1. Important Concepts/terminology and basic techniques

   (a) Systems of linear equations: solutions, consistence, augmented matrix

   (b) Three elementary row operations, row echelon form (REF), reduced row echelon form (RREF), leading entries, leading 1’s, pivot columns

   (c) Basic variables, free variables, solution structures of homogeneous and nonhomogeneous linear systems, vector parametrization of solutions

   (d) Linear combinations, span, linear dependence and linear independence of vectors

   (e) Linear transformations and associated standard matrices, one-to-one and onto

   (f) Matrix operations and properties: sum, scalar multiplication, product of matrices, transpose of a matrix, powers of square matrices, zero matrices, identity matrices

2. True or False questions. Justify each answer.

   (a) If a system of linear equations has two different solutions, it must have infinitely many solutions.

   (b) If a system of linear equations has no free variables, then it always has a unique solution.

   (c) If $A$ is an $m \times n$ matrix and the equation $Ax = b$ is consistent for some $b$, then the columns of $A$ span $\mathbb{R}^m$.

   (d) If $A$ and $B$ are row equivalent, then they have the same reduced row echelon form.

   (e) The equation $Ax = 0$ has only the trivial solution if and only if there is no free variable.

   (f) If $A$ is an $m \times n$ matrix and the equation $Ax = b$ is consistent for every $b \in \mathbb{R}^m$, then $A$ has $m$ pivot columns.

   (g) If an $m \times n$ matrix $A$ has $m$ pivot columns, then $Ax = b$ has a unique solution for each $b \in \mathbb{R}^m$.

   (h) If $3 \times 3$ matrix $A$ has three pivot columns, then the RREF of $A$ is $I_3$.

   (i) If in the set $S = \{v_1, v_2, v_3\} \subset \mathbb{R}^3$ no vector is a scalar multiple of other vector, then $S$ is linearly independent.

   (j) If $\{u, v, w\} \subset \mathbb{R}^n$ is linearly independent, then $n > 2$.

   (k) In some cases, it is possible for four vectors to span $\mathbb{R}^5$.

   (l) If $u, v \in \mathbb{R}^m$, then $-u \in \text{span}\{u, v\}$.
(m) If \( \mathbf{w} \) is a linear combination of \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^n \), then \( \mathbf{u} \) is a linear combination of \( \mathbf{v} \) and \( \mathbf{w} \).

(n) If \( A \) is an \( m \times n \) matrix and the linear transformation \( \mathbf{x} \mapsto A\mathbf{x} \) is onto \( \mathbb{R}^m \), then \( m \leq n \).

(o) If \( A \) is an \( m \times n \) matrix and the linear transformation \( \mathbf{x} \mapsto A\mathbf{x} \) is one-to-one, then \( m \geq n \).

(p) If \( A \) is a \( 4 \times 3 \) matrix, then the linear transformation \( \mathbf{x} \mapsto A\mathbf{x} \) cannot map \( \mathbb{R}^3 \) onto \( \mathbb{R}^4 \).

(q) If \( A \) is a \( 4 \times 3 \) matrix with 3 pivot columns, then the linear transformation \( \mathbf{x} \mapsto A\mathbf{x} \) is one-to-one.

(r) If \( A \) and \( B \) are \( m \times n \) matrices, then \( AB^T \) and \( A^T B \) are defined.

(s) If \( AB = C \) and \( C \) has 2 columns, then \( A \) has 2 columns.

(t) If \( BC = BD \), then \( C = D \).

(u) If \( AC = 0 \), then either \( A = 0 \) or \( C = 0 \).

(v) If \( A \) and \( B \) are \( n \times n \) matrices, then \( A^2 - B^2 = (A + B)(A - B) \).

3. For each of the linear systems,

\[
\begin{align*}
&x + y + z = 0 \\
&x + 2y + 3z = 0 \\
&3x + 5y + 7z = 1
\end{align*}
\]

(a) write a corresponding vector equation,

(b) write a corresponding matrix equation,

(c) determine the corresponding augmented matrix,

(d) find a reduced row echelon matrix of the augmented matrix,

(e) find the general solution or explain why the system is inconsistent.

4. Determine \( h \) and \( k \) such that the system (i) is inconsistent, (ii) has a unique solution, (iii) has infinitely many solutions.

(a) \( \frac{x - y}{hx + 2y} = h \) \( \frac{x + 3y}{4x_1 + hx_2} = k \) \( \frac{-2y_1 + hx_2}{6x_1 + kx_2} = 1 \)

(b) \( \frac{x_1 + 3x_2}{k} = 8 \)

(c) \( \frac{x_1 + 2x_2}{k} = -2 \)

5. Let

\[
A = \begin{bmatrix}
1 & 2 \\
3 & 4 \\
0 & -1
\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}
-2 \\
0 \\
3
\end{bmatrix}.
\]

Is \( \mathbf{b} \) a linear combination of the columns of \( A \)?
6. (a) For
\[ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \]
determine if \( b \in \text{span}\{a_1, a_2\} \).

(b) For
\[ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 1 \\ -1 \\ 10 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 8 \\ 15 \\ -5 \end{bmatrix}, \]
determine if \( b \in \text{span}\{a_1, a_2, a_3\} \).

7. (a) Under what condition(s) on \( b_1, b_2, b_3 \) is the system consistent?
\[
\begin{align*}
2x_1 - 4x_2 - 2x_3 &= b_1 \\
-5x_1 + x_2 + x_3 &= b_2 \\
7x_1 - 5x_2 - 3x_3 &= b_3.
\end{align*}
\]

(b) Find condition(s) for \( b \in \mathbb{R}^3 \) so that \( b \) is a linear combination of the vectors
\[ \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}. \]

(c) Find condition(s) for \( b \in \mathbb{R}^3 \) so that \( b \) is a linear combination of the columns of
\[ A = \begin{bmatrix} 2 & -4 & -2 \\ -5 & 1 & 1 \\ 7 & -5 & -3 \end{bmatrix}. \]

(d) Suppose \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by \( T(x) = Ax \) for \( x \in \mathbb{R}^3 \), where \( A \) is given in (c).
Is \( T \) onto? Is \( T \) one-to-one?

8. Determine the linear dependence or independence of each set of vectors.

(a) \[ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}; \]

(b) \[ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}; \]

\[ \text{(c) } \text{Find condition(s) for } b \in \mathbb{R}^3 \text{ so that } b \text{ is a linear combination of the columns of } \]
(c) \[
\begin{bmatrix}
1 \\
-1 \\
2
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
-1 \\
-2
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
-3 \\
1
\end{bmatrix};
\]
(d) \[
\begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}.
\]

9. Determine the value(s) of \(a\) such that \[
\begin{bmatrix}
1 \\
a
\end{bmatrix}, \quad \begin{bmatrix}
a \\
a + 2
\end{bmatrix}
\]
are linearly dependent.

10. Under what condition(s) on \(a, b, \ldots, f\) are the vectors in each of the following sets linearly independent?

(a) \[
\begin{bmatrix}
a \\
0
\end{bmatrix}, \quad \begin{bmatrix}
b \\
c
\end{bmatrix}, \quad \begin{bmatrix}
d \\
e
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
a \\
1 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
b \\
c \\
1
\end{bmatrix}, \quad \begin{bmatrix}
d \\
e \\
f
\end{bmatrix}.
\]

11. Determine the standard matrix of the linear transformation and verify whether it is onto or one-to-one.

(a) \[
T(x) = \begin{bmatrix}
x_1 + 2x_2 + x_3 \\
x_1 - x_2 + x_3 \\
2x_1 + x_2 - x_3
\end{bmatrix}
\] where \(x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} \in \mathbb{R}^3\) (b) \(T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, 0)\).

12. Determine the standard matrix \(A\) of the linear transformation \(T : \mathbb{R}^2 \to \mathbb{R}^3\) given that

\[
T\left(\begin{bmatrix}
1 \\
2
\end{bmatrix}\right) = \begin{bmatrix}
1 \\
-1
\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}
2 \\
-1
\end{bmatrix}\right) = \begin{bmatrix}
3 \\
0
\end{bmatrix}.
\]

13. Let

\[
A = \begin{bmatrix}
1 - a \\
-a^2 \\
1 + a
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 1 \\
2 & 2
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & a \\
0 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
a \\
0 \\
a - a
\end{bmatrix}, \quad E = \begin{bmatrix}
1 & 2a
\end{bmatrix}.
\]

Compute each of the following matrices or explain why it is not defined.

(a) \(AB, AC, AD, DE, ED\)
(b) \(BC^T D, C^T C + A, D^T C - EB, C^T D + BE^T\)

14. Suppose \(A\) is \(m \times n\) and \(B\) is \(p \times q\). Determine the size of \(C\) so that \(D\) is defined. Also, determine the size of \(D\).

(a) \(D = A^T C B\), \quad (b) \(D = CA^T + I\).
15. Find the formulas for $X, Y, Z$ in terms of $A, B$.

\[
\begin{pmatrix}
X & 0 & 0 \\
Y & 0 & I \\
B & I \\
\end{pmatrix}
\begin{pmatrix}
A & Z \\
0 & 0 \\
0 & I \\
\end{pmatrix}
=\begin{pmatrix}
I & 0 \\
0 & I \\
\end{pmatrix},
\]

(b) \[
\begin{pmatrix}
A & B \\
0 & I \\
0 & 0 & I \\
\end{pmatrix}
\begin{pmatrix}
X & Y & Z \\
0 & 0 & I \\
\end{pmatrix}
=\begin{pmatrix}
I & 0 & 0 \\
0 & 0 & I \\
\end{pmatrix}.
\]

16. Show that if $A$ is $4 \times 3$ and has two pivot columns, then the columns of $A$ are linearly dependent.

17. Prove if $\{u, v\} \subset \mathbb{R}^n$ is linearly dependent, then $\{u, v, w\}$ is linearly dependent for any $w \in \mathbb{R}^n$.

18. Prove if $\{u, v, w\} \subset \mathbb{R}^n$ is linearly independent, then $w \not\in \text{span}\{u, v\}$.

19. Prove $\{u, v\} \subset \mathbb{R}^n$ is linearly independent if and only if $\{u + v, v\}$ is linearly independent.

20. Suppose $\{a_1, a_2, \ldots, a_{10}\}$ is linearly independent. Prove $\{a_1, a_2, a_3\}$ is linearly independent.

21. Construct three matrices $A$, $B$, and $C$ so that $AB = AC$ but $B \neq C$. 