Today I want to talk about polynomials. Polynomials are made of monomials. So I cover both. Spellings first:

“polynomial”, “monomial”.

First, we call each of

1, \( x, \ x^2, \ x^3, \ x^4, \ x^5, \ldots \),

a monomial in \( x \). Here, we include 1 in the list because \( 1 = x^0 \). Next, we also call each of

3, \(-2x, \quad 4x^2, \quad \sqrt{2} x^3, \quad \frac{1}{4} x^4, \quad 7x^5, \quad \ldots \),

a monomial in \( x \).

More generally, suppose \( a \) is a constant real number, and \( n \) is an integer with \( n \geq 0 \). Then we call

\[ a x^n \]

a monomial in \( x \).

Note that neither of

\[
\begin{align*}
x^{-1} \\
\frac{1}{x^2}
\end{align*}
\]

is a monomial, because the exponent is either negative or a non-integer.
Next, a **polynomial** is a finite sum of monomials. Thus

\[ 1 + x, \quad 2x + x^2, \quad -\frac{4}{5} + \sqrt{3}x^3, \quad x + x^4 + x^7, \quad -6 + 2x - 8x^3 + x^5, \]

are examples of polynomials in \( x \). On the other hand, none of

\[ \sqrt{x}, \quad 1 + x^\frac{3}{2}, \quad \frac{1}{x}, \quad \sqrt{2 + x^2}, \quad \frac{3}{4 - x}, \quad 2x^{-2} + x^2 \]

is a polynomial in \( x \).

- **Note.** There is no general rule that we must obey when it comes to the order of terms in polynomial expressions. So, you may write either

\[ 2 + x, \quad \text{or} \quad x + 2. \]

These two are one and the same. Similarly, you may write either

\[ 4 - 3x + x^2, \quad -3x + x^2 + 4, \quad x^2 + 4 - 3x, \]
\[ 4 + x^2 - 3x, \quad x^2 - 3x + 4, \quad \text{or} \quad -3x + 4 + x^2. \]

These six are all one and the same. Usually, though, we prefer to write polynomials either in the **ascending order** or in the **descending order** of exponents. So,

\[ 4 - 3x + x^2 \quad \text{ (ascending order) }, \]

and

\[ x^2 - 3x + 4 \quad \text{ (descending order) }, \]

are equally preferable.
Exercise 1. Permute the order of terms, if necessary, to make each of the given polynomials in the ascending order.

1. \(2x + x^4 - \frac{1}{2}x^2\).
2. \(-\frac{4}{3}x^3 - 5x^2 - 4x^5\).
3. \(x^8 + 5x^6 - 10x^9 + 3\).
4. \(x + \sqrt{5}x^5 + \sqrt{3}x^3 - \sqrt{2}x^2\).
5. \(x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\).

**Answers:**

1. \(2x - \frac{1}{2}x^2 + x^4\).
2. \(-5x^2 - \frac{4}{3}x^3 - 4x^5\).
3. \(3 + 5x^6 + x^8 - 10x^9\).
4. \(x - \sqrt{2}x^2 + \sqrt{3}x^3 + \sqrt{5}x^5\).
5. \(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9\).

Exercise 2. Permute the order of terms, if necessary, to make each of the given polynomials in the descending order.

1. \(6x^2 + x^3\).
2. \(\frac{1}{2}x^3 + \frac{1}{3}x^4 - x\).
3. \(x^7 + 5x^5 - 10x^8 + 4x^6\).
4. \(1 - x^2 + x^4 - x^6 + x^8 - x^{10}\).

**Answers:**

1. \(x^3 + 6x^2\).
2. \(\frac{1}{3}x^4 + \frac{1}{2}x^3 - x\).
3. \(-10x^8 + x^7 + 4x^6 + 5x^5\).
4. \(-x^{10} + x^8 - x^6 + x^4 - x^2 + 1\).
• Polynomial addition.

We may add two polynomials and the outcome is a polynomial. Let me use some examples to illustrate it.

Example 1. Let’s calculate

\[(x + 2) + (x^3 + 4x)\].

This is just uncover the parenthesis and reorder the terms (in either the ascending or descending order). So,

\[(x^2 + 2) + (x^3 + 4x) = x^2 + 2 + x^3 + 4x \]

\[= x^3 + x^2 + 4x + 2.\]

(This answer is clearly written in the descending order. You can write it in the ascending order instead. You don’t have to give both.)

Example 2. Let’s calculate

\[(x^4 + \frac{1}{5}x^2) + (2x^4 - 6x^3 + x)\].

This one looks pretty similar to Example 1 above, but this one involves more than what was required in Example 1. Indeed, let’s just first uncover the parenthesis, and reorder terms:

\[(x^4 + \frac{1}{5}x^2) + (2x^4 - 6x^3 + x)\]

\[= x^4 + \frac{1}{5}x^2 + 2x^4 - 6x^3 + x \]

\[= x^4 + 2x^4 - 6x^3 + \frac{1}{5}x^2 + x.\]
Realize that there are two monomials that involve $x^4$. You have to combine them. $x^4$ and $2x^4$ make $3x^4$. So the above is simplified as

$$3x^4 - 6x^3 + \frac{1}{5}x^2 + x.$$ 

This is the final answer.

* If you like, you can do the above as follows:

$$\begin{align*}
  x^4 & + \frac{1}{5}x^2 \\
  2x^4 & - 6x^3 & + x \\
\hline
  3x^4 & - 6x^3 & + \frac{1}{5}x^2 & + x.
\end{align*}$$

* Let’s do a similar example, but in a different format.

**Example 3.** Let’s find $f(x) + g(x)$, where

$$f(x) = x^4 + \frac{7}{2}x^3 - \frac{5}{2}x^2 - x, \quad \text{and} \quad g(x) = 6x^4 + \frac{1}{2}x^3 - x^2 - 3x + 2.$$ 

This is essentially the same type of a problem as the previous ones, but the difference is that two polynomials are given names as $f(x)$ and $g(x)$. Still, the same method works. Here is how it goes:
\[ f(x) + g(x) \]

\[ \begin{align*}
  &= \left( x^4 + \frac{7}{2} x^3 - \frac{5}{2} x^2 - x \right) + \left( 6x^4 + \frac{1}{2} x^3 - x^2 - 3x + 2 \right) \\
  &= x^4 + \frac{7}{2} x^3 - \frac{5}{2} x^2 - x + 6x^4 + \frac{1}{2} x^3 - x^2 - 3x + 2 \\
  &= x^4 + 6x^4 + \frac{7}{2} x^3 + \frac{1}{2} x^3 - \frac{5}{2} x^2 - x^2 - x - 3x + 2 \\
  &= x^4 + 6x^4 + \left( \frac{7}{2} x^3 + \frac{1}{2} x^3 \right) + \left( - \frac{5}{2} x^2 - x^2 \right) + \left( - x - 3x \right) + 2 \\
  &= 7x^4 + 4x^3 + \left( - \frac{7}{2} x^2 \right) + \left( -4x \right) + 2 \\
  &= 7x^4 + 4x^3 - \frac{7}{2} x^2 - 4x + 2.
\end{align*} \]

\[ \star \] If you like, you can do the above as follows:

\[ \begin{align*}
  x^4 &+ \frac{7}{2} x^3 - \frac{5}{2} x^2 - x \\
 + \left( 6x^4 + \frac{1}{2} x^3 - x^2 - 3x + 2 \right) \\
 &= 7x^4 + 4x^3 - \frac{7}{2} x^2 - 4x + 2
\end{align*} \]
Exercise 3.  Do

(1) \((x^7 + 3x^5 + 2x^3) + (-x^6 - x^4 - 2x^2)\).

(2) \((x^4 + 9x^3 + 1) + (-x^4 - x^3 - 5x^2 + 2x + 3)\).

(3) \((\frac{1}{2} x^3 + \frac{1}{3} x) + (\frac{1}{3} x^3 - \frac{1}{4} x)\).

(4) \(f(x) + g(x)\), where

\[ f(x) = x^6 + 8x^5 + 12x^4 + 36x^3 + 9x^2, \]
\[ g(x) = x^8 - 3x^6 - 8x^4 - 24x^3 + 45x - 120. \]

(5) \(f(x) + g(x)\), where

\[ f(x) = x^7 + x^5 + x^3 + x, \]
\[ g(x) = x^8 + x^6 + x^4 + x^2 + 1. \]

[Answers] :

(1) \(x^7 - x^6 + 3x^5 - x^4 + 2x^3 - 2x^2.\)  (2) \(8x^3 - 5x^2 + 2x + 4.\)

(3) \(\frac{5}{6} x^3 + \frac{1}{12} x.\)

(4) \(x^8 - 2x^6 + 8x^5 + 4x^4 + 12x^3 + 9x^2 + 45x - 120.\)

(5) \(x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.\)
- **Polynomial subtraction.**

  We may subtract one polynomial from another polynomial. The outcome is a polynomial. This is very similar to polynomial additions. Let’s do some example.

**Example 4.** Let’s do

\[
(x^3 + 2x^2) - (3x + 4).
\]

When you uncover the parenteses, you have to be careful. Namely:

\[
(x^3 + 2x^2) - (3x + 4) = x^3 + 2x^2 - 3x - 4.
\]

In the above, notice that all the terms inside the second parenthesis got negated after uncovering that parenthesis. That’s the correct way to do it.

**Example 5.** Let’s do

\[
(x^6 - 8x^4 + 3x^2) - (2x^6 - 3x^4).
\]

Once again, when you uncover the parenteses,

\[
(x^6 - 8x^4 + 3x^2) - (2x^6 - 3x^4)
\]

\[
= x^6 - 8x^4 + 3x^2 - 2x^6 + 3x^4
\]

(all the terms in the second parenthesis got negated)

\[
= x^6 - 2x^6 - 8x^4 + 3x^4 + 3x^2
\]

(re-ordered terms)

\[
= (x^6 - 2x^6) + (-8x^4 + 3x^4) + 3x^2
\]

\[
= -x^6 - 5x^4 + 3x^2.
\]
If you like, you can do it like

\[
x^6 - 8x^4 + 3x^2
\]

\[
- 2x^6 - 3x^4
\]

\[
- x^6 - 5x^4 + 3x^2
\]

Exercise 4. Do

(1) \((x^3 + 11x^2 + 21x) - (-x^2 - x + 4)\).

(2) \((-x^7 + 5x^6 + x^3 - 6) - (-2x^6 - 3x^4 + 7x^3 + 2x + 5)\).

(3) \(\left(\frac{3}{2}x^4 + \frac{7}{4}x^2\right) - \left(\frac{1}{6}x^4 - \frac{1}{4}x^2 + 1\right)\).

(4) \(f(x) - g(x)\), where

\[
f(x) = x^5 + 4x^4 + 16x^3 + 22x^2 + 18x,
\]

\[
g(x) = x^6 - 31x^4 - 62x^2 + 72x + 56.
\]

(5) \(f(x) - g(x)\), where

\[
f(x) = x^{13} + x^9 + x^5 + x,
\]

\[
g(x) = x^{11} + x^7 + x^3.
\]

[**Answers**]:

(1) \(x^3 + 12x^2 + 22x - 4\). (2) \(-x^7 + 7x^6 + 3x^4 - 6x^3 - 2x - 11\).

(3) \(\frac{4}{3}x^4 + 2x^2 - 1\). (4) \(-x^6 + x^5 + 35x^4 + 16x^3 + 84x^2 - 54x - 56\).

(5) \(x^{13} - x^{11} + x^9 - x^7 + x^5 - x^3 + x\).
• **Constant multiplication.**

We may multiply a constant with a polynomial.

**Example 6.** Let’s do

\[ 10 \left( 2x^3 + 3x^2 + 4x + 5 \right). \]

This is simply multiply 10 to each of the terms. So

\[ 10 \left( 2x^3 + 3x^2 + 4x + 5 \right) = 20x^3 + 30x^2 + 40x + 50. \]

**Example 7.** Let’s do

\[ -2 \left( 12x^5 - 21x^3 + 48x \right). \]

This is simply multiply -2 to each of the terms. So

\[ -2 \left( 12x^5 - 21x^3 + 48x \right) = -24x^5 + 42x^3 - 96x. \]

**Example 8.** Sometimes you see something like

\[ \frac{2x^4 - 11x^2 + 14x - 9}{2}. \]

This is the same as

\[ \frac{1}{2} \left( 2x^4 - 11x^2 + 14x - 9 \right). \]

The answer is, of course,

\[ x^4 - \frac{11}{2}x^2 + 7x - \frac{9}{2}. \]
Exercise 5. Simplify:

1. \(6 \left( x^7 + 7 x^6 + 21 x^5 \right)\).

2. \(-4 \left( - x^2 + 5x + 3 \right)\).

3. \(\frac{8x^{10} - 20x^8 + 24x^6 - 12x^4}{4}\).

4. \(\frac{1}{3} \left( \frac{3}{5} x^4 + \frac{3}{7} x^3 + \frac{3}{25} x^2 + \frac{3}{65} x \right)\).

5. \(3f(x)\) where

\[f(x) = x^5 + 4x^4 + 16x^3 + 22x^2 + 18x.\]

[Answers]:

1. \(6x^7 + 42x^6 + 126x^5\).

2. \(4x^2 - 20x - 12\).

3. \(2x^{10} - 5x^8 + 6x^6 - 3x^4\).

4. \(\frac{1}{5} x^4 + \frac{1}{7} x^3 + \frac{1}{25} x^2 + \frac{1}{65} x\).

5. \(3x^5 + 12x^4 + 48x^3 + 66x^2 + 54x\).