Today’s theme is application of antiderivatives/indefinite integrals — not just random applications, but one specific application, in the context of the Bernoulli polynomials/numbers.

Two lectures ago (in “Review of Lectures – XXVII”) we have practiced how to reconstruct the Bernoulli polynomials from the Bernoulli numbers. The pitfall of that method is it required us to refer to the table of Bernoulli numbers. So we needed to supply the method (algorithm) to generate the Bernoulli numbers separately. That part was covered in “Review of Lectures – XXVI”, called ‘Akiyama–Tanigawa diagram’. (See also “Extra Credit Homework – I”.) Today, we learn a completely different method to reconstruct the Bernoulli polynomials/numbers, without relying on the ‘Akiyama–Tanigawa diagram’. This is actually a perfect application of the concept of antiderivatives/indefinite integrals. So, our goal is to recover

\[
B_1^\circ(x) = x - 1, \quad B_1(x) = x - \frac{1}{2},
\]

\[
B_2^\circ(x) = x^2 - x, \quad B_2(x) = x^2 - x + \frac{1}{6},
\]

\[
B_3^\circ(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,
\]

\[
B_4^\circ(x) = x^4 - 2x^3 + x^2, \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},
\]

\[
B_5^\circ(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \quad B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x,
\]

\[
B_6^\circ(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2, \quad B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42},
\]

\[
B_7^\circ(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x, \quad B_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x,
\]

one by one, from the complete scratch.
1. Starting point of construction. \( B_1^\circ (x) \).

We do not rely on any prior knowledge about the shape of Bernoulli polynomials/numbers, except

Initial conditions:

(i) \( B_k^\circ (0) = 0 \) for all \( k \) with \( k \geq 2 \),

(ii) \( B_k^\circ (1) = 0 \) for all \( k \) with \( k \geq 1 \),

and the following:

(1d) \( B_1^\circ (x) = x - 1 \).

(Note that there is no (1a), (1b) and (1c).)

We are going to follow the step-by-step procedure below. Those steps exhibit the same pattern, each step uses the outcome of the previous step. So after three or so steps, you get the firm idea how to proceed. As you can see, in (2d), (3d), and (4d), you see that \( B_2^\circ (x) \), \( B_3^\circ (x) \), \( B_4^\circ (x) \) are recovered. At the same time, In (2c), (3c), and (4c), you see that n \( B_1 \), \( B_2 \), and \( B_3 \) are recovered.

2. Determining \( B_1 \) and \( B_2^\circ (x) \).

Add \( B_1 \) to (1d), and thereby make it \( B_1 (x) \). At this point \( B_1 \) is an unknown constant.

\[ B_1 (x) = x - 1 + B_1. \]

Take its antiderivative:

\[ \int B_1 (x) \, dx = \frac{1}{2} x^2 - x + B_1 \cdot x + C. \]

This is \( \frac{1}{2} B_2^\circ (x) \), so
\[(2a) \quad \frac{1}{2} B_2^\circ (x) = \frac{1}{2} x^2 - x + B_1 \cdot x + C. \]

Substitute \( x = 0 \), and \( x = 1 \) into \((2a)\) independently. Use \( B_2^\circ (0) = 0 \) and \( B_2^\circ (1) = 0 \) (‘initial conditions’):

\[
\begin{align*}
0 &= \frac{1}{2} \cdot 0^2 - 0 + B_1 \cdot 0 + C, \\
0 &= \frac{1}{2} \cdot 1^2 - 1 + B_1 \cdot 1 + C.
\end{align*}
\]

The first of the two equations in \((2b)\) reads \(0 = C\). So \( C = 0 \). Taking this into account, the second of the two equations in \((2b)\) becomes

\[
0 = \frac{1}{2} - 1 + B_1.
\]

Solve it:

\[
B_1 = -\left(\frac{1}{2} - 1\right) = \frac{1}{2}.
\]

So we just found \( B_1 \). Namely,

\[
(2c) \quad B_1 = \frac{1}{2}.
\]

Substitute this and \( C = 0 \) back into \((2a)\) above:

\[
\frac{1}{2} B_2^\circ (x) = \frac{1}{2} x^2 - x + \frac{1}{2} x
= \frac{1}{2} x^2 - \frac{1}{2} x.
\]

Multiply 2 to the both sides:

\[
(2d) \quad B_2^\circ (x) = x^2 - x.
\]

3
3. Determining $B_2$ and $B_3^\circ(x)$.

Add $B_2$ to (2d), and thereby make it $B_2(x)$. At this point $B_2$ is an unknown constant.

\[ B_2(x) = x^2 - x + B_2. \]

Take its antiderivative:

\[ \int B_2(x)\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + B_2 \cdot x + C. \]

This is $\frac{1}{3}B_3^\circ(x)$, so

\[ \frac{1}{3}B_3^\circ(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + B_2 \cdot x + C. \]  

(3a)

Substitute $x = 0$, and $x = 1$ into (3a) independently. Use $B_3^\circ(0) = 0$ and $B_3^\circ(1) = 0$ (‘initial conditions’):

\[ \begin{cases} 
0 = \frac{1}{3} \cdot 0^3 - \frac{1}{2} \cdot 0^2 + B_2 \cdot 0 + C, \\
0 = \frac{1}{3} \cdot 1^3 - \frac{1}{2} \cdot 1^2 + B_2 \cdot 1 + C.
\end{cases} \]  

(3b)

The first of the two equations in (3b) reads $0 = C$. So $C = 0$. Taking this into account, the second of the two equations in (3b) becomes

\[ 0 = \frac{1}{3} - \frac{1}{2} + B_2. \]

Solve it:

\[ B_2 = -\left(\frac{1}{3} - \frac{1}{2}\right) \]

\[ = \frac{1}{6}. \]
So we just found $B_2$. Namely,

\[(3c) \quad B_2 = \frac{1}{6}.
\]

Substitute this and $C = 0$ back into (3a) above:

\[
\frac{1}{3}B_3^\circ(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x.
\]

Multiply 3 to the both sides:

\[(3d) \quad B_3^\circ(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.
\]

4. Determining $B_3$ and $B_4^\circ(x)$.

Add $B_3$ to (3d), and thereby make it $B_3(x)$. At this point $B_3$ is an unknown constant.

\[B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x + B_3.
\]

Take its antiderivative:

\[
\int B_3(x) \, dx = \frac{1}{4}x^4 - \frac{1}{2}x^3 + \frac{1}{4}x^2 + B_3 \cdot x + C.
\]

This is $\frac{1}{4}B_4^\circ(x)$, so

\[(4a) \quad \frac{1}{4}B_4^\circ(x) = \frac{1}{4}x^4 - \frac{1}{2}x^3 + \frac{1}{4}x^2 + B_3 \cdot x + C.
\]

Substitute $x = 0$, and $x = 1$ into (4a) independently. Use $B_4^\circ(0) = 0$ and $B_4^\circ(1) = 0$ ('initial conditions').
\[
\begin{aligned}
0 &= \frac{1}{4} \cdot 0^4 - \frac{1}{2} \cdot 0^3 + \frac{1}{4} \cdot 0^2 + B_3 \cdot 0 + C, \\
0 &= \frac{1}{4} \cdot 1^4 - \frac{1}{2} \cdot 1^3 + \frac{1}{4} \cdot 1^2 + B_3 \cdot 1 + C.
\end{aligned}
\]

The first of the two equations in (4b) reads \(0 = C\). So \(C = 0\). Taking this into account, the second of the two equations in (4b) becomes

\[
0 = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} + B_3.
\]

Solve it:

\[
B_3 = -\left(\frac{1}{4} - \frac{1}{2} + \frac{1}{4}\right)
= 0.
\]

So we just found \(B_3\). Namely,

\[
(4c) \quad B_3 = 0.
\]

Substitute this and \(C = 0\) back into (4a) above:

\[
\frac{1}{4} B_4^\circ (x) = \frac{1}{4} x^4 - \frac{1}{2} x^3 + \frac{1}{4} x^2.
\]

Multiply 4 to the both sides:

\[
(4d) \quad B_4^\circ (x) = x^3 - 2x^3 + x^2.
\]

\* I think you have seen enough so you can take it from here. Namely, add \(B_4\) to (4d) and then integrate it. That is \(\frac{1}{5}\) of \(B_5^\circ (x)\). One of the initial conditions: \(B_5^\circ (0) = 0\) tells you \(C = 0\). The other initial condition: \(B_5^\circ (1) = 0\) allows you to determine \(B_4\). Substituting it in the expression of \(B_5^\circ (x)\) previously obtained which involves \(B_4\) yields the actual shape of \(B_5^\circ (x)\). So, in principle, as you keep going this way, you recover as many Bernoulli polynomials/numbers as you want, though in practice the computation gets longer and longer as you go on. Now you can do the exercise below.
Exercise 1. Assume

(7d) \[ B_7^\circ (x) = x^7 - \frac{7}{2} x^6 + \frac{7}{2} x^5 - \frac{7}{6} x^3 + \frac{1}{6} x, \]

and recover \( B_7 \) and \( B_8^\circ (x) \).

[Solution]: Add \( B_7 \) to (7d), and thereby make it \( B_7^\circ (x) \). At this point \( B_7 \) is an unknown constant.

\[ B_7^\circ (x) = x^7 - \frac{7}{2} x^6 + \frac{7}{2} x^5 - \frac{7}{6} x^3 + \frac{1}{6} x + B_7. \]

Take its antiderivative:

\[ \int B_7^\circ (x) \, dx = \frac{1}{8} x^8 - \frac{1}{2} x^7 + \frac{7}{12} x^6 - \frac{7}{24} x^4 + \frac{1}{12} x^2 + B_7 \cdot x + C. \]

This is \( \frac{1}{8} B_8^\circ (x) \), so

(8a) \[ \frac{1}{8} B_8^\circ (x) = \frac{1}{8} x^8 - \frac{1}{2} x^7 + \frac{7}{12} x^6 - \frac{7}{24} x^4 + \frac{1}{12} x^2 + B_7 \cdot x + C. \]

Substitute \( x = 0 \), and \( x = 1 \) into (8a) independently. Use \( B_8^\circ (0) = 0 \) and \( B_8^\circ (1) = 0 \) ('initial conditions'):

(8b) \[ \begin{cases} 0 = \frac{1}{8} 0^8 - \frac{1}{2} 0^7 + \frac{7}{12} 0^6 - \frac{7}{24} 0^4 + \frac{1}{12} 0^2 + B_7 \cdot 0 + C. \\ 0 = \frac{1}{8} 1^8 - \frac{1}{2} 1^7 + \frac{7}{12} 1^6 - \frac{7}{24} 1^4 + \frac{1}{12} 1^2 + B_7 \cdot 1 + C. \end{cases} \]

The first of the two equations in (8b) reads \( 0 = C \). So \( C = 0 \). Taking this into account, the second of the two equations in (8b) becomes

\[ 0 = \frac{1}{8} - \frac{1}{2} + \frac{7}{12} - \frac{7}{24} + \frac{1}{12} + B_7. \]
Solve it:

\[ B_7 = -\left( \frac{1}{8} - \frac{1}{2} + \frac{7}{12} - \frac{7}{24} + \frac{1}{12} \right) = 0. \]

So we just found \( B_7 = 0 \). Namely,

\[ B_7 = 0. \tag{8c} \]

Substitute this and \( C = 0 \) back into (8a) above:

\[ \frac{1}{8} B_8^{\circ} (x) = \frac{1}{8} x^8 - \frac{1}{2} x^7 + \frac{7}{12} x^6 - \frac{7}{24} x^4 + \frac{1}{12} x^2. \]

Multiply 8 to the both sides:

\[ B_8^{\circ} (x) = x^8 - 4x^7 + \frac{14}{3} x^6 - \frac{7}{3} x^4 + \frac{2}{3} x^2. \tag{8d} \]

**Exercise 2.** Assume

\[ B_{12}^{\circ} (x) = x^{12} - 6 x^{11} + 11 x^{10} - \frac{33}{2} x^8 + 22 x^6 - \frac{33}{2} x^4 + 5 x^2, \tag{12d} \]

and recover \( B_{12} \) and \( B_{13}^{\circ}(x) \).

**Answer:**

\[ B_{12} = \frac{-691}{2730}, \]

\[ B_{13} (x) = x^{13} - \frac{13}{2} x^{12} + 13 x^{11} - \frac{143}{6} x^9 + \frac{286}{7} x^7 - \frac{429}{10} x^5 + \frac{65}{3} x^3 - \frac{691}{210} x. \]