phi = b \phi \]
in \( H^1 \).

**Proof.** By Theorem 4.6, it follows that the kernels of \( \tilde{D}^b_k M_b D_k \) and \( D_k \) are Littlewood-Paley square function kernels of type \( LPK \), that
\[ \tilde{D}^b_k M_b D_k (b) = (\tilde{D}^b_k M_b D_k)^* (b) = D^b_k (b) = D^b_k (b) = 0, \]
and finally that
\[ \sum_{k \in \mathbb{Z}} M_b \tilde{D}^b_k M_b D_k \phi = \sum_{k \in \mathbb{Z}} M_b D_k M_b \phi = b \phi \]
in \( L^p \) for all \( 1 < p < \infty \) when \( \phi \in C^\delta_b \) when \( b \phi \) has mean zero. Therefore it follows from Theorem 1.2 that the formula holds in \( H^1 \) as well. \( \square \)

5. A Square Function-Like Estimate

In this section, we work with Littlewood-Paley type square function kernel adapted to para-accretive functions, but we do not actually prove any square function bounds. Instead we prove an estimate for a sort of “dual pairing” that will be useful to approximate Lebesgue space norms for the singular integral operators in the next section.

**Theorem 5.1.** If \( \{ \Theta_k \} \in SLPK \) and there exist para-accretive functions \( b_0, b_1, b_2 \) such that
\[ \int_{\mathbb{R}^n} \Theta_k (x,y_1,y_2) b_0 (x) dx = \int_{\mathbb{R}^n} \Theta_k (x,y_1,y_2) b_1 (y_1) b_2 (y_2) dy_1 dy_2 = 0 \]
for all \( x, y_1, y_2 \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \), then for all \( 1 < p, p_1, p_2 < \infty \) satisfying (2.2), \( f_i \in L^{p_i} \) for \( i = 0, 1, 2 \) where \( p_0 = p' \)
\[ \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k (f_1, f_2) (x) f_0 (x) dx \right| \lesssim \prod_{i=0}^2 \| f_i \|_{L^{p_i}}. \]

**Proof.** Since \( b_i, b_i^{-1} \in L^{\infty} \), it is sufficient to prove this estimate for \( b_i f_i \) in place of \( f_i \) for \( i = 0, 1, 2 \). Fix \( 1 < p, p_1, p_2 < \infty \) satisfying (2.2), \( f_i \in C^\delta_b \) for \( i = 0, 1, 2 \) and some \( \delta \) where \( b_i f_i \) has mean zero for \( i = 0, 1, 2 \). Define
\[ \Pi^1_f (f_1, f_2) (x) = M_{b_1} D^b_{k+1} M_{b_1} f_1 (x) M_{b_2} S^{b_2}_{k+1} M_{b_2} f_2 (x) \]
\[ \Pi^2_f (f_1, f_2) (x) = M_{b_1} S^{b_1}_{k+1} M_{b_1} f_1 (x) M_{b_2} D^b_{k+1} M_{b_2} f_2 (x). \]
where $S^h_k$ and $D^h_k$ are defined as in Theorem 4.6. Then it follows that

$$\Theta_k(b_1 f_1, b_2 f_2) = \lim_{N \to \infty} \Theta_k(M_{b_1} S^{h_1}_{k_1} M_{b_1} f_1, M_{b_2} S^{h_2}_{k_2} M_{b_2} f_2) - \Theta_k(M_{b_1} S^{h_1}_{k_1} M_{b_1} f_1, M_{b_2} S^{h_2}_{k_2} M_{b_2} f_2)$$

$$= \lim_{N \to \infty} \sum_{j=-N}^{N-1} \Theta_k(M_{b_1} S^{h_1}_{k_1} M_{b_1} f_1, M_{b_2} S^{h_2}_{k_2} M_{b_2} f_2) - \Theta_k(M_{b_1} S^{h_1}_{k_1} M_{b_1} f_1, M_{b_2} S^{h_2}_{k_2} M_{b_2} f_2)$$

$$= \lim_{N \to \infty} \sum_{j=-N}^{N-1} \Theta_k \Pi_j^1(f_1, f_2) + \Theta_k \Pi_j^2(f_1, f_2)$$

where the convergence holds in $L^p$. Then we approximate the above dual pairing in the following way

$$\left| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \Theta_k(b_1 f_1, b_2 f_2)(x) b_0(x) f_0(x) dx \right| \leq \sum_{j,k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^1(f_1, f_2)(x) b_0(x) f_0(x) dx \right| + \sum_{j,k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^2(f_1, f_2)(x) b_0(x) f_0(x) dx \right|.$$

These two terms are symmetric, so we only bound the first one. The bound for the other term follows with a similar argument. By the convergence in Theorem 4.6, we have that

$$\sum_{j,k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^1(f_1, f_2)(x) b_0(x) f_0(x) dx \right| \leq \sum_{j,k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^1(\tilde{D}_{k}^{h_1} M_{b_{1}} D_{k}^{h_1} M_{b_{1}} f_1, f_2)(x) b_0(x) f_0(x) dx \right|$$

$$\leq \sum_{j,k,m \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \tilde{D}_{m}^{h_0} M_{b_{0}} \Theta_k \Pi_j^1(\tilde{D}_{k}^{h_1} M_{b_{1}} D_{k}^{h_1} M_{b_{1}} f_1, f_2)(x) b_0(x) f_0(x) dx \right|$$

By Proposition 3.2 we also have the following three estimates

$$|D_{m}^{h_0} M_{b_{0}} \Theta_k \Pi_j^1(\tilde{D}_{k}^{h_1} M_{b_{1}} D_{k}^{h_1} M_{b_{1}} f_1, f_2)(x)| \lesssim 2^{-\gamma (m-k)} M \left( \Pi_j^1(\tilde{D}_{k}^{h_1} M_{b_{1}} D_{k}^{h_1} M_{b_{1}} f_1, f_2) \right)(x)$$

$$\lesssim 2^{-\gamma (m-k)} M^2 \left( M (D_{k}^{h_1} M_{b_{1}} f_1) \cdot M f_2 \right)(x).$$

$$|D_{m}^{h_0} M_{b_{0}} \Theta_k \Pi_j^1(\tilde{D}_{k}^{h_1} M_{b_{1}} D_{k}^{h_1} M_{b_{1}} f_1, f_2)(x)| \lesssim M (\Theta_k \Pi_j^1(\tilde{D}_{k}^{h_1} M_{b_{1}} D_{k}^{h_1} M_{b_{1}} f_1, f_2))(x)$$

$$\lesssim 2^{-\gamma (m-k)} M^2 \left( M (D_{k}^{h_1} M_{b_{1}} f_1) \cdot M f_2 \right)(x).$$

$$|D_{m}^{h_0} M_{b_{0}} \Theta_k \Pi_j^1(\tilde{D}_{k}^{h_1} M_{b_{1}} D_{k}^{h_1} M_{b_{1}} f_1, f_2)(x)| \lesssim M^2 (\Pi_j^1(\tilde{D}_{k}^{h_1} M_{b_{1}} D_{k}^{h_1} M_{b_{1}} f_1, f_2))(x)$$

$$\lesssim 2^{-\gamma (m-k)} M^2 \left( M (D_{k}^{h_1} M_{b_{1}} f_1) \cdot M f_2 \right)(x).$$

Taking the geometric mean of these three estimates, we have the following pointwise bound

$$|D_{m}^{h_0} M_{b_{0}} \Theta_k \Pi_j^1(\tilde{D}_{k}^{h_1} M_{b_{1}} D_{k}^{h_1} M_{b_{1}} f_1, f_2)(x)| \lesssim 2^{-\gamma \left( \frac{|m-k|}{2} + \frac{|k-j|}{2} + \frac{|j-l|}{2} \right)} M^2 \left( M (D_{k}^{h_1} M_{b_{1}} f_1) \cdot M f_2 \right)(x).$$
Therefore
\[
\sum_{j,k,l,m \in \mathbb{Z}} \int_{\mathbb{R}^n} |D_y^{b_0} M_{b_0} \Theta_j \Pi_{k,l} (\tilde{D}_{m}^{b_0} M_{b_1} D_y^{b_1} M_{b_1} f_1, f_2)(x) \tilde{D}_{m}^{b_0} M_{b_0} f_0(x)| \, dx
\lesssim \int_{\mathbb{R}^n} \sum_{j,k,l,m \in \mathbb{Z}} 2^{-\gamma \left( \frac{|m-k|}{2} + \frac{|k-j|}{2} + \frac{|j-l|}{2} \right)} \mathcal{M}^2 \left( \mathcal{M} (D_y^{b_1} M_{b_1} f_1) \cdot \mathcal{M} f_2 \right)(x) |\tilde{D}_{m}^{b_0} M_{b_0} f_0(x)| \, dx
\lesssim \left\| \mathcal{M}^2 \left( \mathcal{M} (D_y^{b_1} M_{b_1} f_1) \cdot \mathcal{M} f_2 \right)(x) \right\|_{L^p} \left\| \sum_{m \in \mathbb{Z}} |\tilde{D}_{m}^{b_0} M_{b_0} f_0|^2 \right\|_{L^{p'}}
\lesssim \left( \sum_{m \in \mathbb{Z}} \mathcal{M}(D_y^{b_1} M_{b_1} f_1)^2 \right)^{\frac{1}{2}} \left\| \mathcal{M} f_2 \right\|_{L^2} \left\| f_0 \right\|_{L^p} \lesssim \prod_{i=0}^2 \left\| f_i \right\|_{L^p}.
\]

In the last three lines, we apply the Fefferman-Stein vector valued maximal inequality [12], Hölder’s inequality, and the square function bounds for $D_y^{b_1}$ and $\tilde{D}_{m}^{b_0}$ proved by David-Journé-Semmes in [11]. By symmetry and density, this completes the proof. \hfill \Box

6. Singular Integral Operators

In this section, we prove a reduced T(b) theorem, construct a para-accretive paraproduct, and prove a full T(b) theorem all in the bilinear setting. First, we prove a few technical lemmas that relate the work in the preceding sections to singular integral operators.

6.1. Two Technical Lemmas.

**Lemma 6.1.** Let $b_0, b_1, b_2 \in L^\infty$ be para-accretive functions, and assume that $T$ is a bilinear C-Z operator associated to $b_0, b_1, b_2$ such that $M_{b_0} T(M_{b_1}, \cdot, M_{b_2}, \cdot) \in WBP$ for normalized bumps of order $m$. Then for all normalized bumps $\phi_0, \phi_1, \phi_2$, $R > 0$ of order $m$, and $y_0, y_1, y_2 \in \mathbb{R}^n$ such that $|y_0 - y_1| \leq t R$

\[
\left| \langle T(M_{b_0} \phi_1^{y_1,R}, M_{b_2} \phi_2^{y_2,R}, M_{b_0} \phi_0^{y_0,R} \rangle \right| \lesssim (1 + t)^{\eta + 3m} R^\eta.
\]

**Proof.** Let $y_0, y_1, y_2 \in \mathbb{R}^n$, $R > 0$, and define $D = 1 + 2t$. Then it follows that

\[
\left| \langle T(M_{b_0} \phi_1^{y_1,R}, M_{b_2} \phi_2^{y_2,R}, M_{b_0} \phi_0^{y_0,R} \rangle \right| = \left| \langle T(M_{b_0} \hat{\phi}_1^{y_1,R}, M_{b_2} \hat{\phi}_2^{y_2,R}, M_{b_0} \hat{\phi}_0^{y_0,R} \rangle \right|,
\]

where $\hat{\phi}_0(u) = \phi_0(Du)$ and $\hat{\phi}_i(u) = \phi_i(Du + \frac{y_i - y_0}{R})$ for $i = 1, 2$. If $|u| > 1$, then clearly $D|u| > 1$, and

\[
\left| Du + \frac{y_0 - y_1}{R} \right| \geq D|u| - \left| \frac{y_0 - y_1}{R} \right| \geq (1 + 2t)|u| - t \geq 1.
\]

This completes the proof.


So we have that \( \text{supp}(\tilde{\phi}_i) \subset B(0,1) \). It follows that \( D^{-m}\tilde{\phi}_i \in C^m_0 \) are normalized bumps of order \( m \), and it follows that

\[
\left| \left( T(M_{b_1}\tilde{\phi}_i^{\partial,D_R}, M_{b_2}\tilde{\phi}_2^{\partial,D_R}), M_{b_0}\tilde{\phi}_0^{\partial,D_R} \right) \right| \lesssim D^{3m}(DR)^n \lesssim (1+t)^{n+3m}\mathbb{R}^n.
\]

This completes the proof. \( \square \)

**Lemma 6.2.** Let \( b_0, b_1, b_2 \in L^\infty \) be para-accretive functions. Suppose \( T \) is an bilinear \( C-Z \) operator associated to \( b_0, b_1, b_2 \) with compactly supported kernels \( \phi_k \) and \( D^{k_0}_k = S_{k+1} - S_k \) with \( k \in \mathbb{Z} \). Then

\[
\Theta_k(x,y_1,y_2) = \left\langle T \left( b_1s_k^{b_1}(\cdot,y_1), b_2s_k^{b_2}(\cdot,y_2) \right), b_0d_k^{b_0}(x,\cdot) \right\rangle
\]

is a collection of Littlewood-Paley square function kernels of type SBLPK. Furthermore \( \Theta_k \) satisfies

\[
\int_{\mathbb{R}^n} \Theta_k(x,y_1,y_2) b_0(x) dx = 0
\]

for all \( y_1,y_2 \in \mathbb{R}^n \).

**Proof.** Fix \( x,y_1,y_2 \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \). We split estimate (2.6) into two cases: \( |x-y_1| + |x-y_2| \leq 2^{3-k} \) and \( |x-y_1| + |x-y_2| > 2^{3-k} \). Note that

\[
\phi_1(u) = s_k^{b_1}(u+2^ky_1,2^ky_1)
\]

is a normalized bump up to a constant multiple and \( s_k^{b_1}(u,y_1) = 2^{-kn}\phi_1^{2^{2-k}}(u) \). Likewise \( s_k^{b_2}(u,y_2) = 2^{-kn}\phi_2^{2^{2-k}}(u) \) and \( d_k^{b_0}(x,u) = 2^{-kn}\phi_0^{2^{2-k}}(u) \) where \( \phi_0 \) and \( \phi_2 \) are normalize bumps up to a constant multiple. Then

\[
|\Theta_k(x,y_1,y_2)| = \left| \left\langle T \left( b_1s_k^{b_1}(\cdot,y_1), b_2s_k^{b_2}(\cdot,y_2) \right), b_0d_k^{b_0}(x,\cdot) \right\rangle \right|
\]

is \( 2^{2kn} \left| \left\langle T \left( b_1\phi_1^{2^{2-k}}, b_2\phi_2^{2^{2-k}} \right), b_0\phi_0^{2^{2-k}} \right\rangle \right| \lesssim 2^{2kn} \)

Now if we assume that \( |x-y_1| + |x-y_2| > 2^{3-k} \), then it follows that \( |x-y_0| > 2^{2-k} \) for at least one \( i_0 \in \{1,2\} \) and hence

\[
\text{supp}(d_k^{b_0}(x,\cdot)) \cap \text{supp}(s_k^{b_1}(\cdot,y_1)) \cap \text{supp}(s_k^{b_2}(\cdot,y_2)) \subset B(x,2^{-k}) \cap B(y_0,2^{-k}) = \emptyset.
\]

Therefore, we can estimate \( \Theta_k \) the kernel representation of \( T \) in the following way

\[
|\Theta_k(x,y_1,y_2)|
\]

\[
\lesssim \int_{|x-u_0|<2^{-k}} \int_{|y_1-u_1|<2^{-k}} \int_{|y_2-u_2|<2^{-k}} \frac{|u_0-x|^{\gamma} 2^{3nk} du_0 du_1 du_2}{\left( |x-u_1| + |x-u_2| \right)^{2n+\gamma}}
\]

\[
\lesssim \int_{|x-u_0|<2^{-k}} \int_{|y_1-u_1|<2^{-k}} \int_{|y_2-u_2|<2^{-k}} \frac{2^{-k 2^{3nk}} du_0 du_1 du_2}{\left( 2^{-k} |x-y_1| + |x-y_2| \right)^{2n+\gamma}}
\]

\[
\lesssim \left( 2^{-k} + |x-y_1| + |x-y_2| \right)^{2n+\gamma}
\]

\[
\lesssim \Phi^{n+\gamma/2}_k (x-y_1) \Phi^{n+\gamma/2}_k (x-y_2).
\]
For (2.7), note that by the continuity from $b_1 C_0^5 \times b_2 C_0^5$ into $(b_0 C_0^5)'$ and that $S_k^\alpha = P_k M_{(b_0)}^{-1} P_k$ has a $C_0^\infty$ kernel, we have for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$

$$|\partial_\alpha T_k(x,y,z)| = \left| T \left( b_1 s_k^{b_1} (\cdot, y_1), b_2 s_k^{b_2} (\cdot, y_2) \right), \partial_\alpha b_o(x, \cdot) \left( d_k(x, \cdot) \right) \right| \lesssim 2^k 2^{k\alpha}.$$

Estimate (2.7) easily follows in light of Remark 2.8. By symmetry, it follows that $\{T_k\}$ is a collection of smooth bilinear Littlewood-Paley square function kernels. Now we verify that $\theta_k$ has integral 0 in the $x$ variable. By the continuity of $T$ from $b_1 C_0^5 \times b_2 C_0^5$ into $(b_0 C_0^5)'$

$$\int_{\mathbb{R}^n} \theta_k(x,y_1,y_2) b_0(x) dx = \lim_{R \to \infty} \left\langle T \left( b_1 s_k^{b_1} (\cdot, y_1), b_2 s_k^{b_2} (\cdot, y_2) \right), b_0 \int_{|x| < R} d_k^{b_0} (x, \cdot) b_0(x) dx \right\rangle$$

where we take this to be the definition of $\lambda_R$. Now if we take $R > 2 \cdot 2^{-k}$, then for $|u| < R - 2^{-k}$ it follows that

$$\text{supp} (d_k^{b_0} (\cdot, u)) \subset B(u, 2^{-k}) \subset B(0, |u| + 2^{-k}) \subset B(0, R),$$

and hence for $|u| < R - 2^{-k}$ we have that

$$\lambda_R(u) = b_0(u) \int_{|x| < R} d_k^{b_0} (x, u) b_0(x) dx = b_0(u) D_k^{b_0} b_0(u) = 0.$$

Also when $|u| > R + 2^{-k}$, it follows that $\text{supp} (d_k^{b_0} (\cdot, u)) \cap B(0, R) = \emptyset$, and hence that $\lambda_R(u) = 0$. So we have $\lambda_R(x) = 0$ for $|x| < R - 2^{-k}$ and for $|x| > R + 2^{-k}$. Finally $||\lambda_R||_{L^\infty} \leq \text{sup}_{u, \cdot} ||d_k^{b_0} (\cdot, u)||_{L^1} \lesssim 1$. Since $\text{supp} (d_k^{b_0} (x, \cdot)) \subset B(0, R + 2^{-k}) \setminus B(0, R - 2^{-k})$, it follows that for $R > 4(2^{-k} + |y_1|)$, we may use the integral representation

$$\left| \left\langle T \left( b_1 s_k^{b_1} (\cdot, y_1), b_2 s_k^{b_2} (\cdot, y_2) \right), \lambda_R \right\rangle \right|$$

$$\leq \int_{\mathbb{R}^3} |K(u, v_1, v_2) b_1 (v_1) s_k^{b_1} (v_1, y_1) b_2 (v_2) s_k^{b_2} (v_2, y_2) \lambda_R(u)| d u d v_1 d v_2$$

$$\lesssim \int_{|v_2 - y_2| < 2^{-k}} \int_{|v_1 - y_1| < 2^{-k}} \int \supp (\lambda_R) \frac{2^{2k\alpha}}{|u - v_1 + |u - v_2||^{2n}} d u d v_1 d v_2$$

$$\lesssim \int_{|v_2 - y_2| < 2^{-k}} \int_{|v_1 - y_1| < 2^{-k}} \int \supp (\lambda_R) \frac{2^{2k\alpha}}{|u - v_1 - |v_1 - y_1||^{2n}} d u d v_1 d v_2$$

$$\lesssim \int_{|v_2 - y_2| < 2^{-k}} \int_{|v_1 - y_1| < 2^{-k}} \int \supp (\lambda_R) \frac{2^{2k\alpha}}{|R - 2^{-k} - |v_1 - y_1||^{2n}} d u d v_1 d v_2$$

$$\lesssim \frac{2^{2k\alpha}}{R^{2n}} d u d v_1 d v_2$$

$$\lesssim |\supp (\lambda_R)| R^{-2n}$$

$$\lesssim 2^{-k} R^{-(\alpha+1)}.$$

This tends to zero as $R \to \infty$. Hence $\theta_k(x,y_1,y_2)$ has integral zero in the $x$ variable. \hfill \Box

6.2. Reduced Bilinear $T(b)$ Theorem. It has become a standard argument in $T_1$ and $T_b$ theorems to first prove a reduced version, see e.g. [10], [11], and [22]. The general idea of the argument is to first assume a stronger $Tb = 0$ cancellation condition, and then prove that an operator satisfying the weaker $Tb \in BMO$ cancellation condition is a perturbation of an operator satisfying the stronger cancellation condition. More precisely this is done
Let $T$ be an bilinear C-Z operator associated to para-accretive functions $b_0, b_1, b_2$. If $M_{b_0} T(M_{b_1} \cdot, M_{b_2} \cdot) \in WBP$ and

$$M_{b_0} T(b_1, b_2) = M_{b_0} T^{+1}(b_0, b_2) = M_{b_2} T^{+2}(b_1, b_0) = 0,$$

then $T$ can be extended to a bounded linear operator from $L^{p_1} \times L^{p_2}$ into $L^p$ for all $1 < p_1, p_2 < \infty$ satisfying (2.2).

Note that in the hypothesis of Theorem 6.3, we take $M_{b_0} T(b_1, b_2) = 0$ in the sense of Definition 2.5: For appropriate $\eta^1_R, \eta^2_R$ and all $\phi \in \mathcal{C}_0^\infty$ such that $b_0\phi$ has mean zero

$$\lim_{R \to \infty} \langle T(\eta^1_R b_1, \eta^2_R b_2), b_0\phi \rangle = 0.$$

The meaning of $M_{b_1} T^{+1}(b_0, b_2) = M_{b_2} T^{+2}(b_1, b_0) = 0$ are expressed in a similar way.

**Proof.** Let $T$ be as in the hypothesis, $1 < p, p_1, p_2 < \infty$ satisfy (2.2), and $f_0, f_1, f_2 \in \mathcal{C}_0^1$ such that $b_i f_i$ have mean zero. Then by Proposition 4.5 and the continuity of $T$ from $b_1 \mathcal{C}_0^1 \times b_2 \mathcal{C}_0^1$ into $(b_0 \mathcal{C}_0^1)'$, it follows that

$$\left| \langle T(b_1 f_1, b_2 f_2), b_0 f_0 \rangle \right| \leq \lim_{N \to \infty} \left| \sum_{k=-N}^{N-1} \langle T(M_{b_1} S^k f_1, M_{b_2} S^k f_2), M_{b_0} \phi \rangle \right|$$

$$= \lim_{N \to \infty} \sum_{k=-N}^{N-1} \left| \langle T(M_{b_1} S^k f_1, M_{b_2} S^k f_2), M_{b_0} \phi \rangle \right|$$

$$\leq \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k^0(b_1 f_1, b_2 f_2) b_0(x) f_0(x) dx \right|$$

$$+ \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k^1(b_0 f_0, b_2 f_2) b_1(x) f_1(x) dx \right|$$

$$+ \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k^2(b_1 f_1, b_0 f_0) b_2(x) f_2(x) dx \right|,$$

where

$$\Theta_k^0(f_1, f_2) = D_k^0 M_{b_0} T(M_{b_1} S^k f_1, M_{b_2} S^k f_2),$$

$$\Theta_k^1(f_1, f_2) = D_k^1 M_{b_1} T^{+1}(M_{b_0} S^k f_1, M_{b_2} S^k f_2),$$

$$\Theta_k^2(f_1, f_2) = D_k^2 M_{b_2} T^{+2}(M_{b_1} S^k f_1, M_{b_0} S^k f_2).$$

We focus on $\Theta_k^0 = \Theta_k$ to simplify notation; the other terms are handled in the same way. Since $M_{b_0} T(M_{b_1} \cdot, M_{b_2} \cdot) \in WBP$ and $T$ has a standard kernel, it follows from Lemma 6.2 that $\{ \Theta_k \} \in SBLPK$ and $\Theta_k(x, y_1, y_2) b_0(x)$ has mean zero in the $x$ variable for all $y_1, y_2 \in \mathbb{R}^n$.

Now we show that $\Theta_k(b_1, b_2) = 0$, which follows from the assumption that $M_{b_0} T(b_1, b_2) =$
A similar argument holds for $\varepsilon_x^0$:

**Proof.** To prove the size estimate, we take $\varepsilon_x^0$ into $L^1$ and $L^2$, and $\varepsilon_x^1$ in $L^3$.

Suppose $\varepsilon_x^0$ is a bilinear version of an observation made by Benyi-Maldonado-Nahmod-Torres [1] and is a bilinear version of an observation made by Coifman-Meyer [9].

6.3. A Para-Product Construction. In the original proof of the T1 theorem, David-Journé [10] used the Bony paraproduct [4] to pass from their reduced T1 theorem to the full T1 theorem. Following the same idea, David-Journé-Semmes [11] proved the Tb theorem by constructing a para-accretive version of the Bony paraproduct. In [22], we constructed a bilinear Bony-type paraproduct, which allowed us to transition from a reduce bilinear T1 theorem to a full T1 theorem. Here we construct a bilinear paraproduct in a para-accretive function setting. First we prove a quick lemma, which essentially appears in a work by Benyi-Maldonado-Nahmod-Torres [1] and is a bilinear version of an observation made by Coifman-Meyer [9].

**Lemma 6.4.** Suppose $\theta_k \in SBLPK$ with decay parameter $N > 2n$, and define $K : \mathbb{R}^{3n} \setminus \{(x,x,x) : x \in \mathbb{R}^n \} \rightarrow \mathbb{C}$

$$K(x,y_1,y_2) = \sum_{k \in \mathbb{Z}} \theta_k(x,y_1,y_2).$$

Then $K$ is a bilinear standard Calderón-Zygmund kernel.

**Proof.** To prove the size estimate, we take $d = |x - y_1| + |x - y_2| \neq 0$ and compute

$$|K(x,y_1,y_2)| \lesssim \sum_{k \in \mathbb{Z}} \left(1 + 2^k |x - y_1|\right)^{N+\gamma} \left(1 + 2^k |x - y_2|\right)^{N+\gamma} \lesssim d^{2n}.$$

For the regularity in $x$, we take $(x,x',y_1,y_2) \in \mathbb{R}^n$ with $|x-x'| < \max(|x-y_1|,|x-y_2|)/2$ and define $d' = |x'-y_1| + |x'-y_2|$. Then

$$|K(x,y_1,y_2) - K(x',y_1,y_2)| \lesssim \sum_{k \in \mathbb{Z}} \frac{(2^k |x-x'|)^{2+\gamma} 2^{2kn}}{(1 + 2^k |x-y_1|)^{N+\gamma}(1 + 2^k |x-y_2|)^{N+\gamma}}$$

$$+ \sum_{k \in \mathbb{Z}} \frac{(2^k |x-x'|)^{2+\gamma} 2^{2kn}}{(1 + 2^k |x'-y_1|)^{N+\gamma}(1 + 2^k |x'-y_2|)^{N+\gamma}}$$

$$= I + II.$$
We first bound \( I \) by \(|x - x'|^\gamma \) times
\[
\sum_{2^k \leq d^{-1}} 2^{k(2n+\gamma)} + \sum_{2^k > d^{-1}} \frac{2^{k(2n+\gamma)}}{(2^k d^{N+\gamma})} \lesssim d^{-(2n+\gamma)} + d^{-(N+\gamma)} \sum_{2^k > d^{-1}} 2^{k(2n-N)} \lesssim d^{-(2n+\gamma)}.
\]
By symmetry, it follows that \( II \lesssim |x - x'|^\gamma d^{-(2n+\gamma)} \), but since \(|x - x'| < \max(|x-y_1|,|x-y_2|)/2\), without loss of generality say \(|x-y_1| \geq |x-y_2|\) it follows that
\[
II \lesssim \frac{|x-x'|^\gamma}{(|x'-y_1| + |x'-y_2|)^{2n+\gamma}} \lesssim \frac{|x-x'|^\gamma}{(|x-y_1| - |x-x'|)^{2n+\gamma}} \lesssim \frac{|x-x'|^\gamma}{|x-y_1|^{2n+\gamma}} \lesssim \frac{|x-x'|^\gamma}{d^{2n+\gamma}}
\]
With a similar computation for \( y_1,y_2 \), it follows that \( K \) is a standard kernel.

**Theorem 6.5.** Given para-accrrete functions \( b_0,b_1,b_2 \in L^\infty \) and \( \beta \in BMO \), there exists a \( \text{bilinear Calderón-Zygmund operator} \ L \) that is bounded from \( L^{p_1} \times L^{p_2} \) into \( L^p \) for all \( 1 < p_1,p_2 < \infty \) satisfying (2.2) with \( p = 2 \) such that \( M_{b_0}T(b_1,b_2) = \beta, M_{b_0}T^{*1}(b_0,b_2) = M_{b_2}T^{*2}(b_1,b_0) = 0 \).

**Proof.** Let \( b_0,b_1,b_2 \) be para-accrrete functions, and \( S^b_i, D^b_i, \) and \( \widetilde{D}^b_i \) be the approximation to identity and reproducing formula operators with respect to \( b_i \) for \( i = 0,1,2 \) that have compactly supported kernels as defined in Remark 4.4 and Theorem 4.6. Define
\[
L(f_1,f_2) = \sum_{k \in \mathbb{Z}} L_k(f_1,f_2) = \sum_{k \in \mathbb{Z}} \widetilde{D}^b_k M_{b_0} \left( (\widetilde{D}^b_k + M_{b_0} \beta)(S^b_k f_1)(S^b_k f_2) \right)
\]
\[
\ell(x,y) = \sum_{k \in \mathbb{Z}} \ell_k(x,y) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \widetilde{D}^b_k(x,u) b_0(u) \widetilde{D}^b_k b_0 \beta(u,y_1) s^b_k(u,y_2) du.
\]
It follows that \( L \) is bounded from \( L^{p_1} \times L^{p_2} \) into \( L^2 \) for all \( 1 < p_1,p_2 < \infty \) satisfying \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \):
\[
\left| \int_{\mathbb{R}^n} L(f_1,f_2)(x) f_0(x) dx \right| \leq \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \widetilde{D}^b_k M_{b_0} \beta(x) S^b_k f_1(x) S^b_k f_2(x) \widetilde{D}^b_k f_0(x) b_0(x) dx \right|
\]
\[
\lesssim \left\| \left( \sum_{k \in \mathbb{Z}} |M_{\widetilde{D}^b_k + M_{b_0} \beta} S^b_k f_1 S^b_k f_2|^2 \right) \right\|_{L^2} \left\| \left( \sum_{k \in \mathbb{Z}} |D^b_k f_0|^2 \right) \right\|_{L^2}
\]
\[
\lesssim \left( \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} [\Phi \ast |f_1|] \Phi \ast |f_2| \right)^{\frac{1}{p_1}} \left( \int_{\mathbb{R}^n} \left| \widetilde{D}^b_k M_{b_0} \beta(x) \right|^2 dx \right)^{\frac{1}{p_2}}
\]
\[
\leq \left( \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} [\Phi \ast |f_1|] \Phi \ast |f_2| \right)^{\frac{1}{p_1}} \left( \int_{\mathbb{R}^n} \left| \widetilde{D}^b_k M_{b_0} \beta(x) \right|^2 dx \right)^{\frac{1}{p_2}}
\]
\[
\lesssim \| f_0 \|_{L^2} \| f_1 \|_{L^{p_1}} \| f_2 \|_{L^{p_2}}.
\]
Note that in the last line we apply the discrete version of a well-known result relating Carleson measure and square functions due to Carleson [7] and Jones [25] for the Carleson measure
\[
d\mu(x,t) = \sum_{k \in \mathbb{Z}} |\widetilde{D}^b_k b_0 \beta(x)|^2 \delta_{2^{-k}}.
\]
For details of the discrete version of this result, see for example the book by Grafakos [15], Theorems 7.3.7 and 7.3.8(c). This proves that $L$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^{2}$ for all $1 < p_1, p_2 < \infty$ satisfying (2.2) with $p = 2$. It is easy to check that $\{ \ell_k \} \in SBL_{pl}$ with size index $N > 2n$: since $d_k^{b_0}$ and $s_k^{b_0}$ are compactly supported kernels, for $i = 1, 2$ it follows that

$$|\ell_k(x, y_1, y_2)| \leq |b_0 D_k^{b_0} |M_{b_0} \beta|_{L^\infty} \int_{\mathbb{R}^n} |d_k^{b_0} (x - u)s_k^{b_1} (u - y_1)s_k^{b_2} (u - y_2)| du$$

$$\lesssim 2^{kn} \int_{\mathbb{R}^n} \phi_k^{4(n+1)} (x - u) \Phi_k^{4(n+1)} (u - y_1) du$$

$$\lesssim 2^{kn} \phi_k^{4(n+1)} (x - y_1).$$

Hence the size condition (2.6) with size index $N = 2n + 1$ and $\gamma = 1$ follows

$$|\ell_k(x, y_1, y_2)| \lesssim 2^{2n} (x - y_1) \Phi_k^{2n+2} (x - y_2).$$

The regularity estimates (2.7)-(2.9) follow easily from the regularity of $d_k^{b_0}$, $s_k^{b_1}$, and $s_k^{b_2}$. Then by Lemma 6.4, $L$ has a standard Calderón-Zygmund kernel $\ell$. It follows from a result of Grafakos-Torres [17] that $L$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^p$ where $1 < p_1, p_2 < \infty$ satisfy (2.2). Next we compute $M_{b_0} L(b_1, b_2)$: Let $\delta > 0$, $\phi \in C_0^\infty$ such that $\text{supp}(\phi) \subset B(0, N)$ and $b_0 \phi$ has mean zero. Let $\eta_R(x) = \eta(x/R)$ where $\eta \in C_0^\infty$ satisfies $\eta \equiv 1$ on $B(0, 1)$, and $\text{supp}(\eta) \subset B(0, 2)$. Then

$$\langle L(b_1, b_2), b_0 \phi \rangle$$

$$= \lim_{R \to \infty} \sum_{2^{-k} < R/4} \int_{\mathbb{R}^n} \bar{D}_k^{b_0} M_{b_0} \beta(x) S_k^{b_1} M_{b_1} \eta_R(x) S_k^{b_2} M_{b_2} \eta_R(x) M_{b_0} D_k^{b_0} (b_0 \phi)(x) dx$$

$$+ \lim_{R \to \infty} \sum_{2^{-k} > R/4} \int_{\mathbb{R}^n} \bar{D}_k^{b_0} M_{b_0} \beta(x) S_k^{b_1} M_{b_1} \eta_R(x) S_k^{b_2} M_{b_2} \eta_R(x) M_{b_0} D_k^{b_0} (b_0 \phi)(x) dx,$$

where we may write this only if the two limits on the right hand side of the equation exist. As we are taking $R \to \infty$ and $N$ is a fixed quantity determined by $\phi$, without loss of generality assume that $R > 2N$. Note that for $2^{-k} \leq R/4$ and $|x| < N + 2^{-k}$,

$$\text{supp}(s_k^{b_0}(x, \cdot)) \subset B(x, 2^{-k}) \subset B(0, N + 2^{1-k}) \subset B(0, R).$$

Since $\eta_R \equiv 1$ on $B(0, R)$, it follows that $S_k^{b_0} M_{b_0} \eta_R(x) = 1$ for all $|x| < N + 2^{-k}$ when $2^{-k} \leq R/4$. Therefore

$$\lim_{R \to \infty} \sum_{2^{-k} < R/4} \int_{\mathbb{R}^n} \bar{D}_k^{b_0} M_{b_0} \beta(x) M_{b_0} D_k^{b_0} (b_0 \phi)(x) dx = \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} M_{b_0} \bar{D}_k^{b_0} M_{b_0} D_k b_0 \phi(x) \beta(x) dx$$

$$= \langle \beta, b_0 \phi \rangle.$$

Here we use the convergence of the accretive type reproducing formula in $H^1$ from Corollary 4.8. For any $k \in \mathbb{Z}$, we have the estimates

$$\text{(6.1)} \quad \|s_k^{b_0} M_{b_0} \eta_R\|_{L^1} \lesssim \|\eta_R\|_{L^1} \lesssim R^n,$$

$$\text{(6.2)} \quad \|s_k^{b_0} M_{b_0} \eta_R\|_{L^\infty} \lesssim \|\eta_R\|_{L^\infty} = 1,$$
and for any $x \in \mathbb{R}^n$

$$|D_k^{b_0} M_{b_0} \phi(x)| \leq \int_{\mathbb{R}^n} |d_k^{b_0}(x, y) - d_k^{b_0}(x, 0)| |b_0(y)\phi(y)|dy$$

$$\lesssim \int_{\mathbb{R}^n} (2^k|y|)^\gamma |\phi(y)|dy \lesssim N^\gamma |\phi||_{L_1} 2^{k(n+\gamma)}.$$  

Here we know that $\{d_k^{b_0}\} \in LPK$, so without loss of generality we take the corresponding smoothness parameter $\gamma \leq \delta$. Later we will use that $\gamma \leq \delta$ implies $\phi \in C_0^\infty \subset C_0^\infty$, so we have that $|\phi(x) - \phi(y)| \lesssim |x-y|^\gamma$. Therefore

$$\sum_{2^{-k} \geq R/4} \int_{\mathbb{R}^n} |\tilde{D}_k^{b_0} M_{b_0} \beta(x) S_k^{1\beta} M_{b_1} \eta_R(x) S_k^{2\beta} M_{b_2} \eta_R(x) M_{b_0} D_k^{b_0} (b_0 \phi)(x)|dx$$

$$\leq \sum_{2^{-k} \geq R/4} ||\tilde{D}_k^{b_0} M_{b_0} \beta||_{L^\infty} ||S_k^{1\beta} M_{b_1} \eta_R||_{L^1} ||S_k^{2\beta} M_{b_2} \eta_R||_{L^1} ||M_{b_0} D_k^{b_0} (b_0 \phi)||_{L^\infty}$$

$$(6.3) \lesssim R^n \sum_{2^{-k} \geq R/4} 2^{k(n+\gamma)} \lesssim R^{-\gamma}. $$

Hence the second limit above exists and tends to 0 as $R \to \infty$. Then $\langle L(b_1, b_2), b_0 \phi \rangle = \langle \beta, b_0 \phi \rangle$ for all $\phi \in C_0^\infty$ such that $b_0 \phi$ has mean zero and hence $M_{b_0} L(b_1, b_0) = \beta$ as defined in Section 2. Again for any $\phi \in C_0^\infty$ such that $b_1 \phi$ has mean zero and supp($\phi$) $\subset B(0,N)$, we have

$$\langle L^{1\beta}(b_1, b_2), b_1 \phi \rangle$$

$$= \lim_{R \to \infty} \sum_{k \leq R} \int_{\mathbb{R}^n} ||\tilde{D}_k^{b_0} M_{b_0} \beta||_{L^\infty} ||S_k^{1\beta} M_{b_1} \phi||_{L^1} ||S_k^{2\beta} M_{b_2} \eta_R||_{L^1} ||M_{b_0} D_k^{b_0} (b_0 \phi)||_{L^\infty}$$

$$\lesssim \lim_{R \to \infty} \sum_{k \leq R} ||S_k^{1\beta} M_{b_1} \phi||_{L^1} ||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty}. $$

We will now show that $||S_k^{1\beta} M_{b_1} \phi||_{L^1} ||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty}$ bounded by a in integrable function in $k$ (i.e. summable) independent of $R$, so that we can bring the limit in $R$ inside the sum.

To do this we start by estimating

$$||S_k^{1\beta} M_{b_1} \phi(x)|| \leq \int_{\mathbb{R}^n} |S_k^{1\beta}(x, y) - s_k^{1\beta}(x, 0)| |\phi(y)b_1(y)|dy$$

$$\leq N^\gamma |\phi||_{L_1} ||b_1||_{L^1} 2^{k\gamma} (\Phi_0^N(x) + \Phi_k^N(x))$$

and so $||S_k^{1\beta} M_{b_1} \phi||_{L^1} \lesssim 2^k$. We also have that $||S_k^{1\beta} M_{b_1} \phi||_{L^1} \lesssim ||\phi||_{L^1} \lesssim 1$, so $||S_k^{1\beta} M_{b_1} \phi||_{L^1} \lesssim \min(1, 2^k)$. Also

$$||D_k^{b_0} M_{b_0} \eta_R(x)|| \leq \int_{\mathbb{R}^n} |d_k^{b_0}(x, y)| |\eta_R(y) - \eta_R(x)| |b_0(y)|dy$$

$$\lesssim 2^{-k} R^{-\gamma} \int_{\mathbb{R}^n} \Phi_k^{N+\gamma}(x-y)(2^k|x-y|)dy \lesssim 2^{-k} R^{-\gamma}.$$ 

It follows that $||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty} \lesssim ||\eta_R||_{L^\infty} \lesssim 1$, and hence $||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty} \lesssim \min(1, 2^{-k})$. So when $R > 1$, we have

$$||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty} ||S_k^{1\beta} M_{b_1} \phi||_{L^1} \lesssim \min(2^{-k} R^{-\gamma}, 2^{k}) \lesssim 2^{-\gamma k}.$$
and hence by dominated convergence
\[
\left|\langle L^1(b_1, b_2), b_1 \phi \rangle \right| \lesssim \sum_{k \in \mathbb{Z}} \lim_{R \to \infty} \|S_k^{b_1} M_{b_1} \phi\|_{L^1} \|D_k^{b_0} M_{b_0} \eta_R\|_{L^\infty} \lesssim \sum_{k \in \mathbb{Z}} 2^{-k\gamma} R^{-\gamma} = \mathcal{O}
\]
Then \(M_{b_1} L^{s_1}(b_1, b_2) = 0\) and a similar argument shows that \(M_{b_2} L^{s_2}(b_1, b_0) = 0\), which concludes the proof.

Now to complete the proof of Theorem 1.1 is a standard argument using the reduced \(Tb\) Theorem 6.3 and paraproduct construction in Theorem 6.5.

**Proof.** Assume that \(M_{b_0} T(M_{b_1} \cdot, M_{b_2} \cdot)\) satisfies the weak boundedness property and
\[
M_{b_0} T(b_1, b_2), M_{b_1} T^{s_1}(b_0, b_2), M_{b_2} T^{s_2}(b_1, b_0) \in BMO.
\]
By Theorem 6.5, there exist bounded bilinear Calderón-Zygmund operators \(L_i\) such that
\[
M_{b_0} L_0(b_1, b_2) = M_{b_0} T(b_1, b_2), \quad M_{b_1} L_0^{s_1}(b_0, b_2) = M_{b_1} L_0^{s_1}(b_1, b_0) = 0,
\]
\[
M_{b_1} L_1^{s_1}(b_1, b_2) = M_{b_1} T^{s_1}(b_0, b_2), \quad M_{b_0} L_1(b_1, b_2) = M_{b_2} L_1^{s_2}(b_1, b_0) = 0,
\]
\[
M_{b_2} L_2^{s_2}(b_1, b_0) = M_{b_2} T^{s_2}(b_1, b_0), \quad M_{b_1} L_2^{s_1}(b_0, b_2) = M_{b_0} L_2(b_1, b_2) = 0.
\]
Now define the operator
\[
S = T - \sum_{i=0}^2 L_i,
\]
which is continuous from \(b_1 C_0^\alpha \times b_2 C_0^\beta\) into \((b_0 C_0^\beta)'\). Also \(M_{b_0} S(M_{b_1} \cdot, M_{b_2} \cdot)\) satisfies the weak boundedness property since \(M_{b_0} T(M_{b_1} \cdot, M_{b_2} \cdot)\) and \(M_{b_0} L_i(M_{b_1} \cdot, M_{b_2} \cdot)\) for \(i = 0, 1, 2\) do. Finally we have
\[
M_{b_0} S(b_1, b_2) = M_{b_0} T(b_1, b_2) - \sum_{i=0}^2 M_{b_0} L_i(b_1, b_2) = 0
\]
\[
M_{b_1} S^{s_1}(b_0, b_2) = M_{b_1} T^{s_1}(b_0, b_2) - \sum_{i=0}^2 M_{b_1} L_i^{s_1}(b_0, b_2) = 0
\]
\[
M_{b_2} S^{s_2}(b_1, b_0) = M_{b_2} T^{s_2}(b_1, b_0) - \sum_{i=0}^2 M_{b_2} L_i^{s_2}(b_1, b_0) = 0
\]
Then by Theorem 6.3, \(S\) can be extended to a bounded linear operator from \(L^{p_1} \times L^{p_2}\) into \(L^p\) for all \(1 < p, p_1, p_2 < \infty\) satisfying (2.2). Therefore \(T\) is bounded on the same indices, and by results from [17], \(T\) is also bounded without restriction on \(p\). The converse is also a well-known result from [17].

7. **Application to a Bilinear Reisz Transforms Defined on Lipschitz Curves**

In this section, we apply the bilinear \(Tb\) theorem proved above to a bilinear version of the Reisz transforms along Lipschitz curves in the complex plane. We prove bounds of the form
\[
\|T(f_1, f_2)\|_{L^p(\Gamma)} \lesssim \|f_1\|_{L^{p_1}(\Gamma)} \|f_2\|_{L^{p_2}(\Gamma)}
\]
for parameterized Lipschitz curves \(\Gamma\) and \(p, p_1, p_2\) satisfying Hölder.
We fix some notation for this section. Let \( L \) be a Lipschitz function with small Lipschitz constant \( \lambda < 1 \), and define the parameterization \( \gamma(x) = x + iL(x) \) of the curve \( \Gamma = \{ \gamma(x) : x \in \mathbb{R} \} \). Define \( L^p(\Gamma) \) to be the collection of all measurable functions \( f : \Gamma \to \mathbb{C} \) such that

\[
||f||_{L^p(\Gamma)} = \left( \int_{\Gamma} |f(z)|^p |dz| \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}} |f(\gamma(x))|^p |\gamma'(x)| \, dx \right)^{\frac{1}{p}} < \infty.
\]

The applications in this section are in part motivated by the proof of \( L^p \) bounds for the Cauchy integral using the Tb theorem of David-Journé-Semmes \cite{11}. We define the Cauchy integral operator for appropriate \( g : \Gamma \to \mathbb{C} \) and \( z \in \Gamma \)

\[
\mathcal{C}_r g(z) = \lim_{\epsilon \to 0^+} \int_{\Gamma} \frac{g(\xi)}{\xi - z} \, d\xi,
\]

and parameterized Cauchy integral operator for \( f : \mathbb{R} \to \mathbb{C} \) and \( x \in \mathbb{R} \)

\[
\mathcal{C}_r f(z) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{f(y) \, dy}{(\gamma(y) + i\epsilon) - \gamma(x)}.
\]

The bounds of \( \mathcal{C}_r \) on \( L^p(\Gamma) \) can be reduced to the bounds of \( \mathcal{C}_r \) on \( L^p(\mathbb{R}) \). We formally check the Tb conditions for \( \mathcal{C}_r \) with \( b_0 = b_1 = \gamma \) needed to apply the Tb theorem of David-Journé-Semmes: We check (1) \( \mathcal{C}_r(\gamma) \in BMO \)

\[
\mathcal{C}_r \gamma(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{\gamma'(y) \, dy}{(\gamma(y) + i\epsilon) - \gamma(x)} = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{d\xi}{(\xi + i\epsilon) - \gamma(x)} = 2\pi i
\]

and (2) \( \mathcal{C}_r(\gamma) \in BMO \), for appropriate \( \phi \in C_0^\infty \)

\[
\mathcal{C}_r^\phi \gamma(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{\gamma'(y) \, dy}{(\gamma(y) + i\epsilon) - \gamma(x)} = \lim_{\epsilon \to 0^+} - \int_{\mathbb{R}} \frac{d\xi}{(\xi - i\epsilon) - \gamma(x)} = 0.
\]

The crucial role that Cauchy’s formula plays in this argument is to be able to evaluate the limit from the definition of \( \mathcal{C}_r \) for nice enough input functions \( \gamma(x) \delta(x) \). In our application, we use a similar argument except the role of Cauchy’s integral formula is replaced with an integration by parts identity to verify the WBP and Tb conditions.

To further motivate looking at bilinear Reisz transforms along Lipschitz curve, we look at the “flat” bilinear Riesz transforms, which we generate from a potential function perspective. Consider the potential function

\[
F(x,y_1,y_2) = \frac{1}{((x-y_1)^2+(x-y_2)^2)^{1/2}},
\]

and the kernels that it generates:

\[
K_0(x,y_1,y_2) = \partial_x F(x,y_1,y_2) = \frac{2x-y_1-y_2}{((x-y_1)^2+(x-y_2)^2)^{3/2}},
\]

\[
K_1(x,y_1,y_2) = \partial_{y_1} F(x,y_1,y_2) = \frac{x-y_1}{((x-y_1)^2+(x-y_2)^2)^{3/2}},
\]

\[
K_2(x,y_1,y_2) = \partial_{y_2} F(x,y_1,y_2) = \frac{x-y_2}{((x-y_1)^2+(x-y_2)^2)^{3/2}}.
\]

We define the bilinear Reisz transforms as principle value integrals for \( f_1,f_2 \in C_0^\infty \),

\[
R_j(f_1,f_2)(x) = p.v. \int_{\mathbb{R}^2} K(x,y_1,y_2)f_1(y_1)f_2(y_2) \, dy_1 \, dy_2.
\]
Here is it only interesting to study two of these three operators since \( R_0 = R_1 + R_2 \). The bilinear T1 theorem of Christ-Journé [8] or Grafakos-Torres [17] can be applied to the bilinear Riesz transforms \( R_1 \): We formally check \((1)\) \( R_1(1, 1) \in BMO \)

\[
R_1(1, 1)(x) = -\int_{\mathbb{R}^2} F(x, y_1, y_2) \partial y_1(1)dy_1 dy_2 = 0,
\]

\(2\) \( R_1^1(1, 1) \in BMO \)

\[
R_1^1(1, 1)(y_1) = \int_{\mathbb{R}^2} \partial y_1 F(x, y_1, y_2) dx dy_2
\]

\[
= \int_{\mathbb{R}^2} (\partial y_1 F(x, y_1, y_2) - \partial y_2 F(x, y_1, y_2)) dx dy_2
\]

\[
= -\int_{\mathbb{R}^2} F(x, y_1, y_2)(\partial y_1(1) - \partial y_2(1)) dx dy_2 = 0,
\]

and \(3\) \( R_1^2(1, 1) \in BMO \)

\[
R_1^2(1, 1)(y_2) = -\int_{\mathbb{R}^2} F(x, y_1, y_2) \partial y_1(1) dx dy_1 = 0.
\]

Here we observe that the conditions \( R_1(1, 1), R_1^2(1, 1) = 0 \) are identical arguments and rely on the cancellation of the kernel \( K_1 \). The \( R_1^1(1, 1) \) condition relies on more than just the cancellation \( K_1 \); it also exploits the symmetry of the kernel via the identity \( \partial y_1 F(x, y_1, y_2) = \partial y_2 F(x, y_1, y_2) \). This is the general argument that we will use to prove \( L^p \) bounds for bilinear Riesz transforms defined along Lipschitz curves, which we define now.

For \( z, \xi_1, \xi_2 \in \Gamma \), define the potential function

\[
F_1(z, \xi_1, \xi_2) = \frac{1}{((z - \xi_1)^2 + (z - \xi_2)^2)^{1/2}},
\]

and the Riesz kernels generated by \( F_1 \):

\[
K_{\Gamma, 0}(z, \xi_1, \xi_2) = \partial_j F_1(\xi_1, \xi_1, \xi_2) = \frac{2z - \xi_1 - \xi_2}{((z - \xi_1)^2 + (z - \xi_2)^2)^{3/2}},
\]

\[
K_{\Gamma, 1}(z, \xi_1, \xi_2) = \partial_j F_1(\xi_1, z, \xi_2) = \frac{z - \xi_1}{((z - \xi_1)^2 + (z - \xi_2)^2)^{3/2}},
\]

\[
K_{\Gamma, 2}(z, \xi_1, \xi_2) = \partial_j F_1(z, \xi_1, \xi_2) = \frac{z - \xi_2}{((z - \xi_1)^2 + (z - \xi_2)^2)^{3/2}}.
\]

In the remainder of this section, we will keep the notation \( z = (x, y), \xi_1 = (y_1), \xi_2 = (y_2), y_0 = x \), and \( \xi_0 = z \). Here we define \( \sqrt{\cdot} \) on \( \mathbb{C} \) with the negative real axis for a branch cut, i.e. for \( \omega = re^{i\theta} \in \mathbb{C} \) with \( r > 0 \) and \( \theta \in (-\pi, \pi] \), we define \( \sqrt{\omega} = \sqrt{r}e^{i\theta/2} \). We make this definition to be precise, but it will not cause any issues with computations since we will only evaluate \( \sqrt{\omega} \) for \( \omega \in \mathbb{C} \) with positive real part.

For appropriate \( g_1, g_2 : \Gamma \to \mathbb{C} \) and \( z \in \Gamma \), we define

\[
C_{\Gamma, j}(g_1, g_2)(z) = p.v. \int_{\mathbb{R}^2} K_{\Gamma, j}(z, \xi_1, \xi_2)g_1(\xi_1)g_2(\xi_2)d\xi_1 d\xi_2
\]

\[
= \lim_{\varepsilon \to 0^+} \int_{|Re(z - \xi_1)|, |Re(z - \xi_2)| > \varepsilon} K_{\Gamma, j}(z, \xi_1, \xi_2)g_1(\xi_1)g_2(\xi_2)d\xi_1 d\xi_2.
\]

Initially we take this definition for \( g_j = f_j \circ \gamma^{-1} \) for \( f_j \in C_0^\infty(\mathbb{R}) \), \( j = 1, 2 \), but even for such \( g_j \) it is not yet apparent that this limit exists. We will establish that this limit exists, and furthermore that \( C_{\Gamma, j} \) can be continuously extended to a bilinear operator from \( L^p(\Gamma) \times L^p(\Gamma) \) into \( L^p(\Gamma) \). To prove these things, we will pass through “parameterized” versions
of $F$, $K$, and $C_{\Gamma,j}$ for $j = 0, 1, 2$ in the same way that David-Journé-Semmes did to apply their Tb theorem to the Cauchy integral operator in [11]: For $x, y_1, y_2 \in \mathbb{R}$, define

$$
\tilde{F}_\Gamma(x, y_1, y_2) = F_{\Gamma}(\gamma(x), \gamma(y_1), \gamma(y_2)), \quad \tilde{K}_{\Gamma,j}(x, y_1, y_2) = K_{\Gamma}(\gamma(x), \gamma(y_1), \gamma(y_2)),
$$

and for $f_1, f_2 \in C_0^\infty(\mathbb{R})$, define for $x \in \mathbb{R}$

$$
M_\gamma \tilde{C}_{\Gamma,j}(\gamma f_1, \gamma f_2)(x) = \text{p.v.} \int_{\mathbb{R}^2} \tilde{K}_j(x, y_1, y_2) f_1(y_1) f_2(y_2) \gamma'(y_1) \gamma'(y_2) dy_1 dy_2
$$

(7.2)

$$
= \lim_{\varepsilon \to 0^+} \int_{|x-y_1|, |x-y_2| > \varepsilon} \tilde{K}_j(x, y_1, y_2) f_1(y_1) f_2(y_2) \gamma'(y_1) \gamma'(y_2) dy_1 dy_2.
$$

for $j = 0, 1, 2$. We begin by proving that $\tilde{C}_{\Gamma,j}$ for $j = 0, 1, 2$ is well defined, and find an absolutely convergent integral representation for it that depends on derivatives of the input functions $f_1, f_2 \in C_0^\infty(\mathbb{R})$.

**Proposition 7.1.** Let $L$ be a Lipschitz function with Lipschitz constant $\lambda < 1$ such that for almost every $x \in \mathbb{R}$ the limits

$$
\lim_{\varepsilon \to 0^+} \gamma'(x+\varepsilon) = \gamma'(x+) \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \gamma'(x-\varepsilon) = \gamma'(x-)
$$

exist. Then $M_\gamma \tilde{C}_{\Gamma,j}(\gamma f_1, \gamma f_2)$ is an almost everywhere well defined function for $f_1, f_2 \in C_0^\infty(\mathbb{R})$ and $j = 0, 1, 2$. More precisely, for $f_1, f_2 \in C_0^\infty(\mathbb{R})$ the limit in (7.2) converges for almost every $x \in \mathbb{R}$ and

$$
M_\gamma \tilde{C}_{\Gamma,0}(\gamma f_1, \gamma f_2)(x) = -\int_{\mathbb{R}^2} \tilde{F}_\Gamma(x, y_1, y_2) f'_1(y_1) f'_2(y_2) \gamma'(y_1) \gamma'(y_2) dy_1 dy_2,
$$

$$
M_\gamma \tilde{C}_{\Gamma,1}(\gamma f_1, \gamma f_2)(x) = -\int_{\mathbb{R}^2} \tilde{F}_\Gamma(x, y_1, y_2) f_1(y_1) f'_2(y_2) \gamma'(y_1) \gamma'(y_2) dy_1 dy_2,
$$

$$
M_\gamma \tilde{C}_{\Gamma,2}(\gamma f_1, \gamma f_2)(x) = -\int_{\mathbb{R}^2} \tilde{F}_\Gamma(x, y_1, y_2) f_1(y_1) f'_2(y_2) \gamma'(y_1) \gamma'(y_2) dy_1 dy_2.
$$

Furthermore $\tilde{C}_{\Gamma,j}$ is continuous from $\gamma' C_0^\infty(\mathbb{R}) \times \gamma' C_0^\infty(\mathbb{R})$ into $(\gamma' C_0^\infty(\mathbb{R}))'$, and for $f_0, f_1, f_2 \in C_0^\infty(\mathbb{R})$,

$$
\left< \tilde{C}_{\Gamma,0}(\gamma f_1, \gamma f_2), \gamma f_0 \right> = -\int_{\mathbb{R}^3} \tilde{F}_\Gamma(x, y_1, y_2) f'_0(x) f_1(y_1) f_2(y_2) \gamma'(y_1) \gamma'(y_2) dx dy_1 dy_2
$$

$$
\left< \tilde{C}_{\Gamma,1}(\gamma f_1, \gamma f_2), \gamma f_0 \right> = -\int_{\mathbb{R}^3} \tilde{F}_\Gamma(x, y_1, y_2) f_0(x) f'_1(y_1) f_2(y_2) \gamma'(y_1) \gamma'(y_2) dx dy_1 dy_2
$$

$$
\left< \tilde{C}_{\Gamma,2}(\gamma f_1, \gamma f_2), \gamma f_0 \right> = -\int_{\mathbb{R}^3} \tilde{F}_\Gamma(x, y_1, y_2) f_0(x) f_1(y_1) f'_2(y_2) \gamma'(y_1) \gamma'(y_2) dx dy_1 dy_2.
$$

**Proof.** Fix $f_1, f_2 \in C_0^\infty$, and we start by showing that $M_\gamma \tilde{C}_{\Gamma,1}(\gamma f_1, \gamma f_2)$ can be realized as a bounded function. Define for $\varepsilon > 0$ and $x \in \mathbb{R}$

$$
C_\varepsilon(x) = \int_{|x-y_1|, |x-y_2| > \varepsilon} \tilde{K}_{\Gamma,1}(x, y_1, y_2) f_1(y_1) f_2(y_2) \gamma'(y_1) \gamma'(y_2) dy_1 dy_2.
$$
Note that \( \partial_y \tilde{F}_1(x, y_1, y_2) = \tilde{K}_{\Gamma,1}(x, y_1, y_2) \gamma'(y_1) \), and we integrate by parts to rewrite \( C_\epsilon \)

\[
C_\epsilon(x) = \int_{|x-y_1|,|x-y_2|>\epsilon} \partial_y \tilde{F}_1(x, y_1, y_2) f_1(y_1) f_2(y_2) \gamma'(y_2) dy_1 dy_2 
\]

\[
= -\int_{|x-y_1|,|x-y_2|>\epsilon} \tilde{F}_1(x, y_1, y_2) f_1'(y_1) f_2(y_2) \gamma'(y_2) dy_1 dy_2 
\]

\[
+ \int_{|x-y_2|>\epsilon} \tilde{F}_1(x, x-\epsilon, y_2) f_1(x-\epsilon) f_2(y_2) \gamma'(y_2) dy_2 
\]

\[
- \int_{|x-y_2|>\epsilon} \tilde{F}_1(x, x+\epsilon, y_2) f_1(x+\epsilon) f_2(y_2) \gamma'(y_2) dy_2 
\]

\[
= -\int_{|x-y_1|,|x-y_2|>\epsilon} \tilde{F}_1(x, y_1, y_2) f_1'(y_1) f_2(y_2) \gamma'(y_2) dy_1 dy_2 
\]

\[
+ \int_{|x-y_2|>\epsilon} \left( \tilde{F}_1(x, x-\epsilon, y_2) - \tilde{F}_1(x, x+\epsilon, y_2) \right) f_1(x-\epsilon) f_2(y_2) \gamma'(y_2) dy_2 
\]

\[
+ \int_{|x-y_2|>\epsilon} \tilde{F}_1(x, x+\epsilon, y_2) \left( f_1(x-\epsilon) - f_1(x+\epsilon) \right) f_2(y_2) \gamma'(y_2) dy_2 
\]

\[
= I_\epsilon(x) + H_\epsilon(x) + II_\epsilon(x). 
\]

We use that \( f_1 \in C^\infty_0(\mathbb{R}) \) to conclude that when integrating by parts, the boundary terms at \( y_1 = \pm\infty \) vanish, leaving the \( y_1 = x \pm \epsilon \) above. We now verify that the limits of \( I_\epsilon(x), \)

\( H_\epsilon(x), \) and \( II_\epsilon(x) \) each exist as \( \epsilon \to 0. \)

**I_\epsilon** converges: To compute this limit, we verify that the integrand of \( I_\epsilon \) is an integrable function. Note that \( ||L'||_{L^\infty} = \lambda < 1 \) implies

\[
|\tilde{F}_1(x, y_1, y_2)| \leq \frac{1}{|\text{Re}((x-y_1)^2 - (L(x) - L(y_1))^2 + (x-y_2)^2 - (L(x) - L(y_2))^2)|^{1/2}} \]

\[
\leq \frac{1}{(1-\lambda)^{1/2} |x-y_1| + |x-y_2|}. 
\]

Now let \( R_0 > 0 \) be large enough so that \( \text{supp}(f_j) \subset B(0, R_0) \) for \( j = 1, 2, \) and it follows that

\[
\int_{\mathbb{R}^2} |\tilde{F}_1(x, y_1, y_2) f_1'(y_1) f_2(y_2) \gamma'(y_2)| dy_1 dy_2 
\]

\[
\leq \frac{||\gamma'||_{L^\infty}^2}{(1-\lambda)^{1/2}} \int_{|y_1|,|y_2| \leq R_0} ||f_1'||_{L^\infty} ||f_2||_{L^\infty} dy_1 dy_2 \leq \frac{R_0}{(1-\lambda)^{1/2}} ||f_1'||_{L^\infty} ||f_2||_{L^\infty}. 
\]

Therefore \( I_\epsilon \) converges to an absolutely convergent integral as \( \epsilon \to 0. \)

**H_\epsilon** converges: First we make a change of variables in \( H_\epsilon \) to rewrite

\[
H_\epsilon = \int_{|y_2|>1} h_\epsilon(x, y_2) f_1(x-\epsilon) f_2(x-\epsilon y_2) \gamma'(x-\epsilon y_2) dy_2, 
\]

where \( h_\epsilon(x, y_2) = \gamma'(x) \left( \tilde{F}_1(x, x-\epsilon, x-\epsilon y_2) - \tilde{F}_1(x, x+\epsilon, x-\epsilon y_2) \right). \)

We wish to apply dominated convergence to \( H_\epsilon \) as it is written in (7.3). First we show that the integrand converges to zero almost everywhere (in particular for every \( y_2 \neq 0 \) and \( x \) such that \( \gamma'(x) \) exists): For \( y_2 > 0 \) it follows that \( f_2(x-\epsilon y_2) \gamma'(x-\epsilon y_2) \to f_2(x) \gamma'(x) \) as \( \epsilon \to 0^+ \). When \( y_2 < 0 \), it follows that \( f_2(x-\epsilon y_2) \gamma'(x-\epsilon y_2) \to f_2(x) \gamma'(x+) \) as \( \epsilon \to 0^+. \) So either way, the limit exists for \( y_2 \neq 0 \) and almost every \( x \). Now we show that \( h_\epsilon(x, y_2) \to 0 \).
for almost every \(x, y_2 \in \mathbb{R}\). For any \(x \in \mathbb{R}\) such that \(\gamma'(x)\) exists and \(y_2 \neq 0\), we compute

\[
\lim_{\epsilon \to 0} \epsilon \gamma'(x) F_1(x, x - \epsilon, x - \epsilon y_2) = \lim_{\epsilon \to 0} \gamma'(x) \left(\frac{(\gamma(x) - \gamma(x - \epsilon))^2}{\epsilon^2} + y_2^2 \frac{(\gamma(x) - \gamma(x - \epsilon y_2))^2}{(\epsilon y_2)^2}\right)^{1/2} = \left(\frac{\gamma'(x)}{\gamma'(x)^2 + y_2^2 \gamma'(x)^2}\right)^{1/2} = (1 + y_2^2)^{-1/2}.
\]

It follows that \(\epsilon \gamma'(x) F_1(x, x + \epsilon, x - \epsilon y_2) \to (1 + y_2^2)^{-1/2}\) as \(\epsilon \to 0\) for all \(x, y_2 \in \mathbb{R}\) such that \(\gamma'(x)\) exists and \(y_2 \neq 0\). Now in order to apply dominated convergence to (7.3), we need only to show that \(h_\epsilon(x, y_2)\) is integrable in \(y_2\) independent of \(\epsilon\). Define \(g_\epsilon = \gamma(x) - \gamma(x - t)\), which satisfies for all \(s, t \in \mathbb{R}\)

\[
|g_\epsilon| = |\gamma(x) - \gamma(x - |t|)| \leq ||\gamma||_{L^\infty} |t| \\
Re(g_\epsilon^2 + g_\epsilon) = Re\left[(\gamma(x) - \gamma(x - s))^2\right] + Re\left[(\gamma(x) - \gamma(x - t))^2\right] \geq (1 - \lambda)(s^2 + t^2).
\]

Also it is easy to verify that if \(\omega = re^{i\theta}, \zeta = re^{i\phi} \in \mathbb{C}\) both have positive real part, i.e. \(\theta, \phi \in (-\pi/2, \pi/2)\), then

\[
|\sqrt{\omega} + \sqrt{\zeta}| \geq \sqrt{r} \cos(\theta/2) + \sqrt{r} \cos(\phi/2) \\
\geq \sqrt{r} \cos(\theta) + \sqrt{r} \cos(\phi) = \sqrt{\Re(\omega)} + \sqrt{\Re(\zeta)}.
\]

Here we use that \(\sqrt{\cos(\theta)} \leq \cos(\theta/2)\) for \(\theta \in (-\pi/2, \pi/2)\). Using these properties, we bound \(h_\epsilon\)

\[
|h_\epsilon(x, y_2)| = \epsilon \left|\frac{1}{(g_\epsilon^2 + g_\epsilon y_2)^{1/2}} - \frac{1}{(g_\epsilon^2 + g_\epsilon y_2)^{1/2}}\right| \\
= \epsilon \left|\frac{g_\epsilon^2 + g_\epsilon y_2}{(g_\epsilon^2 + g_\epsilon y_2)^{1/2}}\right| \left(\frac{1}{(g_\epsilon^2 + g_\epsilon y_2)^{1/2}} - \frac{1}{(g_\epsilon^2 + g_\epsilon y_2)^{1/2}}\right) \\
\leq \epsilon \left\{\frac{|g_\epsilon^2 + g_\epsilon y_2|}{(g_\epsilon^2 + g_\epsilon y_2)^{1/2}}\right\} \left(\frac{1}{(g_\epsilon^2 + g_\epsilon y_2)^{1/2}} - \frac{1}{(g_\epsilon^2 + g_\epsilon y_2)^{1/2}}\right) \\
\leq \epsilon \left\{\frac{2(2\epsilon)^2}{(1 - \lambda)^3 (1 + |y_2|)^3}\right\} \\
\leq \epsilon \left\{\frac{1}{(1 - \lambda)^3 (1 + |y_2|)^3}\right\}.
\]

Therefore \(|h_\epsilon(x, y_2)| \leq (1 + |y_2|)^{-3}\), and we can apply dominated convergence to (7.3). Hence \(I_\epsilon \to 0\) as \(\epsilon \to 0\).

\(III_\epsilon \to 0\): For this term, we use the regularity and compact support of \(f_1\) to directly bound

\[
|III_\epsilon| \leq \int_{x < |y_2| < |x| + R_0 + \epsilon} |F_\epsilon(x, x + \epsilon, y_2)||f_1(x - \epsilon) - f_1(x + \epsilon)||f_2(y_2)|dy_2 \\
\leq \frac{1}{(1 - \lambda)^{1/2}} \int_{x < |y_2| < |x| + R_0 + \epsilon} \frac{1}{|x - y_2|} (2\epsilon||f_1'||L^-)||f_2||L^-dy_2 \\
\leq \frac{||f_1'||L^-||f_2||L^-}{(1 - \lambda)^{1/2}} \epsilon|\log(|x| + R_0 + \epsilon) - \log(\epsilon)|.
\]
Recall we chose $R_0 > 0$ such that $\text{supp}(f_j) \subset B(0,R_0)$ for $j = 1, 2$. Hence $III_e \to 0$ as $\varepsilon \to 0$, and so

$$\lim_{\varepsilon \to 0} C_\varepsilon(x) = C(x) = -\int_{\mathbb{R}^2} \tilde{F}_T(x,y_1,y_2)f_1'(y_1)f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1dy_2,$$

which is an absolutely convergent integral. This verifies the absolutely convergent integral representation for $\tilde{C}_{\Gamma,1}$ in Proposition 7.1. It also follows from our estimate of $I_\varepsilon$ that $C_\varepsilon(x)$ is bounded uniformly in $x$; hence for $f_0 \in \mathcal{C}_0^\infty(\mathbb{R})$ and $\varepsilon > 0$

$$|C_\varepsilon(x)f_0(x)\gamma'(x)| \preceq (1 - \lambda)^{-1/2}||f_1'||_{L^\infty}||f_2||_{L^2}R_0|f_0(x)|,$$

and by dominated convergence

$$-\int_{\mathbb{R}^3} \tilde{F}_T(x,y_1,y_2)f_0(x)f_1'(y_1)f_2(y_2)\gamma'(y_1)\gamma'(y_2)dxdy_1dy_2 = \lim_{\varepsilon \to 0} \int_\mathbb{R} C_\varepsilon(x)\gamma'(x)f_0(x)dx = \int_\mathbb{R} C(x)f_0(x)\gamma'(x)dx = \left\langle \tilde{C}_{\Gamma,1}\gamma f_1, \gamma f_2\right\rangle.$$

Furthermore the bounds of $I_\varepsilon$, $II_\varepsilon$, and $III_\varepsilon$ are in terms of $||f_j||_{L^\infty}$, $||f_j'||_{L^\infty}$, and $R_0$ for $j = 0, 1, 2$ it follows that $\tilde{C}_{\Gamma,1}$ is continuous from $\mathcal{C}_0^\infty(\mathbb{R}) \times \mathcal{C}_0^\infty(\mathbb{R})$ into $(\mathcal{C}_0^\infty(\mathbb{R}))'$. By symmetry, the properties of $\tilde{C}_{\Gamma,2}$ follow as well. Also

$$\left\langle \tilde{C}_{\Gamma,0}\gamma f_1, \gamma f_2, \gamma f_0 \right\rangle = \left\langle \lim_{\varepsilon \to 0} \int_{|x-y_1|,|x-y_2| > \varepsilon} \tilde{K}_{\Gamma,0}(x,y_1,y_2)f_0(x)f_1(y_1)f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1dy_2dx \right\rangle$$

$$= \left\langle \lim_{\varepsilon \to 0} \int_{|x-y_1|,|x-y_2| > \varepsilon} \partial_x \tilde{K}_{\Gamma}(x,y_1,y_2)f_0(x)f_1(y_1)f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1dy_2dx \right\rangle$$

$$= \left\langle \lim_{\varepsilon \to 0} \int_{|x-y_1|,|x-y_2| > \varepsilon} \partial_y \tilde{K}_{\Gamma}(x,y_1,y_2)f_0(x)f_1(y_1)f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1dy_2dx \right\rangle$$

$$+ \left\langle \lim_{\varepsilon \to 0} \int_{|x-y_1|,|x-y_2| > \varepsilon} \partial_{y_1} \tilde{K}_{\Gamma}(x,y_1,y_2)f_0(x)f_1(y_1)f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1dy_2dx \right\rangle$$

$$= \left\langle \tilde{C}_{\Gamma,1}\gamma f_1, \gamma f_2, \gamma f_0 \right\rangle + \left\langle \tilde{C}_{\Gamma,2}\gamma f_1, \gamma f_2, \gamma f_0 \right\rangle.$$

By the absolutely integrable representations of $\tilde{C}_{\Gamma,1}$ and $\tilde{C}_{\Gamma,2}$, it follows that

$$\left\langle \tilde{C}_{\Gamma,0}\gamma f_1, \gamma f_2, \gamma f_0 \right\rangle = \left\langle \tilde{C}_{\Gamma,1}\gamma f_1, \gamma f_2, \gamma f_0 \right\rangle + \left\langle \tilde{C}_{\Gamma,2}\gamma f_1, \gamma f_2, \gamma f_0 \right\rangle$$

$$= -\int_{\mathbb{R}^3} \tilde{F}_T(x,y_1,y_2)f_0(x)f_1'(y_1)f_2(y_2)\gamma'(x)\gamma'(y_2)dxdy_1dy_2$$

$$-\int_{\mathbb{R}^3} \tilde{F}_T(x,y_1,y_2)f_0(x)f_1(y_1)f_2'(y_2)\gamma'(x)\gamma'(y_1)dxdy_1dy_2.$$
The boundary terms of the integration by parts here (the last 4 terms) tend to zero as $\varepsilon \to 0$ in the same way as they did for $II_\varepsilon$ and $III_\varepsilon$ above. Now we will integrate by parts one more time in $x$ here to obtain an integral representation for $C_{\Gamma,0}$:

$$\lim_{\varepsilon \to 0} \int_{[x-y_1],[x-y_2]} \partial_x \tilde{F}_T(x,y_1,y_2) f_0(x) f_1(y_1) f_2(y_2) \gamma(x) \gamma(y_2) dx dy_1 dy_2$$

$$= \lim_{\varepsilon \to 0} \int_{[x-y_1],[x-y_2]} \partial_x \tilde{F}_T(x,y_1,y_2) f_0(x) f_1(y_1) f_2(y_2) \gamma(x) \gamma(y_2) dx dy_1 dy_2$$

$$+ \int_{[x-y_1],[x-y_2]} \partial_x \tilde{F}_T(x,y_1,y_2) f_0(x) f_1(y_1) f_2(y_2) \gamma(x) \gamma(y_1) dx dy_1 dy_2$$

$$+ \int_{[x-y_1],[x-y_2]} \tilde{F}_T(x,y_1,y_2) f_0(x) f_1(y_1) f_2(y_2) \gamma(x) \gamma(y_1) dx dy_1 dy_2$$

$$- \int_{[x-y_1],[x-y_2]} \tilde{F}_T(x,y_1,y_2) f_0(x) f_1(y_1) f_2(y_2) \gamma(x) \gamma(y_1) dx dy_1 dy_2$$

$$\lim_{\varepsilon \to 0} \int_{[x-y_1],[x-y_2]} \partial_x \tilde{F}_T(x,y_1,y_2) f_0(x) f_1(y_1) f_2(y_2) \gamma(x) \gamma(y_2) dx dy_1 dy_2$$

$$= \lim_{\varepsilon \to 0} \int_{[x-y_1],[x-y_2]} \partial_x \tilde{F}_T(x,y_1,y_2) f_0(x) f_1(y_1) f_2(y_2) \gamma(x) \gamma(y_2) dx dy_1 dy_2$$

$$+ \int_{[y_1,y_2]} \tilde{F}_T(x,y_1,y_2) f'_0(x) f_1(y_1) f_2(y_2) \gamma(x) \gamma(y_2) dx dy_1 dy_2$$

$$- \int_{[y_1,y_2]} \tilde{F}_T(x,y_1,y_2) f'_0(x) f_1(y_1) f_2(y_2) \gamma(x) \gamma(y_2) dx dy_1 dy_2$$

$$+ \int_{[y_1,y_2]} \tilde{F}_T(x,y_1,y_2) f'_0(x) f_1(y_1) f_2(y_2) \gamma(x) \gamma(y_2) dx dy_1 dy_2$$

$$- \int_{[y_1,y_2]} \tilde{F}_T(x,y_1,y_2) f'_0(x) f_1(y_1) f_2(y_2) \gamma(x) \gamma(y_2) dx dy_1 dy_2$$

Once again we use the same argument for the $II_\varepsilon$ and $III_\varepsilon$ terms to verify that these boundary terms (the last 4 terms) tend to zero as $\varepsilon \to 0$. Then the pairing identity for $C_{\Gamma,0}$ holds as well. This completes the proof of Proposition 7.1.
In the next proposition we extend $\widetilde{C}_\Gamma$ and $C_\Gamma$ to product Lebesgue spaces.

**Proposition 7.2.** Let $L$ be a Lipschitz function with Lipschitz constant $\lambda < 1$ such that for almost every $x \in \mathbb{R}$ the limits

$$\lim_{\varepsilon \to 0^+} \gamma'(x + \varepsilon) = \gamma'(x+) \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \gamma'(x - \varepsilon) = \gamma'(x-)$$

exist. If $L$ is differentiable off of some compact set and there exists $c_0 \in \mathbb{R}$ such that

$$\lim_{|x| \to \infty} L'(x) = c_0,$$

then $\widetilde{C}_{\Gamma,j}$ is bounded $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^p(\mathbb{R})$ for all $1 < p_1, p_2 < \infty$ satisfying (2.2) for each $j = 0, 1, 2$. Furthermore, $C_{\Gamma,j}$ is bounded $L^{p_1}(\Gamma) \times L^{p_2}(\Gamma)$ into $L^p(\Gamma)$ for all $1 < p_1, p_2 < \infty$ satisfying (2.2) for each $j = 0, 1, 2$.

**Proof.** We will apply Theorem 1.1 to $\tilde{C}_{\Gamma,1}$ with $b_0 = b_1 = b_2 = \gamma$. Note that $\gamma'$ is para-accretive since $Re(\gamma') = 1$ and $\gamma \in L^\infty$. It is not hard to see that $\tilde{K}_{\Gamma,1}$ is the kernel function associated to $\widetilde{C}_{\Gamma,1}$. It also follows from $||L'||_{L^\infty} = \lambda < 1$ that $\tilde{K}_{\Gamma,1}$ is a standard bilinear kernel:

$$|\tilde{K}_{\Gamma,1}(x, y_1, y_2)| \leq \frac{1}{(1 - \lambda)^{3/2}} \frac{||\gamma(x - \gamma(y_1))||}{(x - y_1)^2 + (x - y_2)^2} \leq \frac{1}{(1 - \lambda)^{3/2}} \frac{||\gamma'||_{L^\infty}}{(x - y_1)^2 + (x - y_2)^2},$$

and

$$\partial_{y_2} \tilde{K}_{\Gamma,1}(x, y_1, y_2) \leq \frac{3}{(1 - \lambda)^{5/2}} \frac{|\gamma(x - \gamma(y_1))||\gamma(x - \gamma(y_2))||\gamma'(y_2)|}{((x - y_1)^2 + (x - y_2)^2)^{5/2}} \leq \frac{3}{(1 - \lambda)^{3/2}} \frac{||\gamma'||_{L^\infty}^3}{((x - y_1)^2 + (x - y_2)^2)^{3/2}}.$$

A similar estimate holds for $\partial_{y_1} \tilde{K}_{\Gamma,1}(x, y_1, y_2)$ and $\partial_{y_1} \tilde{K}_{\Gamma,1}(x, y_1, y_2)$, which implies that $\tilde{K}_{\Gamma,1}(x, y_1, y_2)$ is a standard bilinear kernel. Now it remains to verify that $C_{\Gamma,1}$ satisfies the WBP and the BMO testing conditions for $b_0 = b_1 = b_2 = \gamma$. Let $\phi_0, \phi_1, \phi_2$ be normalized bumps of order $1$, $u \in \mathbb{R}$, and $R > 0$. By Proposition 7.1, we have

$$\left| \left\langle \tilde{C}_{\Gamma,1}(\gamma^R \phi^R_1, \gamma^R \phi^R_2), \gamma^R \phi^R_0 \right\rangle \right| \leq \int_{\mathbb{R}^3} \left| \tilde{F}_{\Gamma}(x, y_1, y_2) \phi^R_0(x)(\phi^R_1(y_1))\phi^R_2(y_2) \gamma'(y_2) \phi(x) \right| dxdy_1dy_2$$

$$\leq \frac{1}{R(1 - \lambda)^{1/2}} \int_{|x-y_1| + |x-y_2| < 1} \frac{dx dy_1 dy_2}{(1 - \lambda)^{1/2}} \leq \frac{R}{(1 - \lambda)^{1/2}}$$

So $C_{\Gamma,1}$ satisfies the WBP. Now we check the three BMO conditions of Theorem 1.1:

$M_\gamma \tilde{C}_{\Gamma,1}$$ \gamma, \gamma' = 0$ in BMO: Let $\phi \in C_0^\infty(\mathbb{R})$ such that $\gamma' \phi$ has mean zero and $\eta \in C_0^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $[-1, 1]$, $\text{supp}(\eta) \subset [-2, 2]$, and $\eta R(x) = \eta(x/R)$. Again we use Proposition 7.1 and make a change of variables,

$$\left\langle \tilde{C}_{\Gamma,1}(\gamma \eta R, \gamma \eta R), \gamma \phi \right\rangle = -\int_{\mathbb{R}^3} \tilde{F}_{\Gamma}(x, y_1, y_2)(\eta R)'(y_1) \gamma(y_2) \eta R(y_2) \gamma'(y_2) \phi(x) dxdy_1dy_2$$

$$= -\int_{\mathbb{R}^3} R\tilde{F}_{\Gamma}(x, Ry_1, Ry_2) \phi(x) \eta'(y_1) \eta(y_2) \gamma'(y_2) dxdy_1dy_2.$$
Then for $y_1, y_2 \neq 0$, we can compute the pointwise limit

$$\lim_{R \to \infty} R \bar{F}_T(x, R y_1, R y_2) \gamma'(R y_2) = \frac{R \gamma'(R y_2)}{\left((\gamma(x) - \gamma(R y_1))^2 + (\gamma(x) - \gamma(R y_2))^2\right)^{1/2}} \gamma'(R y_2)$$

$$= \frac{1}{\left(\frac{1}{1+ic_0} + \frac{y_2^2(\gamma(x) - \gamma(R y_2))^2}{(R y_2)^2}\right)^{1/2}} = \frac{1}{(\gamma^2 + y_2^2)^{1/2}}.$$

Here we use that $L$ is differentiable off of a compact set, that $L'(x) \to c_0$ as $|x| \to \infty$, and L'Hospital's rule to conclude that $L(x)/x \to c_0$ as $|x| \to \infty$. Now let $R > 0$ be large enough so that $\text{supp}(\phi) \subset B(0, R/4)$, and using that $\text{supp}(\eta') \subset [-2, 2] \setminus [-1, 1]$, we have the estimate

$$|R \bar{F}_T(x, R y_1, R y_2)\gamma'(x)\gamma'(R y_2)\phi(x)\eta'(y_1)\eta(y_2)| \lesssim \frac{1}{(1-\lambda)^{1/2}} \frac{R \phi(x)\eta'(y_1)}{|x-R y_1|+|x-R y_2|} \lesssim \frac{1}{(1-\lambda)^{1/2}} \frac{R \phi(x)\eta'(y_1)}{|y_1|+|y_2|}.$$

Then by dominated convergence

$$\lim_{R \to \infty} \left\langle \bar{C}_{\Gamma,1}(\gamma \eta_R, \gamma \eta_R), \gamma \phi \right\rangle = \int_\mathbb{R} \left( -\int_{\mathbb{R}^2} \eta'(y_1)\eta(y_2) \frac{1}{(y_1^2+y_2^2)^{1/2}} dy_1 dy_2 \right) \phi(x)\gamma'(x)dx = 0.$$

Here we also use that $\gamma \phi$ has mean zero. Therefore $M_\gamma \bar{C}_{\Gamma,1}(\gamma', \gamma') = 0 \in BMO$ in the sense of Definition 2.5.

$M_\gamma \bar{C}_{\Gamma,1}^\dagger(\gamma', \gamma') = 0 \in BMO$ : Note that for every $x, y_1, y_2 \in \mathbb{R}$ such that $|x-y_1|+|x-y_2| \neq 0$ we can write

$$K_{\Gamma,1}(x, y_1, y_2)\gamma'(x)\gamma'(y_1)\gamma'(y_2) = \partial_{y_1} F_{\Gamma,1}(x, y_1, y_2)\gamma'(x)\gamma'(y_2)$$

$$= -\partial_{y_1} F_{\Gamma,1}(x, y_1, y_2)\gamma'(y_1)\gamma'(y_2) - \partial_{y_2} F_{\Gamma,1}(x, y_1, y_2)\gamma'(x)\gamma'(y_1)$$

$$= \left(\bar{K}_{\Gamma,0}(x, y_1, y_2) - \bar{K}_{\Gamma,2}(x, y_1, y_2)\right)\gamma'(x)\gamma'(y_1)\gamma'(y_2).$$

Then it follows that $M_\gamma \bar{C}_{\Gamma,1}(M_\gamma \cdot, M_\gamma \cdot) = M_\gamma \bar{C}_{\Gamma,0}(M_\gamma \cdot, M_\gamma \cdot) - M_\gamma \bar{C}_{\Gamma,2}(M_\gamma \cdot, M_\gamma \cdot)$, and so by Proposition 7.1

$$\left\langle \bar{C}_{\Gamma,1}^\dagger(\gamma \eta_R, \gamma \eta_R), \gamma \phi \right\rangle = \left\langle \bar{C}_{\Gamma,1}(\gamma \phi, \gamma \eta_R), \gamma \eta_R \right\rangle$$

$$= \left\langle \bar{C}_{\Gamma,0}(\gamma \phi, \gamma \eta_R), \gamma \eta_R \right\rangle - \left\langle \bar{C}_{\Gamma,2}(\gamma \phi, \gamma \eta_R), \gamma \eta_R \right\rangle$$

$$= -\int_{\mathbb{R}^3} \bar{F}_T(x, y_1, y_2)\phi(y_1)\gamma'(y_1)(\eta(y_2))(\eta_R)(y_2) dy_1 dy_2 dx$$

$$+ \int_{\mathbb{R}^3} \bar{F}_T(x, y_1, y_2)\phi(y_1)(\eta_R)(y_2)\gamma'(y_1)(\eta_R)(y_2) dy_1 dy_2 dx.$$

These two expressions tend to zero by the same argument that (7.4) tends to zero as $R \to \infty$ in the proof of the $M_\gamma \bar{C}_{\Gamma,1}(\gamma', \gamma') = 0$ condition. Therefore $M_\gamma \bar{C}_{\Gamma,1}^\dagger(\gamma', \gamma') = 0 \in BMO$ as well.
By Proposition 7.1, we can compute
\[
M_f \tilde{C}_{\Gamma}^{\mathbb{I}^2}(\gamma', \gamma') = 0 \in BMO.
\]
By the argument. Therefore \(M_f \tilde{C}_{\Gamma}^{\mathbb{I}^2}(\gamma', \gamma') = 0 \in BMO\).

Then by Theorem 1.1, \(\tilde{C}_{\Gamma, 1}\) can be extended to a bounded operator from \(L^{p_1} \times L^{p_2}\) into \(L^p\) for appropriate \(p, p_1, p_2\). Now it is easy to prove that \(C_{\Gamma, 1}\) can also be extended to a bounded operator: Let \(1 < p_1, p_2 < \infty\) and \(1/2 < p < \infty\) satisfy (2.2). For \(g_1 \in L^{p_1}(\Gamma)\) and \(g_2 \in L^{p_2}(\Gamma)\), and it follows that
\[
\|C_{\Gamma, 1}(g_1, g_2)\|_{L^p(\Gamma)} = \left( \int_{\mathbb{R}} |C_{\Gamma, 1}(g_1, g_2)(\gamma(x))|^p |\gamma'(x)| dx \right)^{1/p}
\leq ||\gamma'\|_{L^p} ||\tilde{C}_{\Gamma, 1}\|_{L^1(\mathbb{R}) \rightarrow L^p(\mathbb{R})} ||g_1 \circ \gamma^{-1}\|_{L^{p_1}(\mathbb{R})} ||g_2 \circ \gamma^{-1}\|_{L^{p_2}(\mathbb{R})}
\leq ||\gamma'\|_{L^p} ||\gamma^{-1}\|_{L^p} ||\tilde{C}_{\Gamma, 1}\|_{L^1(\mathbb{R}) \rightarrow L^p(\mathbb{R})} ||g_1\|_{L^{p_1}(\Gamma)} ||g_2\|_{L^{p_2}(\Gamma)}.
\]
The bounds for \(\tilde{C}_{\Gamma, 0}, \tilde{C}_{\Gamma, 2}, C_{\Gamma, 0},\) and \(C_{\Gamma, 2}\) follow in the same way. 

REFERENCES