[I] (Take-home; 20pts) You open a bank account, deposit a dollar in that account.

(1) Your bank offers 100 percent interest annually.

After one year, your balance is \$2.

(2) Suppose your bank offers a compound interest with 100 percent rate annually.

(2a) After two years, your balance is \$4.

(2b) After three years, your balance is \$8.

(3) Suppose the compounding takes place five times annually. So every \(\frac{1}{5}\)-th of a year \(\frac{1}{5} \cdot 100 = 20\) percent of your balance will be accrued as an interest.

(3a) After the \(\frac{1}{5}\)-th of a year, your balance is \$1.20.

(3b) After the \(\frac{2}{5}\)-th of a year, your balance is \$1.44.

(3c) After one year, your balance is \$ \left(1 + \frac{1}{5}\right)^5.$
Suppose the compounding takes place $10^{100}$ times annually. So every $\frac{1}{10^{100}}$-th of a year, $\frac{1}{10^{100}}$ times 100 percent of your balance will be accrued as an interest.

After one year, your balance is $\left( 1 + \frac{1}{10^{100}} \right)^{10^{100}}$.

Which one does your answer in (4) fall into?

- □ between $1$ and $2$.
- □ between $2$ and $2.5$.
- □ between $2.5$ and $3.0$.
- □ more than $3$.

**Answer**: Between $2.5$ and $3.0$.

Indeed, the answer in (4) is $2.718281828459045...$.

[II] (Take-home; 20pts) (a) Use calculator to pull the decimal expressions of the numbers in each of (a5) through (a10).

(a1) $1 + \frac{1}{1!} = 2.0000000000$, 

(a2) $1 + \frac{1}{1!} + \frac{1}{2!} = 2.5000000000$, 

(a3) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = 2.6666666666...$, 

(a4) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.7083333333...$, 

(a5) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$

$= 2.7166666666...$,
(a6) \[ 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} \]
\[= 2.7 \ 1\ 8\ 0\ 5\ 5 \ldots , \]

(a7) \[ 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} \]
\[= 2.7 \ 1\ 8\ 2\ 5\ 3 \ldots , \]

(a8) \[ 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} \]
\[+ \frac{1}{9!} = 2.7 \ 1\ 8\ 2\ 7\ 8 \ldots , \]

(a9) \[ 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} \]
\[+ \frac{1}{9!} + \frac{1}{10!} = 2.7 \ 1\ 8\ 2\ 8\ 1 \ldots . \]

(b) Use calculator to find the smallest positive integer \( n \) such that \((1 + \frac{1}{n})^n\) is bigger than the value in (a4) above \((= 2.7083333\ldots)\).

\[ \text{[Answer]}: \quad n = 136. \quad \text{Indeed,} \]

3
\[
\left(1 + \frac{1}{135}\right)^{135} = 2.7082819990... , \quad \text{whereas}
\]
\[
\left(1 + \frac{1}{136}\right)^{136} = 2.708350352... .
\]

(c) **True or false**:

“Let \( k \) be an arbitrarily chosen positive integer, and fixed. If you choose a large enough \( n \), then

\[
\left(1 + \frac{1}{n}\right)^n > 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots + \frac{1}{k!}.
\]

**Answer**: True.

(d1) \[
e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.
\]

(d2) \[
e = \lim_{n \to \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}\right).
\]

(d3) The decimal expression of \( e \) up to the first six place under the decimal point

\[
2.718281 ...
\]

[III] **Take-home; 20pts** Prove that \( \sqrt{3} \) is an irrational number.

**Proof.** Proof by contradiction. Suppose \( \sqrt{3} \) is written as

\[
\sqrt{3} = \frac{k}{m}
\]

using some integers \( k \) and \( m \) (where \( m \neq 0 \)).
First, if both $k$ and $m$ are divisible by $3$, then we may simultaneously divide both the numerator and the denominator by $3$ (and the value of the fraction stays the same). After that procedure, suppose both the numerator and the denominator still remain to be divisible by $3$, then we repeat the same procedure as many times as necessary until at least one of the numerator and the denominator is not divisible by $3$. Thus we may assume, without loss of generality, that at least one of $k$ and $m$ is not divisible by $3$.

Under this assumption, square the both sides of the identity $\sqrt{3} = \frac{k}{m}$, thus

$$3 = \frac{k^2}{m^2}.$$ 

This is the same as $3m^2 = k^2$.

The left-hand side of this last identity is clearly divisible by $3$, so this last identity forces its right-hand side to be divisible by $3$.

That in turn implies $k$ is divisible by $3$, because if $k$ is not divisible by $3$, then $k^2$ is not divisible by $3$.

But then $k$ being divisible by $3$ implies $k^2$ is divisible by $9$.

So by virtue of the above last identity $3m^2$ is divisible by $9$, or the same to say, $m^2$ is divisible by $3$. This implies that $m$ is divisible by $3$.

In short, both $k$ and $m$ are divisible by $3$.

This contradicts our assumption. The proof is complete. □