Existence of strong solutions and uniqueness in law for stochastic differential equations driven by fractional Brownian motion

Tyrone Duncan *  David Nualart †
Department of Mathematics
University of Kansas,
Lawrence, Kansas, 66045

March 25, 2009

Abstract

In this paper we establish the existence of path-wise solutions and the uniqueness in law for multidimensional stochastic differential equations driven by a multi-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

Keywords: Fractional Brownian motion; Stochastic differential equations.

2000 Mathematics Subject Classification: Primary 60H10; Secondary 60H05,

1 Introduction

The aim of this paper is to study the $d$-dimensional stochastic differential equation given in integral form

$$X_t = X_0 + \int_0^t \sigma(X_s) dB_t^H + \int_0^t b(X_s) ds,$$  (1.1)

where $B^H$ is a $d$-dimensional standard fractional Brownian motion (fBm) with Hurst parameter $H > \frac{1}{2}$. The stochastic integral is interpreted as a

* T. Duncan is supported by the NSF grand DMS 0505706
† D. Nualart is supported by the NSF grant DMS0604207.
path-wise Riemann-Stieltjes integral. It is well-known (e.g. [5, 9]) that there exists a unique strong solution if the drift \( b \) has linear growth and is locally Lipschitz, and the diffusion coefficient \( \sigma \) is continuously differentiable with bounded partial derivatives and the derivative \( \sigma' \) is locally Hölder continuous of order \( \gamma > \frac{1}{H} - 1 \). Under these assumptions, the unique solution \( X \) has Hölder continuous trajectories of order \( \delta \), for any fixed \( \delta < H \). This result has been proved by Lyons [5], in the case \( b = 0 \), using Young integrals [12] and the \( p \)-variation norm. An extension of this result to time dependent coefficients based on the techniques of the classical fractional calculus developed by Zähle in [13] can be found in [9].

The aim of this paper is two fold: First we establish the existence of a solution under weaker assumptions than above and without any local regularity requirement on the drift \( b \). More precisely, we assume that \( b \) has linear growth, and \( \sigma \) is Hölder continuous of order \( \gamma > \frac{1}{H} - 1 \). Secondly, we show the uniqueness in law again without regularity on the drift \( b \), assuming some nondegeneracy conditions on \( \sigma \). The proof of the existence of a solution is based on some estimates similar to those obtained by Hu and Nualart in [4]. The proof of the uniqueness in law is based on an extension of the Girsanov theorem to the fractional Brownian motion (see Decreusefond and Üstünel [2]). This approach has been used by Nualart and Ouknine in [8] to show the existence of strong solutions to the one-dimensional version of Equation (1.1) in the case of an additive fBm with Hurst parameter \( H \in (0, 1) \) (see also [6] and [1]).

The paper is organized as follows. The next section contains some preliminaries on fractional integrals and derivatives and the Girsanov theorem for a fBm. Section 3 deals with the existence of solutions with nonsmooth drift and Section 4 is devoted to the uniqueness in law.

\section{Preliminaries}

In this section we review some preliminaries on fractional operators and on the Girsanov theorem for the fractional Brownian motion.

\subsection{Fractional integrals and derivatives}

Let \( a, b \in \mathbb{R} \) with \( a < b \). Let \( f \in L^1(a, b) \) and \( \alpha > 0 \). The left-sided and the right-sided fractional Riemann-Liouville integrals of \( f \) of order \( \alpha \) are defined
for almost all \( x \in (a, b) \) by

\[
I^\alpha_{a+} f(t) = \frac{1}{\Gamma (\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds
\]

and

\[
I^\alpha_{b-} f(t) = \frac{(-1)^\alpha}{\Gamma (\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) \, ds,
\]

respectively, where \((-1)^{-\alpha} = e^{-i\pi\alpha}\) and \(\Gamma (\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} \, dr\) is the Euler gamma function. Let \(I^\alpha_{a+} (L^p)\) (resp. \(I^\alpha_{b-} (L^p)\)) be the image of \(L^p(a, b)\) by the operator \(I^\alpha_{a+}\) (resp. \(I^\alpha_{b-}\)). If \(f \in I^\alpha_{a+} (L^p)\) (resp. \(f \in I^\alpha_{b-} (L^p)\)) and \(0 < \alpha < 1\) then the Weyl derivatives are defined as

\[
D^\alpha_{a+} f(t) = \frac{1}{\Gamma (1-\alpha)} \left( \frac{f(t)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} \, ds \right) \tag{2.1}
\]

and

\[
D^\alpha_{b-} f(t) = \frac{(-1)^\alpha}{\Gamma (1-\alpha)} \left( \frac{f(t)}{(b-t)^\alpha} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} \, ds \right) \tag{2.2}
\]

where \(a \leq t \leq b\) (the convergence of the integrals at the singularity \(s = t\) holds point-wise for almost all \( t \in (a, b) \) if \( p = 1 \) and moreover in the \(L^p\)-sense if \( 1 < p < \infty \)).

For any \( \lambda \in (0, 1) \), let \(C^\lambda(a, b)\) be the space of \(\lambda\)-Hölder continuous functions on the interval \([a, b]\). The following notation for some norms is used

\[
\| x \|_{a, b, \beta} = \sup_{a \leq \theta < r \leq b} \frac{|x_r - x_\theta|}{|r - \theta|^{\beta}},
\]

and

\[
\| x \|_{a, b, \infty} = \sup_{a \leq r \leq b} |x_r|,
\]

where \( x : [a, b] \to \mathbb{R} \) is a given continuous function. We also write \(\|x\|_\beta = \|x\|_{0, 0, \beta}\).

Recall from [11] that

- If \( \alpha < \frac{1}{p} \) and \( q = \frac{p}{1-\alpha p} \) then
  \[
  I^\alpha_{a+} (L^p) = I^\alpha_{b-} (L^p) \subset L^q (a, b).
  \]

- If \( \alpha > \frac{1}{p} \) then
  \[
  I^\alpha_{a+} (L^p) \cup I^\alpha_{b-} (L^p) \subset C^{\alpha - \frac{1}{p}} (a, b).
  \]
The following inversion formulas hold:

\[ I^\alpha_{a+} (D^\alpha_{a+} f) = f, \quad \forall f \in I^\alpha_{a+} (L^p) \] (2.3)

\[ I^\alpha_{b-} (D^\alpha_{b-} f) = f, \quad \forall f \in I^\alpha_{b-} (L^p) \] (2.4)

and

\[ D^\alpha_{a+} (I^\alpha_{a+} f) = f, \quad D^\alpha_{b-} (I^\alpha_{b-} f) = f, \quad \forall f \in L^1 (a,b) . \] (2.5)

On the other hand, for any \( f,g \in L^1 (a,b) \) by fractional integration by parts there is the equality

\[ \int_a^b I^\alpha_{a+} f(t)g(t)dt = (-1)^\alpha \int_a^b f(t)I^\alpha_{b-} g(t)dt , \] (2.6)

and for \( f \in I^\alpha_{a+} (L^p) \) and \( g \in I^\alpha_{b-} (L^p) \)

\[ \int_a^b D^\alpha_{a+} f(t)g(t)dt = (-1)^{-\alpha} \int_a^b f(t)D^\alpha_{b-} g(t)dt . \] (2.7)

Suppose that \( f \in C^\lambda (a,b) \) and \( g \in C^\mu (a,b) \) with \( \lambda + \mu > 1 \). Then, from the classical paper by Young [12], the Riemann-Stieltjes integral \( \int_a^b f dg \) exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral \( \int_a^b f dg \) in terms of fractional derivatives (see [13]).

Proposition 1 Suppose that \( f \in C^\lambda (a,b) \) and \( g \in C^\mu (a,b) \) with \( \lambda + \mu > 1 \). Let \( \lambda > \alpha \) and \( \mu > 1 - \alpha \). Then the Riemann-Stieltjes integral \( \int_a^b f dg \) exists and it can be expressed as

\[ \int_a^b f dg = (-1)^\alpha \int_a^b D^\alpha_{a+} f (t) D^{1-\alpha}_{b-} g_b (t) dt, \] (2.8)

where \( g_b (t) = g (t) - g (b) \).

2.2 Girsanov theorem for the fractional Brownian motion

Let \( B^H = \{ B^H_t, t \in [0,T] \} \) be a standard \( d \)-dimensional fractional Brownian motion with the Hurst parameter \( H \in (\frac{1}{2},1) \) defined on a complete probability space \( (\Omega, \mathcal{F}, P) \), that is, the components of \( B^H \) are independent centered Gaussian processes with covariance

\[ R_H(t,s) = E(B^H_t \cdot B^H_s) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right) . \]
For each \( t \in [0, T] \) let \( \mathcal{F}_t^{BH} \) be the \( \sigma \)-field generated by the random variables \( \{ B^H_s, s \in [0, t] \} \) and the sets of probability zero.

We denote by \( \mathcal{E} \) the set of step functions on \([0, T] \). Let \( \mathcal{H} \) be the Hilbert space defined as the closure of \( \mathcal{E} \) with respect to the scalar product

\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).
\]

For each \( i = 1, \ldots, d \), the mapping \( 1_{[0,t]} \rightarrow B^H_i(t) \) can be extended to an isometry between \( \mathcal{H} \) and the Gaussian space \( H_1(B^H_i) \) associated with \( B^H_i \).

We will denote this isometry by \( \phi \rightarrow B^H_i(\phi) \).

The covariance kernel \( R_H(t, s) \) can be written as

\[
R_H(t, s) = \int_0^{t \wedge s} K_H(t, r)K_H(s, r)dr,
\]

where \( K_H \) is a square integrable kernel given by:

\[
K_H(t, s) = \Gamma(H + \frac{1}{2})^{-1}(t - s)^{H - \frac{1}{2}}F(H - \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - \frac{t}{s}),
\]

where \( F(a, b, c, z) \) is the Gauss hypergeometric function. Consider the linear operator \( K^*_H \) from \( \mathcal{E} \) to \( L^2([0, T]) \) defined by

\[
(K^*_H 1_{[0,t]})(s) = K_H(t, s)1_{[0,t]}(s).
\]

The operator \( K^*_H \) provides an isometry between the Hilbert spaces \( \mathcal{H} \) and \( L^2([0, T]) \). Hence, the \( d \)-dimensional process \( W = \{ W_t, t \in [0, T] \} \) defined by

\[
W_t^i = B^H_i((K^*_H)^{-1}(1_{[0,t]})) \quad (2.9)
\]

is a standard \( d \)-dimensional Wiener process, and the process \( B^H \) has an integral representation of the form

\[
B^H_t = \int_0^t K_H(t, s)dW_s. \quad (2.10)
\]

On the other hand, the operator \( K_H \) on \( L^2([0, T]) \) associated with the kernel \( K_H \) is an isomorphism from \( L^2([0, T]) \) onto \( I_{0+}^{H+1/2}(L^2([0, T])) \) and it can be expressed in terms of fractional integrals as follows (see [2]):

\[
(K_H h)(t) = \int_0^t s^{H-\frac{1}{2}}I_{0+}^{H-\frac{1}{2}}(x^{\frac{1}{2}-H}h(x))(s)ds, \quad (2.11)
\]

where \( h \in L^2([0, T]) \).

The following definition of an \( \mathcal{F}_t \)-fractional Brownian motion is used.
Definition 2 Let \( \{ \mathcal{F}_t, t \in [0, T] \} \) be a right-continuous increasing family of \( \sigma \)-fields on the complete probability space \( (\Omega, \mathcal{F}, P) \) such that \( \mathcal{F}_0 \) contains all the sets of probability zero. A standard \( d \)-dimensional fractional Brownian motion \( B^H = \{ B^H, t \in [0, T] \} \) is called a standard \( d \)-dimensional \( \mathcal{F}_t \)-fractional Brownian motion if the process \( W \) defined in (2.9) is a standard \( d \)-dimensional \( \mathcal{F}_t \)-Wiener process.

Given a \( d \)-dimensional \( \mathcal{F}_t \)-adapted process with integrable trajectories \( u = \{ u_t, t \in [0, T] \} \) consider the transformation

\[
\tilde{B}^H_t = B^H_t + \int_0^t u_s ds. \tag{2.12}
\]

The process \( \tilde{B}^H \) can be expressed in terms of \( \tilde{W} \) as

\[
\tilde{B}^H_t = B^H_t + \int_0^t u_s ds = \int_0^t K_H(t, s) dW_s + \int_0^t u_s ds = \int_0^t K_H(t, s) d\tilde{W}_s,
\]

where

\[
\tilde{W}_t = W_t + \int_0^t \left( K^{-1}_H \left( \int_0^r u_s ds \right) (r) \right) dr. \tag{2.13}
\]

Note that \( K^{-1}_H \left( \int_0^r u_s ds \right) \) belongs to \( L^2([0, T]) \) almost surely if and only if \( \int_0^r u_s ds \in \mathcal{F} \) almost surely. As a consequence the following version of the Girsanov theorem for the fractional Brownian motion can be verified (see [2], [8]).

Theorem 3 Let \( H \in \left( \frac{1}{2}, 1 \right) \) be fixed. Consider the translated process \( \tilde{B}^H \) in (2.12) defined by a \( d \)-dimensional process \( u = \{ u_t, t \in [0, T] \} \) with integrable trajectories. Assume that:

i) \( \int_0^r u_s ds \in \mathcal{F} \) almost surely, for each \( i \), and set \( v^i_s = \left( K^{-1}_H \int_0^r u_s ds \right) (s) \), where \( u_s = (u^1_s, \ldots, u^d_s) \).

ii) The random variable \( \xi_T \) satisfies \( E(\xi_T) = 1 \), where

\[
\xi_T = \exp \left( -\sum_{i=1}^d \int_0^T v^i_s dW^i_s - \frac{1}{2} \sum_{i=1}^d \int_0^T (v^i_s)^2 ds \right).
\]
Then the translated process $\bar{B}^H$ given by (2.12) is a standard $d$-dimensional $\mathcal{F}_t^{B^H}$-fractional Brownian motion with Hurst parameter $H$ under the new probability $\tilde{P}$ defined by $\frac{d\tilde{P}}{dP} = \xi_T$.

From (2.11) the inverse operator $K^{-1}_H$ is given by

$$K^{-1}_H h = s^{H - \frac{1}{2}} D_{0+}^{H - \frac{1}{2}} s^{\frac{1}{2} - H} h', \quad (2.14)$$

for all $h \in I^{H + \frac{1}{2}}_0 (L^2([0,T]))$, where $h'$ denotes the derivative of $h$.

Equation (2.14) implies that it is necessary for $s^{\frac{1}{2} - H} u^i \in I^{H - 1/2}_0 (L^2([0,T]))$ (i) to hold, and a sufficient condition is the fact that the trajectories of $u^i$ are Hölder continuous of order $H - \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$.

### 3 Existence of strong solutions

Let $B^H = \{B^H_t, t \geq 0\}$ be a standard $d$-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Consider the stochastic differential equation:

$$X_t = X_0 + \int_0^t \sigma(X_s) dB^H_t + \int_0^t b(X_s) ds, \quad 0 \leq t \leq T, \quad (3.1)$$

where $\sigma : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$, $b : \mathbb{R}^d \to \mathbb{R}^d$ are Borel functions, and $X_0$ is a $d$-dimensional random variable on $(\Omega, \mathcal{F}, P)$. The stochastic integral is interpreted as a path-wise Riemann-Stieltjes integral. This integral exists if $\sigma$ is Hölder continuous of order $\gamma > \frac{1}{H} - 1$, and the process $X$ has Hölder continuous trajectories of order $\delta$ for any fixed $\delta < H$ (see Proposition 1 above).

The existence of a solution can be proved by means of a compactness argument, under weaker assumptions on the coefficients.

**Theorem 4** Suppose that $b$ has linear growth, and $\sigma$ is Hölder continuous of order $\gamma > \frac{1}{H} - 1$. Then, there exists a solution $X$ which has Hölder continuous trajectories of order $\beta$, for any $\beta < H$.

In order to verify this theorem, some preliminary estimates are necessary. Let $K$ be the random operator defined by

$$(KX)_t = X_0 + \int_0^t \sigma(X_s) dB^H_s + \int_0^t b(X_s) ds,$$

whenever $X$ is a stochastic process with $\beta$-Hölder continuous trajectories and $\beta \in \left(\frac{1-H}{\gamma}, H\right)$. 7
Lemma 5 Suppose $\sigma$ is Hölder continuous of order $\gamma > \frac{1}{H} - 1$, and let $\beta$ be such that $H > \beta > \frac{1-H}{\gamma}$. The following inequality is satisfied

$$\|K X\|_{s,t,\beta} \leq \frac{1}{(t-s)^{\gamma}}$$

$$+ C \|B^H\|_{\beta} \left[ |\sigma(X_s)| + \|\sigma\|_{\gamma} \|X\|_{s,t,\beta}^{\gamma}(t-s)^{\beta}\right],$$

where the constant $C$ depends only on the parameters $\beta$, $\gamma$ and $H$.

Proof. By definition of $K$

$$(KX)_t - (KX)_s = \int_s^t \sigma(X_s) dB^H_s + \int_s^t b(X_s) ds.$$  

Using (2.8) and taking $\beta \alpha > \alpha > 1 - H$ it follows that

$$\left| \int_s^t \sigma(X_s) dB^H_t \right| = \left| \int_s^t D_{s+}^\alpha \sigma(X)(r) \left(D_{t-}^{1-\alpha} B_t^H(r)\right) dr \right|,$$

where $B_t^H(r) = B_r^H - B_t^H$. From (2.1) and (2.2) it is easy to see that

$$|D_{t-}^{1-\alpha} B_t^H(r)| \leq C |B_t^H|_{\beta} (t-r)^{\alpha + \beta - 1}$$  

(3.2)

and

$$|D_{s+}^\alpha \sigma(X)(r)| \leq C \left[ |\sigma(X_s)|(r-s)^{-\alpha} + \|\sigma\|_{\gamma} \|X\|_{s,t,\beta}^{\gamma}(r-s)^{\gamma - \alpha}\right].$$  

(3.3)

Therefore, (3.2) and (3.3) imply

$$\left| \int_s^t \sigma(X_s) dB^H_t \right| \leq C |B_t^H|_{\beta} \int_s^t \left[ |\sigma(X_s)|(r-s)^{-\alpha}(t-r)^{\alpha + \beta - 1} + \|\sigma\|_{\gamma} \|X\|_{s,t,\beta}^{\gamma}(r-s)^{\gamma - \alpha}(t-r)^{\alpha + \beta - 1}\right] dr$$

$$\leq C |B_t^H|_{\beta} \left[ |\sigma(X_s)|(t-s)^{\beta} + \|\sigma\|_{\gamma} \|X\|_{s,t,\beta}^{\gamma}(t-s)^{\beta(\gamma + 1)}\right].$$  

(3.4)

Then, from (3.4) it follows that

$$|(KX)_t - (KX)_s| \leq \int_s^t |b(X_r)| dr + C |B_t^H|_{\beta} \left[ |\sigma(X_s)|(t-s)^{\beta} + \|\sigma\|_{\gamma} \|X\|_{s,t,\beta}^{\gamma}(t-s)^{\beta(\gamma + 1)}\right],$$
which implies the desired result. ■

Now we proceed with the proof of Theorem 4.

**Proof of Theorem 4.** Consider the Picard iteration scheme defined by $X_t^0 = X_0$, and for $n \geq 1$,

$$X_t^n = X_0 + \int_0^t \sigma(X_{s}^{n-1})dB^H_t + \int_0^t b(X_{s}^{n-1})ds.$$  

Fix $\beta \gamma > \alpha > 1 - H$. Clearly, if $|b(x)| \leq k(1 + |x|)$ the following inequality is satisfied

$$\left| \int_s^t b(X_{s}^{n-1})ds \right| \leq k \int_s^t (1 + |X_{s}^{n-1}|) ds \leq k(t-s) \left( 1 + \|X^{n-1}\|_{s,t,\infty} \right).$$

Then, by Lemma 5

$$\|X^n\|_{s,t,\beta} \leq k(t-s)^{1-\beta} \left( 1 + \|X^{n-1}\|_{s,t,\infty} \right)$$

$$+ C \|B^H\|_{\beta} \left[ |\sigma(X_{s}^{n-1})| + \|\sigma\|_{\gamma} \|X^{n-1}\|_{s,t,\beta} (t-s)^{\beta \gamma} \right].$$

Using the inequality $x^{\gamma} \leq 1 + x$ for $\gamma \in (0,1)$ and $x \geq 0$, and the fact that $\sigma$ has also linear growth ($|\sigma(x)| \leq k(1 + |x|)$), it follows that

$$\|X^n\|_{s,t,\beta} \leq k(t-s)^{1-\beta} \left( 1 + |X_{s}^{n-1}| + (t-s)^{\beta} \|X^{n-1}\|_{s,t,\beta} \right)$$

$$+ C \|B^H\|_{\beta} \left[ k(1 + |X_{s}^{n-1}|) + \|\sigma\|_{\gamma} \left( 1 + \|X^{n-1}\|_{s,t,\beta} \right) (t-s)^{\beta \gamma} \right]$$

$$= A_1 + A_2 |X_{s}^{n-1}| + A_3 \|X^{n-1}\|_{s,t,\beta},$$

where

$$A_1 = k \left[ (t-s)^{1-\beta} + C \|B^H\|_{\beta} \right] + C \|\sigma\|_{\gamma} (t-s)^{\beta \gamma},$$

$$A_2 = k \left[ (t-s)^{1-\beta} + C \|B^H\|_{\beta} \right]$$

and

$$A_3 = k(t-s) + C \|B^H\|_{\beta} \|\sigma\|_{\gamma} (t-s)^{\beta \gamma}.$$

Fix a trajectory of the fractional Brownian motion $B^H(\cdot, \omega_0)$, and suppose that $\Delta$ satisfies $k\Delta + C \|B^H(\cdot, \omega_0)\|_{\beta} \|\sigma\|_{\gamma} \Delta^{\beta \gamma} < \frac{1}{2}$. For notational convenience $\omega_0$ is suppressed subsequently. Then, for any $s < t$ such that $t-s \leq \Delta$ we have $A_3 \leq \frac{1}{2}$,

$$\|X^n\|_{s,t,\beta} \leq 2A_1 + 2A_2 |X_{s}^{n-1}| \quad (3.5)$$

9
and
\[ \|X_n\|_{s,t,\infty} \leq |X_n| + \Delta^\beta (2A_1 + 2A_2 |X_n|). \]
From the latter equality, and proceeding by induction on each interval \([j-1)\Delta, j\Delta]\), it follows easily that
\[ \sup_n \sup_{0 \leq t \leq T} X_n^t < \infty. \]
Then, from (3.5) it follows that
\[ \sup_n \|X_n\|_{\beta} < \infty. \]
As a consequence, for any \(\beta' < \beta\), and for each \(\omega_0\), the sequence of functions \(X_n(\cdot, \omega_0)\) is relatively compact in \(C^{\beta'}([0,T])\). Thus, we can choose a subsequence \(X^N(\cdot, \omega_0)\) which converges in \(C^{\beta'}([0,T])\) to a limit \(X(\cdot, \omega_0)\). The subsequence \(n_k\) might depend on \(\omega_0\), but the selection of the subsequence can be made in such a way that the limit \(X(\cdot, \omega_0)\) is a measurable function of \(\omega_0\). Finally, using estimates similar to those of Lemma 5, it is not difficult to show that the limit \(X(\cdot, \omega_0)\) satisfies the integral equation (3.1).

The following estimate is similar to one obtained in [4].

**Proposition 6** Suppose that \(b\) and \(\sigma\) are bounded, and \(\sigma\) is Lipschitz. Let \(X\) be a stochastic process, with Hölder continuous trajectories of order \(\beta \in (1 - H, H)\), which satisfies Equation (3.1). Then,
\[ |X_s| \leq |X_0| + 2T \|b\|_{\infty} + \|\sigma\|_{\infty} \left( \|\sigma'\|_{-1}^{-1} + T(2C)^{1/\beta} \|\sigma'\|_{-1}^{-1} \|B^H\|_{1/\beta} \right). \]  (3.6)

**Proof.** From Lemma 5 with \(\gamma = 1\) we obtain
\[ \|X\|_{s,t,\beta} \leq (t - s)^{1-\beta} \|b\|_{\infty} + C \|B^H\|_{\beta} \left[ \|\sigma\|_{\infty} + \|\sigma'\|_{\infty} \|X\|_{s,t,\beta(t - s)^{\beta}} \right]. \]
Take \(\Delta = (2C\|\sigma'\|_{\infty} \|B^H\|_{\beta})^{-1/\beta}\). Then, for any \(s < t\) such that \(t - s \leq \Delta\)
\[ \|X\|_{s,t,\beta} \leq 2 \left( \Delta^{1-\beta} \|b\|_{\infty} + C \|B^H\|_{\beta} \|\sigma\|_{\infty} \right). \]  (3.7)
Hence,
\[ \|X\|_{s,t,\infty} \leq |X_s| + 2(t - s) \|b\|_{\infty} + 2C \Delta^\beta \|B^H\|_{\beta} \|\sigma\|_{\infty} \leq |X_s| + 2(t - s) \|b\|_{\infty} + \|\sigma\|_{\infty} \|\sigma'\|_{-1}^{-1}. \]

10
Divide the interval $[0, T]$ into $n = \lceil T/\Delta \rceil + 1$ random subintervals of length less or equal than $\delta$. Applying the previous inequality on the intervals $[0, \Delta], [\Delta, 2\Delta], \ldots, [(n-1)\Delta, n\Delta \wedge T]$, recursively, we obtain

$$\sup_{0 \leq t \leq T} |X_t| \leq |X_0| + 2T\|b\|_\infty + n\|\sigma\|_\infty\|\sigma'\|_\infty^{-1},$$

which implies (3.6).

4 Uniqueness in law

First the notion of weak solution for the stochastic differential equation (3.1) is introduced.

**Definition 7** A weak solution to equation (3.1) means that there is a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \in [0, T]\})$ and a couple of $d$-dimensional $\mathcal{F}_t$-adapted continuous processes $(B^H, X)$, such that:

i) $B^H$ is a standard $d$-dimensional $\mathcal{F}_t$-fractional Brownian motion in the sense of Definition 2.

ii) $X$ and $B^H$ are Hölder continuous of order $\beta$, for all $\beta < H$, and these processes satisfy (3.1), where the stochastic integral is a path-wise Riemann-Stieltjes integral.

The Girsanov theorem introduced in the last section provides a method to verify the following result on the uniqueness in law for weak solutions.

**Theorem 8** Suppose that the coefficients $b$ and $\sigma$ in (3.1) satisfy the following assumptions:

1. On $\mathbb{R}^d$ $\det \sigma \neq 0$, $\sigma$ is bounded, continuously differentiable and $\sigma'$ is bounded and locally Hölder continuous of order $\gamma > \frac{1}{H} - 1$; and $b$ is also bounded.

2. The following inequalities are satisfied

$$|\sigma^{-1}b(x)| \leq k(1 + |x|^\alpha),$$

$$|\sigma^{-1}b(x) - \sigma^{-1}b(y)| \leq k|x - y|^{\lambda \left(\frac{1}{H} \vee \frac{1}{H} \right)} \delta,$$

for all $x, y \in \mathbb{R}^d$, and some $\alpha < H$, $\lambda > 1 - \frac{1}{2H}$ and $\lambda \left(1 + \frac{1}{H}\right) + \frac{\delta}{H} < 1$.

Then, any two weak solutions to Equation (3.1) have the same law.
Proof. Let \((X, B^H)\) be a weak solution of the stochastic differential equation (3.1) defined on the filtered probability space \((\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \in [0, T]\})\).

Fix \(\beta < H\) such that \(\alpha < \beta, \lambda > \frac{1}{\beta}(H - \frac{1}{2})\) and \(\lambda(1 + \frac{1}{H}) + \frac{2}{\beta} < 1\). Set \(g = \sigma^{-1}b\), and

\[
 u_s = K_H^{-1} \left( \int_0^s g(X_r) dr \right) (s).
\]

Equation (2.14) implies that the process \(u\) is adapted and

\[
 u_s = s^{H - \frac{1}{2}} D_{0+}^{H - \frac{1}{2}} s^{\frac{1}{2} - H} g(X_s) =: C(\alpha(s) + \beta(s)),
\]

where

\[
 \alpha(s) = g(X_s) \left[ s^{\frac{1}{2} - H} + (H - \frac{1}{2}) s^{H - \frac{1}{2}} \int_0^s \frac{s^{\frac{1}{2} - H} - r^{\frac{1}{2} - H}}{(s - r)^{\frac{1}{2} + H}} dr \right],
\]

and

\[
 \beta(s) = (H - \frac{1}{2}) s^{H - \frac{1}{2}} \int_0^s \frac{g(X_s) - g(X_r)}{(s - r)^{\frac{1}{2} + H}} r^{\frac{1}{2} - H} dr.
\]

Then, Proposition 6 implies that there exists a constant \(K\) such that

\[
 \sup_{0 \leq s \leq T} |Y_s| \leq K \left( 1 + \|B^H\|_{\frac{1}{\beta}}^\frac{1}{\alpha} \right). \tag{4.1}
\]

Using the estimate (4.1) and the equality

\[
 \int_0^s \frac{r^{\frac{1}{2} - H} - s^{\frac{1}{2} - H}}{(s - r)^{\frac{1}{2} + H}} dr = c_H s^{1 - 2H},
\]

It follows that

\[
 |\alpha(s)| \leq K s^{\frac{1}{2} - H} \left( 1 + \|B^H\|_{\frac{1}{\beta}}^\frac{1}{\alpha} \right).
\]

As a consequence, taking into account that \(\frac{\alpha}{\beta} < 1\), and using Fernique’s theorem (see [3]), for any \(c > 1\)

\[
 E\left( \exp\left( c \int_0^T |\alpha(s)|^2 ds \right) \right) < \infty. \tag{4.2}
\]

In order to estimate the term \(\beta(s)\), apply the Hölder continuity condition on \(g\) to obtain, using that \(\lambda > \frac{1}{\beta}(H - \frac{1}{2})\),

\[
 |\beta(s)| \leq K s^{H - \frac{1}{2}} \|X\|_\infty^\delta \int_0^s \frac{|X_s - X_r|^\lambda}{(s - r)^{H + \frac{1}{2}}} r^{\frac{1}{2} - H} dr \\
 \leq K s^{\gamma \beta} \|X\|_\infty^\delta \|X\|_{\beta}^\lambda.
\]
Then, from (3.7) in the proof of Proposition 6 the following estimate for the \( \beta \)-Hölder norm of the process is satisfied
\[
\|X\|_\beta \leq C \left( 1 + \|B^H\|^{1+\frac{1}{\beta}} \right).
\]
Hence,
\[
|\beta(s)| \leq C \left( 1 + \|B\|^{\lambda(1+\frac{1}{\beta})+\frac{4}{\beta}} \right)
\]
By Fernique’s Theorem it follows that
\[
E \left( \exp \left( c \int_0^T \beta(s)^2 ds \right) \right) < \infty.
\] (4.3)
for any \( c > 1 \).

Let \( \tilde{P} \) be defined by
\[
\frac{d\tilde{P}}{dP} = \exp \left( -\sum_{i=1}^d \int_0^T u_s^i dW_s^i - \frac{1}{2} \int_0^T |u_s|^2 ds \right).
\] (4.4)
It is classical that the process \( u_s \) satisfies conditions i) and ii) of Theorem 3. In fact, \( u_s \) is an adapted process such that \( \int_0^T |u_s|^2 ds < \infty \) almost surely. Finally, taking into account (4.2) and (4.3) it follows by the Novikov Theorem that \( E \left( \frac{d\tilde{P}}{dP} \right) = 1 \).

By the classical Girsanov theorem the process
\[
\tilde{W}_t = W_t + \int_0^t u_sdr
\]
is a standard \( d \)-dimensional \( \mathcal{F}_t \)-Brownian motion under the probability \( \tilde{P} \).

Hence,
\[
\tilde{B}_t^H = \int_0^t K_H(t, s) d\tilde{W}_s = B_t^H + \int_0^t \sigma^{-1} b(X_s) ds
\]
is a standard \( d \)-dimensional \( \mathcal{F}_t \)-fractional Brownian motion under the probability \( \tilde{P} \). In terms of the process \( \tilde{B}_t^H \) the process \( X_t \) satisfies
\[
X_t = x + \int_0^t \sigma(X_s) dB_t^H + \int_0^t b(X_s) ds
= x + \int_0^t \sigma(X_s) d\tilde{B}_t^H.
\]
Hence, the process $X$ under the probability $\tilde{P}$ has the same distribution as $Y$ under $P$, where $Y$ is the unique solution to the Equation

$$Y_t = x + \int_0^t \sigma(Y_s) dB^H_t.$$ 

As a consequence, both processes have the same probability law. In fact, if $\Psi$ is a bounded measurable functional on $C([0,T])$, the following equalities are satisfied

$$E_P(\Psi(X)) = \int_\Omega \Psi(X) \frac{dP}{d\tilde{P}} d\tilde{P}$$

$$= E_\tilde{P} \left( \Psi(X) \exp \left( \int_0^T \left( K^{-1}_H \int_0^r \sigma^{-1}(X) dr \right) (s) dW_s \right) \right)$$

$$+ \frac{1}{2} \int_0^T \left( K^{-1}_H \int_0^r \sigma^{-1}(X) dr \right)^2 (s) ds \right) \right)$$

$$= E_P \left( \Psi(Y) \left( \exp \int_0^T \left( K^{-1}_H \int_0^r \sigma^{-1}(Y) dr \right) (s) dW_s \right) \right)$$

$$- \frac{1}{2} \int_0^T \left( K^{-1}_H \int_0^r \sigma^{-1}(Y) dr \right)^2 (s) ds \right) \right).$$

This implies the uniqueness in law. 

**Remark** In the one-dimensional case, we can establish the uniqueness and the existence of a strong solution for the equation

$$X_t = x + \int_0^t \sigma(X_s) dB^H_s + \int_0^t b(X_s) ds, \quad 0 \leq t \leq T, \quad (4.5)$$

where $\sigma$ is such that $|\sigma(z)| \geq c > 0$, and assuming that $\sigma^{-1}b$ is Hölder continuous of order $\gamma > 1 - \frac{1}{2H}$. In fact, set $F(x) = \int_0^x \frac{1}{\sigma(z)} \, dz$. Then, using the change-of-variables formula for the fractional Brownian motion (see, for instance, [13, Theorem 4.3.1]) it follows that a process $X$ is a solution to Equation (4.5) if and only if the process $Y_t = F(X_t)$ is a solution of

$$Y_t = F(x) + B^H_t + \int_0^t \frac{b(F^{-1}(Y_s))}{\sigma(F^{-1}(Y_s))} ds.$$
It follows that there is a unique strong solution to Equation (4.5) using the results of [8].

References


