§34. MEAN VALUES.

Question 1. What is the mean of

1 and 9?

The sum of these numbers is 10. So, the answer is 10 divided by 2, which is 5. To summarize:

\[
\text{the mean of 1 and 9} = \frac{1 + 9}{2} = 5.
\]

* More generally:

The mean of \(a\) and \(b\) is \(\frac{a + b}{2}\).

Question 2. What is the mean of

7, \(-1\) and 12?

The sum of these numbers is 18. This time, there are three numbers involved. So, the answer is 18 divided by 3, which is 6. To summarize:

\[
\text{the mean of 7, } -1 \text{ and 12} = \frac{7 + (-1) + 12}{3} = 6.
\]

* More generally:

The mean of \(a\), \(b\) and \(c\) is \(\frac{a + b + c}{3}\).
When four or more numbers are involved, it is the same. Namely, agree:

(1) the mean of \( a_1 \) is \( \frac{a_1}{1} \).

(2) the mean of \( a_1 \) and \( a_2 \) is \( \frac{a_1 + a_2}{2} \).

(3) the mean of \( a_1, a_2 \) and \( a_3 \) is \( \frac{a_1 + a_2 + a_3}{3} \).

(4) the mean of \( a_1, a_2, a_3 \) and \( a_4 \) is \( \frac{a_1 + a_2 + a_3 + a_4}{4} \).

(5) the mean of \( a_1, a_2, a_3, a_4 \) and \( a_5 \) is \( \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} \).

(6) the mean of \( a_1, a_2, a_3, a_4, a_5 \) and \( a_6 \) is \( \frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{6} \).

Exercise 1. (1) Find the mean of 2.

(2) Find the mean of 3 and \(-7\).

(3) Find the mean of 3, 5 and 17.

(4) Find the mean of 4, 6, 8 and 10.

(5) Find the mean of \(-1, 2, -3, 4 \) and \(-5\).

(6) Find the mean of 96, 48, 24, 12, 6 and 3.

[Answers]: (1) 2. (2) \(-2\). (3) \(\frac{25}{3}\). (4) 7.

(5) \(-\frac{3}{5}\). (6) \(\frac{63}{2}\).
In the above, for no compelling reason, let's rewrite $a_1$ as $f(1)$;
rewrite $a_2$ as $f(2)$;
rewrite $a_3$ as $f(3)$;
rewrite $a_4$ as $f(4)$;
rewrite $a_5$ as $f(5)$;
rewrite $a_6$ as $f(6)$;

Thus

(1) the mean of $f(1)$ is $\frac{f(1)}{1}$.

(2) the mean of $f(1)$ and $f(2)$ is $\frac{f(1)+f(2)}{2}$.

(3) the mean of $f(1)$, $f(2)$ and $f(3)$ is $\frac{f(1)+f(2)+f(3)}{3}$.

(4) the mean of $f(1)$, $f(2)$, $f(3)$ and $f(4)$ is $\frac{f(1)+f(2)+f(3)+f(4)}{4}$.

(5) the mean of $f(1)$, $f(2)$, $f(3)$, $f(4)$ and $f(5)$ is $\frac{f(1)+f(2)+f(3)+f(4)+f(5)}{5}$.

(6) the mean of $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$ and $f(6)$ is $\frac{f(1)+f(2)+f(3)+f(4)+f(5)+f(6)}{6}$.

;
Example 1. Let $f(x) = x^2$. Then

$$f(1) = 1^2,$$
$$f(2) = 2^2,$$
$$f(3) = 3^2,$$
$$f(4) = 4^2,$$
$$f(5) = 5^2,$$
$$f(6) = 6^2,$$

; \\

Accordingly

(1) the mean of $f(1)$ is $\frac{1^2}{1}$. \\

(2) the mean of $f(1)$ and $f(2)$ is $\frac{1^2 + 2^2}{2}$. \\

(3) the mean of $f(1)$, $f(2)$ and $f(3)$ is $\frac{1^2 + 2^2 + 3^2}{3}$. \\

(4) the mean of $f(1)$, $f(2)$, $f(3)$ and $f(4)$ is $\frac{1^2 + 2^2 + 3^2 + 4^2}{4}$. \\

(5) the mean of $f(1)$, $f(2)$, $f(3)$, $f(4)$ and $f(5)$ is $\frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2}{5}$. \\

(6) the mean of $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$ and $f(6)$ is $\frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6}$. ;
• In a similar vein, suppose \( f(x) \) is given. Somehow the following kind of means play important roles in the subsequent discussion:

- the mean of \( f\left(\frac{x}{2}\right) \) and \( f\left(\frac{2x}{2}\right) \).

\[
\begin{array}{c|c|c|c}
0 & \frac{x}{2} & \frac{2x}{2} \\
\hline
x
\end{array}
\]

- the mean of \( f\left(\frac{x}{3}\right) \), \( f\left(\frac{2x}{3}\right) \) and \( f\left(\frac{3x}{3}\right) \).

\[
\begin{array}{c|c|c|c|c}
0 & \frac{x}{2} & \frac{2x}{3} & \frac{3x}{3} \\
\hline
x
\end{array}
\]

- the mean of \( f\left(\frac{x}{4}\right) \), \( f\left(\frac{2x}{4}\right) \), \( f\left(\frac{3x}{4}\right) \) and \( f\left(\frac{4x}{4}\right) \).

\[
\begin{array}{c|c|c|c|c|c}
0 & \frac{x}{4} & \frac{2x}{4} & \frac{3x}{4} & \frac{4x}{4} \\
\hline
x
\end{array}
\]

Example 2. Once again, let \( f(x) = x^2 \). Then

\[
f\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2, \quad f\left(\frac{2x}{2}\right) = \left(\frac{2x}{2}\right)^2,
\]

\[
f\left(\frac{x}{3}\right) = \left(\frac{x}{3}\right)^2, \quad f\left(\frac{2x}{3}\right) = \left(\frac{2x}{3}\right)^2, \quad f\left(\frac{3x}{3}\right) = \left(\frac{3x}{3}\right)^2,
\]

\[
f\left(\frac{x}{4}\right) = \left(\frac{x}{4}\right)^2, \quad f\left(\frac{2x}{4}\right) = \left(\frac{2x}{4}\right)^2, \quad f\left(\frac{3x}{4}\right) = \left(\frac{3x}{4}\right)^2, \quad f\left(\frac{4x}{4}\right) = \left(\frac{4x}{4}\right)^2,
\]

\[
\vdots \quad \vdots \quad \vdots
\]
Accordingly

- the mean of \( f\left(\frac{x}{2}\right) \) and \( f\left(\frac{2x}{2}\right) \) is \( \frac{\left(\frac{x}{2}\right)^2 + \left(\frac{2x}{2}\right)^2}{2} \),

- the mean of \( f\left(\frac{x}{3}\right) \), \( f\left(\frac{2x}{3}\right) \) and \( f\left(\frac{3x}{3}\right) \) is \( \frac{\left(\frac{x}{3}\right)^2 + \left(\frac{2x}{3}\right)^2 + \left(\frac{3x}{3}\right)^2}{3} \),

- the mean of \( f\left(\frac{x}{4}\right) \), \( f\left(\frac{2x}{4}\right) \), \( f\left(\frac{3x}{4}\right) \) and \( f\left(\frac{4x}{4}\right) \) is
  \[
  \frac{\left(\frac{x}{4}\right)^2 + \left(\frac{2x}{4}\right)^2 + \left(\frac{3x}{4}\right)^2 + \left(\frac{4x}{4}\right)^2}{4},
  \]

These are clearly rewritten as

\[
\left(1^2 + 2^2\right) \frac{x^2}{2^3},
\]

\[
\left(1^2 + 2^2 + 3^2\right) \frac{x^2}{3^3},
\]

\[
\left(1^2 + 2^2 + 3^2 + 4^2\right) \frac{x^2}{4^3},
\]

\[
\vdots
\]

As you extrapolate the patterns, you agree that the following is true:

- the mean of \( f\left(\frac{x}{n}\right) \), \( f\left(\frac{2x}{n}\right) \), \( f\left(\frac{3x}{n}\right) \), \ldots \( f\left(\frac{nx}{n}\right) \) is
  \[
  \left(1^2 + 2^2 + 3^2 + \ldots + n^2\right) \frac{x^2}{n^3}.
  \]

As for the underlined part, let’s recall the formula:
Formula. (from “Review of Lectures – XXVII”).

\[
1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.
\]

So the above is paraphrased as follows:

- the mean of \( f\left(\frac{x}{n}\right), f\left(\frac{2x}{n}\right), f\left(\frac{3x}{n}\right), \ldots, f\left(\frac{nx}{n}\right) \) is

\[
\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right)\frac{x^2}{n^3},
\]

that is,

\[
\left(\frac{1}{3} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{6} \cdot \frac{1}{n^2}\right)x^2.
\]

It is useful for later purpose to examine the state of this quantity when \( n \) grows arbitrarily large. When \( n \) grows, like

\[
n = 1000,
\]

\[
n = 1000000,
\]

\[
n = 1000000000,
\]

\[\vdots\]

then accordingly \( \frac{1}{n} \) and \( \frac{1}{n^2} \) become negligible:

\[
n = 1000 \quad \Rightarrow \quad \frac{1}{n} = 0.001, \quad \frac{1}{n^2} = 0.000001,
\]

\[
n = 1000000 \quad \Rightarrow \quad \frac{1}{n} = 0.000001, \quad \frac{1}{n^2} = 0.000000000001,
\]

\[
n = 1000000000 \quad \Rightarrow \quad \frac{1}{n} = 0.000000001, \quad \frac{1}{n^2} = 0.0000000000000001,
\]

\[\vdots\]
More precisely,

\[
\lim_{n \to \infty} \frac{1}{n} = 0, \quad \lim_{n \to \infty} \frac{1}{n^2} = 0.
\]

Hence

\[
\lim_{n \to \infty} \left( \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{6} \cdot \frac{1}{n^2} \right) x^2 = \frac{1}{3} x^2.
\]

• Summary of Example 2.

“For

\[ f(x) = x^2, \]

the mean of \( f\left( \frac{x}{n} \right), f\left( \frac{2x}{n} \right), f\left( \frac{3x}{n} \right), \cdots, f\left( \frac{nx}{n} \right) \), as \( n \to \infty \), is

\[ \frac{1}{3} x^2. \]

Example 3. Let \( f(x) = x^3 \). Once again, let’s consider

\[ \text{o the mean of } f\left( \frac{x}{2} \right) \text{ and } f\left( \frac{2x}{2} \right), \]

\[ \text{o the mean of } f\left( \frac{x}{3} \right), f\left( \frac{2x}{3} \right) \text{ and } f\left( \frac{3x}{3} \right), \]

\[ \text{o the mean of } f\left( \frac{x}{4} \right), f\left( \frac{2x}{4} \right), f\left( \frac{3x}{4} \right) \text{ and } f\left( \frac{4x}{4} \right), \]

\[ \vdots \]

\[ \text{o the mean of } f\left( \frac{x}{n} \right), f\left( \frac{2x}{n} \right), f\left( \frac{3x}{n} \right) \cdots, f\left( \frac{nx}{n} \right). \]
The answers are as follows:

\[
\left(1^3 + 2^3\right) \frac{x^3}{2^4},
\]

\[
\left(1^3 + 2^3 + 3^3\right) \frac{x^3}{3^4},
\]

\[
\left(1^3 + 2^3 + 3^3 + 4^3\right) \frac{x^3}{4^4},
\]

\vdots

\[
\left(1^3 + 2^3 + 3^3 + \cdots + n^3\right) \frac{x^3}{n^4}.
\]

Once again, as for the underlined part, recall the formula:

**Formula.** (from “Review of Lectures – XXVII”).

\[
1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2,
\]

We may accordingly rewrite the last one as

\[
\left(\frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2\right) \frac{x^3}{n^4},
\]

which equals

\[
\left(\frac{1}{4} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{4} \cdot \frac{1}{n^2}\right) x^3.
\]

Now, limit-wise

\[
\lim_{n \to \infty} \left(\frac{1}{4} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{4} \cdot \frac{1}{n^2}\right) x^3 = \frac{1}{4} x^3.
\]

In sum:
• Summary of Example 3.

“For

\[ f(x) = x^3, \]

the mean of \[ f\left(\frac{x}{n}\right), f\left(\frac{2x}{n}\right), f\left(\frac{3x}{n}\right), \ldots, f\left(\frac{nx}{n}\right), \] as \( n \to \infty, \)

is

\[ \frac{1}{4}x^3. \]

Now, nothing stops us from considering the same problem for

\[ f(x) = x^4, \]

\[ f(x) = x^5, \]

\[ f(x) = x^6, \]

\vdots

By extrapolation, we conclude:

the mean of \[ f\left(\frac{x}{n}\right), f\left(\frac{2x}{n}\right), f\left(\frac{3x}{n}\right), \ldots, f\left(\frac{nx}{n}\right), \] as \( n \to \infty, \)

is

\[ \frac{1}{5}x^4, \quad \text{when} \quad f(x) = x^4, \]

\[ \frac{1}{6}x^5, \quad \text{when} \quad f(x) = x^5, \]

\[ \frac{1}{7}x^6, \quad \text{when} \quad f(x) = x^6, \]

\vdots
• If you utilize the formula in “Review of Lectures – XXVII”, page 1, then you will be able to pull more precise results (below). Before presenting them, we introduce the following notation, which is convenient.

**Notation.** For a given \( f(x) \), the notation \( M_n(f)(x) \) stands for the mean of

\[
  f\left(\frac{x}{n}\right), f\left(\frac{2x}{n}\right), f\left(\frac{3x}{n}\right), \ldots, f\left(\frac{nx}{n}\right).
\]

Also, the notation \( M(f)(x) \) stands for the limit

\[
  \lim_{n \to \infty} M_n(f)(x).
\]

**Results.**

(1) \( f(x) = x \implies M_n(f)(x) = \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{n}\right) x, \)

\[
  M(f)(x) = \frac{1}{2} x.
\]

(2) \( f(x) = x^2 \implies M_n(f)(x) = \left(\frac{1}{3} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{6} \cdot \frac{1}{n^2}\right) x^2, \)

\[
  M(f)(x) = \frac{1}{3} x^2.
\]

(3) \( f(x) = x^3 \implies M_n(f)(x) = \left(\frac{1}{4} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{4} \cdot \frac{1}{n^2}\right) x^3, \)

\[
  M(f)(x) = \frac{1}{4} x^3.
\]
(4) \( f(x) = x^4 \implies M_n(f)(x) = \left( \frac{1}{5} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{3} \cdot \frac{1}{n^2} - \frac{1}{30} \cdot \frac{1}{n^4} \right) x^4, \)

\[ M(f)(x) = \frac{1}{5} x^4. \]

(5) \( f(x) = x^5 \implies M_n(f)(x) = \left( \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{n} + \frac{5}{12} \cdot \frac{1}{n^2} - \frac{1}{12} \cdot \frac{1}{n^4} \right) x^5, \)

\[ M(f)(x) = \frac{1}{6} x^5. \]

**Exercise 2.** Find \( M_n(f)(x) \) and \( M(f)(x) \) for

(6) \( f(x) = x^6. \)

Use the following, if necessary.

**Formula.**

\[
1^6 + 2^6 + 3^6 + \cdots + n^6 = \frac{1}{7} n^7 + \frac{1}{2} n^6 + \frac{1}{2} n^5 - \frac{1}{6} n^3 + \frac{1}{42} n.
\]

**Answer:**

\[ M_n(f)(x) = \left( \frac{1}{7} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{2} \cdot \frac{1}{n^2} - \frac{1}{6} \cdot \frac{1}{n^4} + \frac{1}{42} \cdot \frac{1}{n^6} \right) x^6, \]

\[ M(f)(x) = \frac{1}{7} x^6. \]