HITTING TIMES FOR GAUSSIAN PROCESSES

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We establish a general formula for the Laplace transform of the hitting times of a Gaussian process. Some consequences are derived, and particular cases like the fractional Brownian motion are discussed.

1. Introduction. Consider a zero mean continuous Gaussian process \((X_t, t \geq 0)\), and for any \(a > 0\), we denote by \(\tau_a\) the hitting time of the level \(a\) defined by

\[
\tau_a = \inf \{t \geq 0 : X_t = a\} = \inf \{t \geq 0 : X_t \geq a\}.
\]

Thus the map \((a \mapsto \tau_a)\) is left-continuous and increasing hence with right limits. The map \((a \mapsto \tau_{a+})\) is right continuous where

\[
\tau_{a+} = \lim_{b \downarrow a, b > a} \tau_a = \inf \{t \geq 0 : X_t > a\}.
\]

Little is known about the distribution of \(\tau_a\). It is explicitly known in particular cases like the Brownian motion. If \(X\) is a fractional Brownian motion with Hurst parameter \(H\), there is a result by Molchan [6] which stands that

\[
P(\tau_a > t) = t^{-(1-H)+o(1)}
\]

as \(t\) goes to infinity.

When \(X\) is a standard Brownian motion, it is well-known that

\[
E(\exp(-\alpha \tau_a)) = \exp(-a\sqrt{2\alpha})
\]

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for all $\alpha > 0$. This result is easily proved using the exponential martingale

$$M_t = \exp(\lambda B_t - \frac{1}{2} \lambda^2 t).$$

If we consider a general Gaussian process $X_t$, the exponential process

$$M_t = \exp(\lambda X_t - \frac{1}{2} \lambda^2 V_t),$$

where $V_t = E(X_t^2)$ is no longer a martingale. However, it is equal to 1 plus a divergence integral in the sense of Malliavin Calculus. The aim of this paper is to take advantage of this fact in order to derive a formula for $E(\exp(-\frac{1}{2} \lambda^2 V_\tau a))$. We derive an equation involving this expectation in Theorem 3.4, under rather general conditions on the covariance of the process. As a consequence, we show that if the partial derivative of the covariance is nonnegative, then $E(\exp(-\frac{1}{2} \lambda^2 V_\tau a)) \leq 1$, which implies that $V_\tau a$ has infinite moments of order $p$ for all $p \geq \frac{1}{2}$ and finite negative moments of all orders. In particular, for the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$,

$$E(\exp(-\alpha \tau^2 H a)) \leq \exp(-a \sqrt{2 \alpha}),$$

for all $\alpha, a > 0$.

The paper is organized as follows. In Section 2 we present some preliminaries on Malliavin Calculus, and the main results are proved in Section 3.

2. Preliminaries on Malliavin Calculus. Let $(X_t, t \geq 0)$ be a zero mean Gaussian process such that $X_0 = 0$ and with covariance function

$$R(s, t) = E(X_t X_s).$$

We denote by $\mathcal{E}$ the set of step functions on $[0, +\infty)$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left< 1_{[0,t]}, 1_{[0,s]} \right>_{\mathcal{H}} = R(t, s).$$

The mapping $1_{[0,t]} \longrightarrow X_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(X)$ associated with $X$. We will denote this isometry by $\varphi \longrightarrow X(\varphi)$.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(X(\phi_1), \ldots, X(\phi_n)),$$
where \( n \geq 1, f \in C_0^\infty (\mathbb{R}^n) \) (\( f \) and all its partial derivatives are bounded), and \( \phi_i \in \mathcal{H} \).

The \textit{derivative operator} \( D \) of a smooth and cylindrical random variable \( F \) of the form (2.1) is defined as the \( \mathcal{H} \)-valued random variable

\[
DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (X(\phi_1), \ldots, X(\phi_n)) \phi_i.
\]

The derivative operator \( D \) is then a closable operator from \( L^2(\Omega) \) into \( L^2(\Omega; \mathcal{H}) \). The Sobolev space \( \mathbb{D}^{1,2} \) is the closure of \( S \) with respect to the norm

\[
\|F\|_{1,2}^2 = E(F^2) + E(\|DF\|_{\mathcal{H}}^2).
\]

The \textit{divergence operator} \( \delta \) is the adjoint of the derivative operator. We say that a random variable in \( L^2(\Omega; \mathcal{H}) \) belongs to the domain of the divergence operator, denoted by \( \text{Dom} \delta \), if

\[
|E(\langle DF, u \rangle_{\mathcal{H}})| \leq c_u \|F\|_{L^2(\Omega)}
\]

for any \( F \in S \). In this case \( \delta(u) \) is defined by the duality relationship

\[
(2.2) \quad E(F \delta(u)) = E(\langle DF, u \rangle_{\mathcal{H}}),
\]

for any \( F \in \mathbb{D}^{1,2} \).

Set \( V_t = R(t, t) \). For any \( \lambda > 0 \), we define

\[
M_t = \exp(\lambda X_t - \frac{1}{2} \lambda^2 V_t).
\]

Formally, the Itô’s formula for the divergence integral, proved for instance in [1] implies that

\[
(2.3) \quad M_t = 1 + \lambda \delta(M \cdot 1_{[0,t]}).
\]

However, the process \( M \cdot 1_{[0,t]} \) does not belong, in general, to the domain of the divergence operator. This happens, for instance, in the following basic example.

\textbf{Example 1.} Fractional Brownian motion with Hurst parameter \( H \in (0, 1) \) is a zero mean Gaussian process \( (B_t^H, t \geq 0) \) with the covariance

\[
(2.4) \quad R_H(t, s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right).
\]
In this case, the processes $B^H 1_{[0,t]}$ and \((\exp(\lambda B^H_s - \frac{1}{2} \lambda^2 s^{2H}) 1_{[0,t]}(s), s \geq 0)\) do not belong to $L^2(\Omega; \mathcal{H})$ if $H \leq \frac{1}{4}$ (see [2]).

In order to define the divergence of $M 1_{[0,t]}$ and to establish formula (2.3) we introduce the following additional property on the covariance function of the process $X$.

**H0)** The covariance function $R(t,s)$ is continuous, the partial derivative $\frac{\partial R}{\partial s}(s,t)$ exists in the region $\{0 < s, t, s \neq t\}$, and for all $T > 0$

$$\sup_{t \in [0,T]} \int_0^T \left| \frac{\partial R}{\partial s}(s,t) \right| ds < \infty.$$ 

Notice that this property is satisfied by the covariance (2.4) for all $H \in (0,1)$.

Define

$$\delta_t M = \frac{1}{\lambda} (M_t - 1).$$

The following proposition asserts that $\delta_t M$ satisfies an integration by parts formula, and in this sense, it coincides with an extension of the divergence of $M 1_{[0,t]}$.

**Proposition 2.1.** Suppose that **H0**. Then, for any $t > 0$, and for any smooth and cylindrical random variable of the form $F = f(X_{t_1}, \ldots, X_{t_n})$, we have

$$E(F \delta_t M) = E \left( \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{t_1}, \ldots, X_{t_n}) \int_0^t M_s \frac{\partial R}{\partial s}(s,t_i) ds \right).$$

**Proof.** First notice that condition **H0** implies that the right-hand side of Equation (2.6) is well defined. Then, it suffices to show Equation (2.6) for a function of the form

$$f(x_1, \ldots, x_n) = \exp \left( \sum_{i=1}^n \lambda_i x_i \right),$$
where \( \lambda_i \in \mathbb{R} \). In this case we have for all \( 0 < t_1 < \cdots < t_n \)

\[
\frac{1}{\lambda} E(F(M_t - 1)) = \frac{1}{\lambda} e^{\frac{1}{2} \sum_{i=1}^{n} \lambda_i R(t_i, t_j)} \left( e^{\frac{1}{2} \sum_{i=1}^{n} \lambda_i R(t, t_i)} - 1 \right)
\]

\[
= \sum_{i=1}^{n} \int_{0}^{t} e^{\frac{1}{2} \sum_{i=1}^{n} \lambda_i \lambda_j R(t_i, t_j) + \lambda \sum_{i=1}^{n} \lambda_i R(s, t_i)} \lambda_i \frac{\partial R}{\partial s}(s, t_i) \, ds
\]

\[
= \int_{0}^{t} E \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{t_i}, \ldots, X_{t_n}) M \frac{\partial R}{\partial s}(s, t_i) \right) \, ds,
\]

which completes the proof of the proposition.

In many cases like in the Example 1 with \( H > \frac{1}{4} \), the process \( M \mathbf{1}_{[0,t]} \) belongs to the space \( L^2(\Omega; \mathcal{H}) \), and then the right-hand side of Equation (2.6) equals to

\[
E(DF, M \mathbf{1}_{[0,t]})_{\mathcal{H}}.
\]

In this situation, taking into account the duality formula (2.2), Equation (2.6) says that \( M \mathbf{1}_{[0,t]} \) belongs to the domain of the divergence and \( \delta(M \mathbf{1}_{[0,t]}) = \delta M \).

3. Hitting times. In this section we will assume the following conditions:

**H1)** The partial derivative \( \frac{\partial R}{\partial s}(s, t) \) exists and it is continuous in \([0, +\infty)^2\).

**H2)** \( \lim \sup_{t \to \infty} X_t = +\infty \) almost surely.

**H3)** For any \( 0 \leq s < t \) we have \( E(|X_t - X_s|^2) > 0 \).

Under these conditions the process \( X \) has a continuous version, because

\[
E(|X_t - X_s|^2) = R(t, t) + R(s, s) - 2R(s, t)
\]

\[
= \int_{s}^{t} \left[ \frac{\partial R}{\partial u}(u, t) - \frac{\partial R}{\partial u}(u, s) \right] \, du
\]

\[
\leq 2|t - s| \sup_{s \leq u \leq t} \left| \frac{\partial R}{\partial u}(u, t) \right|.
\]

For any \( a > 0 \), we define the hitting time \( \tau_a \) by (1.1). We know that \( P(\tau_a < \infty) = 1 \) by condition **H2**. Set

\[
(3.1) \quad S_t = \sup_{s \in [0,t]} X_s.
\]
From the results of [8], it follows that for all \( t > 0 \), the random variable \( S_t \) belongs to the space \( \mathbb{D}^{1,2} \). Furthermore, condition H3) allows us to compute the derivative of this random variable.

**Lemma 3.1.** For all \( t > 0 \) with probability one the maximum of the process \( X \) in the interval \([0, t]\) is attained in a unique point, that is \( \tau_{S_t} = \tau_{S_t^+} \), and \( DS_t = \mathbf{1}_{[0, \tau_{S_t}]} \).

**Proof.** The fact that the maximum is attained in a unique point follows from condition H3) and Lemma 2.6 in Kim and Pollard [5]. The formula for the derivative of \( S_t \) follows easily by an approximation argument. \( \Box \)

We need the following regularization of the stopping time \( \tau_a \). Suppose that \( \varphi \) is a nonnegative smooth function with compact support in \((0, +\infty)\), and define for any \( T > 0 \),

\[
Y = \int_0^\infty \varphi(a) (\tau_a \wedge T) \, da.
\]

The next result states the differentiability of the random variable \( Y \) in the sense of Malliavin calculus and provides an explicit formula for its derivative.

**Lemma 3.2.** The random variable \( Y \) defined in (3.2) belongs to the space \( \mathbb{D}^{1,2} \), and

\[
D_r Y = -\int_0^{S_T} \varphi(y) \mathbf{1}_{[0, \tau_y]}(r) \, d\tau_y.
\]

**Proof.** Clearly, \( Y \) is bounded. On the other hand, for any \( r > 0 \) we have \( \{\tau_a > r\} = \{S_r < a\} \). Therefore, we can write using Fubini’s theorem

\[
Y = \int_0^\infty \varphi(a) \left( \int_0^{\tau_a \wedge T} d\theta \right) \, da = \int_0^T \left( \int_0^\infty \varphi(a) \, da \right) \, d\theta
\]

which implies that \( Y \in \mathbb{D}^{1,2} \) because \( S_\theta \in \mathbb{D}^{1,2} \), and

\[
D_r Y = -\int_0^T \varphi(S_\theta) D_r S_\theta \, d\theta = -\int_0^T \varphi(S_\theta) \mathbf{1}_{[0, \tau_{S_\theta}]}(r) \, d\theta.
\]

Finally, making the change of variable \( S_\theta = y \) yields

\[
D_r Y = -\int_0^{S_T} \varphi(y) \mathbf{1}_{[0, \tau_y]}(r) \, d\tau_y.
\]

\( \Box \)
Notice that $M_Y = \exp(\lambda X_Y - \frac{1}{2}\lambda^2 V_Y)$. Hence, letting $t = Y$ in equation (2.5) and taking the mathematical expectation of both members of the equality yields

$$(3.4) \quad E(M_Y) = 1 + \lambda E(\delta_t M \mid t = Y).$$

We are going to show the following result which provides a formula for the left-hand side of Equation (3.4).

**Lemma 3.3.** Assume conditions $\textbf{H1}$, $\textbf{H2}$, and $\textbf{H3}$. Then, we have

$$(3.5) \quad E(M_Y) = 1 - \lambda E(\int_0^T \phi(y) \frac{\partial R}{\partial s}(Y, \tau_y) \, d\tau_y).$$

**Proof.** The proof will be done in two steps.

**Step 1.** We claim that for any function $p(x)$ in $C_0^\infty(\mathbb{R})$ we have

$$(3.6) \quad E(\delta_t M \, p(Y)) = -E\left(\int_0^t M_s p'\left(Y\right) \int_0^{S_\theta} \phi(y) \frac{\partial R}{\partial s}(s, \tau_y) \, ds \, d\theta\right).$$

We can write $Y = \int_0^T \psi(S_\theta) \, d\theta$, where $\psi(x) = \int_x^\infty \phi(a) \, da$. Consider an increasing sequence $D_n$ of finite subsets of $[0, T]$ such that their union is dense in $[0, T]$. Set $Y_n = \int_0^T \psi(S_\theta^n) \, d\theta$, and $S_\theta^n = \max\{X_t, t \in D_n \cap [0, \theta]\}$. Then, $Y_n$ is a Lipschitz function of $\{X_t, t \in D_n\}$. Hence, formula (2.6), which holds for Lipschitz functions, implies that

$$E(\delta_t M \, p(Y_n)) = -E\left(\int_0^t M_s p'\left(Y_n\right) \int_0^{S_\theta^n} \phi(y) \frac{\partial R}{\partial s}(s, \tau_y) \, ds \, d\theta\right).$$

The function $r \to \int_0^t M_s \frac{\partial R}{\partial s}(s, r) \, ds$ is continuous and bounded by condition $\textbf{H1}$. As a consequence, we can take the limit of the above expression as $n$ tends to infinity and we get

$$E(\delta_t M \, p(Y)) = -E\left(\int_0^T p'(Y) \, ds \int_0^{S_\theta} \phi(y) \frac{\partial R}{\partial s}(s, \tau_y) \, ds \, d\theta\right).$$

Finally, making the change of variable $S_\theta = y$ yields (3.6).

**Step 2.** We write

$$E(\delta_t M \mid t = Y) = E\left(\lim_{\varepsilon \to 0} \int_{-\infty}^\infty \delta_t M p_e(Y - t) \, dt\right),$$
where $p_\varepsilon(x)$ is an approximation of the identity, and by convention we assume that $\delta_t M = 0$ if $t$ is negative. We can commute the expectation with the above limit by the dominated convergence theorem because
\[
\int_{-\infty}^{\infty} |\delta_t M| p_\varepsilon(Y - t) \, dt = \int_{-\infty}^{\infty} \frac{1}{\lambda} |M_t - 1| p_\varepsilon(Y - t) \, dt \\
\leq \frac{1}{\lambda} \sup_{0 \leq t \leq T+1} (|M_t| + 1),
\]
if the support of $p_\varepsilon(x)$ is included in $[-\varepsilon, \varepsilon]$, and $\varepsilon \leq 1$. Hence,
\begin{equation}
(3.7) \quad E(\delta_t M|_{t=Y}) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} E(\delta_t M p_\varepsilon(Y - t)) \, dt.
\end{equation}

Using formula (3.6) yields
\begin{equation}
(3.8) \quad E(\delta_t M p_\varepsilon(Y - t)) = - \int_0^t E\left(p_\varepsilon'(Y - t) M_s \left( \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(s, \tau_y) \, d\tau_y \right) \right) \, ds.
\end{equation}

Hence substituting (3.8) into (3.7) and integrating by parts we obtain
\[
E(\delta_t M|_{t=Y}) = - \lim_{\varepsilon \to 0} E\left( \int_{-\infty}^{\infty} p_\varepsilon'(Y - t) M_t \left( \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial t}(t, \tau_y) \, d\tau_y \right) \right) \, dt.
\]

Notice that
\[
\left| \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(s, \tau_y) \, d\tau_y \right| \leq T \sup_{0 \leq s, u \leq T} \left| \frac{\partial R}{\partial s}(s, u) \right| \|\varphi\|_\infty.
\]

Hence, applying the dominated convergence theorem we get
\[
E(M_Y) = 1 + \lambda E(\delta_t M|_{t=Y}) \\
= 1 - \lambda \lim_{\varepsilon \to 0} E\left( \int_{-\infty}^{\infty} p_\varepsilon(Y - t) M_t \left( \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial t}(t, \tau_y) \, d\tau_y \right) \right) \, dt \\
= 1 - \lambda E\left( M_Y \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(Y, \tau_y) \, d\tau_y \right). \]
The next step will be to replace the function \( \varphi(x) \) by an approximation of the identity and let \( T \) tend to infinity. Notice that (3.5) still holds for \( \varphi(x) = 1_{[0,b]}(x) \) for any \( b \geq 0 \). In this way we can establish the following result.

**Theorem 3.4.** Assume conditions \( H1), H2), and H3). For any \( a > 0 \) and \( \lambda \in \mathbb{R} \) we have

\[
\int_{0}^{a} E(M_{\tau_a}) \, dy = a - \lambda E \left( \int_{0}^{a} \int_{0}^{1} M_{z\tau_y + (1-z)\tau_y} \frac{\partial R}{\partial s} (z\tau_y + (1 - z)\tau_y, \tau_y) \, dz \, d\tau_y \right).
\]

**Proof.** Fix \( a > 0 \). We first replace the function \( \varphi(x) \) by an approximation of the identity of the form \( \varphi_\varepsilon(x) = \varepsilon^{-1} 1_{[0,1]}(x/\varepsilon) \) in formula (3.5). We will make use of the following notation

\[
Y_{\varepsilon,a} = \int_{0}^{\infty} \varphi_\varepsilon(x - a) (\tau_x \wedge T) \, dx.
\]

At the same time we fix a nonnegative smooth function \( \psi(x) \) with compact support such that \( \int_{\mathbb{R}} \psi(a) \, da = c \) and we set

\[
\int_{\mathbb{R}} E(M_{Y_{\varepsilon,a}}) \psi(a) \, da = c - \lambda \int_{\mathbb{R}} E \left( M_{Y_{\varepsilon,a}} \int_{0}^{S_T} \varphi_\varepsilon(y - a) \frac{\partial R}{\partial s} (Y_{\varepsilon,a}, \tau_y) \, d\tau_y \right) \psi(a) \, da.
\]

The increasing property of the function \( x \rightarrow \tau_x \) implies that \( \tau_{a+} \wedge T \leq Y_{\varepsilon,a} \leq \tau_{a+} + T \). Hence, \( Y_{\varepsilon} \) converges to \( \tau_{a+} \wedge T \) as \( \varepsilon \) tends to zero. Thus, almost surely we have

\[
\lim_{\varepsilon \to 0} M_{Y_{\varepsilon,a}} = \exp(\lambda X_{\tau_{a+} \wedge T} - \frac{1}{2} \lambda^2 V_{\tau_{a+} \wedge T}).
\]

By the dominated convergence theorem,

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}} E(M_{Y_{\varepsilon,a}}) \psi(a) \, da = \int_{\mathbb{R}} E \left( \exp(\lambda X_{\tau_{a+} \wedge T} - \frac{1}{2} \lambda^2 V_{\tau_{a+} \wedge T}) \right) \psi(a) \, da.
\]
Now, set \( F(t) = M_t \frac{\partial R}{\partial s}(t, \tau_y) \). Then, assuming that \( \varphi_\varepsilon(x) = \varepsilon^{-1} 1_{[0,1]}(x/\varepsilon) \) we have

\[
\int_{y-\varepsilon}^{y} \varphi_\varepsilon(y - a) M_{Y_{\varepsilon,a}} \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) \psi(a) \, da
\]

\[
= \frac{1}{\varepsilon^2} \int_{y-\varepsilon}^{y} 1_{[0,1]}(\frac{y-a}{\varepsilon}) F \left( \int_{a}^{a+\varepsilon} 1_{[0,1]}(\frac{x-a}{\varepsilon}) (\tau_x \land T) \, dx \right) \psi(a) \, da
\]

\[
= \int_{0}^{1} F \left( \int_{0}^{\eta} (\tau_{y+\varepsilon \eta} \land T) \, d\xi \right) \psi(y - \varepsilon \eta) \, d\eta
\]

\[
= \int_{0}^{1} F \left( \int_{0}^{\eta} (\tau_{y+\varepsilon \eta} \land T) \, d\xi + \int_{\eta}^{1} (\tau_{y+\varepsilon \eta} \land T) \, d\xi \right) \psi(y - \varepsilon \eta) \, d\eta.
\]

As \( \varepsilon \) tends to zero this expression clearly converges to

\[
\psi(y) \int_{0}^{1} F \left( \eta (\tau_y \land T) + (1 - \eta) \cap (\tau_{y+} \land T) \right) \, d\eta.
\]

So, we have proved that

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}} M_{Y_{\varepsilon,a}} \varphi_\varepsilon(y - a) \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) \psi(a) \, da = \psi(y) \int_{0}^{1} M_{\tau_{y+} + (1-z)\tau_y} \frac{\partial R}{\partial s}(z\tau_{y+} + (1-z)\tau_y, \tau_y) \, dz.
\]

(3.10)

In order to complete the proof of the theorem we will apply the dominated convergence theorem. We have the following estimate for \( y \leq S_T \)

\[
\left| \int_{\mathbb{R}} M_{Y_{\varepsilon,a}} \varphi_\varepsilon(y - a) \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) \psi(a) \, da \right| \leq \|\psi\|_{\infty} \sup_{t \leq T} \left| \frac{\partial R}{\partial s}(s,t) \right| \sup_{t \leq T} |M_t|,
\]

which allows us to commute the limit (3.10) with the integral with respect to the measure \( P \times d\tau_y \) on the set \( \{(\omega, y) : y \leq S_T(\omega)\} \). In this way we get

\[
\int_{\mathbb{R}} E(M_{\tau_y}) \psi(y) \, dy = \int_{\mathbb{R}} \psi(y) \, dy
\]

\[
- \lambda E \left( \int_{0}^{S_T} \psi(y) \int_{0}^{1} M_{\tau_{y+} + (1-z)\tau_y} \frac{\partial R}{\partial s}(z\tau_{y+} + (1-z)\tau_y, \tau_y) \, dz \, d\tau_y \right).
\]

Approximating \( 1_{[0,a]} \) by a sequence of smooth functions \( (\psi_n, n \geq 1) \) and letting \( T \) tend to infinity completes the proof.

If we assume that the partial derivative \( \frac{\partial R}{\partial t}(t, s) \) is nonnegative, then we can derive the following result.
Proposition 3.5. Assume that $X$ satisfies Hypotheses $H1)$, $H2)$ and $H3)$. If $\frac{\partial R}{\partial s}(s,t) \geq 0$, then, for all $\alpha, a > 0$ we have

(3.11) $E(\exp(-\alpha V_{\tau_a})) \leq e^{-a\sqrt{2\alpha}}.$

Proof. Since $\frac{\partial R}{\partial t}(t,s) \geq 0$, we obtain

$E(M_{\tau_a}) \leq 1,$

that is,

$E(\exp(\lambda a - \frac{1}{2} \lambda^2 V_{\tau_a})) \leq 1,$

or

$E(\exp(-\alpha V_{\tau_a})) \leq e^{-a\sqrt{2\alpha}}.$

The result follows. \qed

The above proposition means that the Laplace transform of the random variable $V_{\tau_a}$ is dominated by the Laplace transform of $\tau_a$, where $\tau_a$ is the hitting time of the level $a$ for the ordinary Brownian motion. This domination implies some consequences on the moments of $V_{\tau_a}$. In fact, for any $r > 0$ we have multiplying (3.11) by $\alpha^r$

(3.12) $E(V_{\tau_a}^{-r}) = \frac{1}{\Gamma(r)} \int_0^\infty E(e^{-\alpha V_{\tau_a}})\alpha^{r-1} d\alpha \leq \frac{1}{\Gamma(r)} \int_0^\infty e^{-a\sqrt{2\alpha}}\alpha^{r-1} d\alpha = \frac{2^r \Gamma(r + 1/2)}{\sqrt{\pi} a^{2r}}.$

On the other hand, for $0 < r < 1$

(3.13) $E(V_{\tau_a}^r) = \frac{r}{\Gamma(1-r)} \int_0^\infty (1 - E(e^{-\alpha V_{\tau_a}}))\alpha^{r-1} d\alpha \geq \frac{r}{\Gamma(1-r)} \int_0^\infty (1 - e^{-a\sqrt{2\alpha}})\alpha^{r-1} d\alpha.$

In particular, for $r \in [1/2, 1)$, $E(V_{\tau_a}^r) = +\infty$. 

Remark 3.6. If \( X \) is the standard Brownian motion, its covariance \( s \wedge t \) does not satisfy condition \( H1 \), but we still can apply our approach. It is known from [4] that \( d\tau_a \) has no absolutely continuous part and that \( \{a, \tau_a = \tau_a^+\} \) is a Cantor set hence of zero Lebesgue measure. It follows from this observation and from (3.10) that

\[
\int E(M_{\tau_y})\psi(y) \, dy = \int \psi(y) \, dy.
\]

Choosing \( \psi = 1_{[0, a]} \) yields to the expected result:

\[
E\left( \int_0^a e^{\lambda \tau_y - \frac{\lambda^2}{2} V(\tau_y)} \, dy \right) = a.
\]

If \( X \) has independent increments and satisfies \( H3 \), then

\[
E\left( e^{-\frac{\lambda^2}{2} V(\tau_a)} \right) = e^{-\lambda a}.
\]

This follows easily from the fact that \( X \) can be written as a time-changed Brownian motion.

Remark 3.7. Consider that \( X \) is a fractional Brownian motion of Hurst index \( H = 1 \). Then \( R(s, t) = st \), and consequently, \( X_t = Yt \) where \( Y \) is a one-dimensional standard Gaussian random variable. Then, \( \tau_a = \tau_a^+ = a/Y^+ \). It is then easy to compute the Laplace transform of \( \tau_a \) and we obtain

\[
E\left( \exp(-\alpha \tau_a^2) \right) = \frac{1}{2} e^{-a\sqrt{2\alpha}}.
\]

We show now that our formula also yields to the right answer. We just note that \( (y \mapsto \tau_y) \) is continuous. This entails that

\[
\frac{\partial R}{\partial s}(2z\tau_y + (1 - z)\tau_y, \tau_y) = \frac{\partial R}{\partial s}(\tau_y, \tau_y) = \frac{1}{2} V'(\tau_y)
\]

and

\[
\int_0^a E\left( \exp(\lambda y - \frac{\lambda^2}{2} V(\tau_y)) \right) \, dy =
\]

\[
a - \frac{\lambda}{2} E\left( \int_0^a \exp(\lambda y - \frac{\lambda^2}{2} V(\tau_y)) V'(\tau_y) \, d\tau_y \right).
\]

Set

\[
\Psi(a, \lambda) = E\left( \exp(\lambda a - \frac{\lambda^2}{2} V(\tau_a)) \right),
\]
then

\begin{equation}
\frac{\partial \Psi}{\partial a}(a, \lambda) = \lambda \Psi(a, \lambda) - \frac{\lambda^2}{2} E(M_{\tau_0} \frac{\partial V(\tau_0)}{\partial a}).
\end{equation}

Substitute (3.15) into (3.16) to obtain

\[ \frac{\partial \Psi}{\partial a} = 2\lambda \Psi - \lambda. \]

Then, there exists a function \( C(\lambda) \) such that

\[ \Psi(a, \lambda) = \frac{1}{2} + C(\lambda) e^{2\lambda a} \]

so that \( E(\exp(-\frac{\lambda^2}{2} \tau_0^2)) = \frac{1}{2} e^{-\lambda a} + C(\lambda) e^{\lambda a} \).

By dominated convergence, it is clear that, for any \( \lambda \),

\[ E\left(\exp(-\frac{\lambda^2}{2} \tau_0^2)\right) \xrightarrow{a \to \infty} 0, \]

thus \( C(\lambda) = 0 \) and

\[ E\left(\exp(-\frac{\lambda^2}{2} \tau_0^2)\right) = \frac{1}{2} e^{-\lambda a}. \]

Changing \( \lambda^2/2 \) into \( \alpha \) gives (3.14).

**Remark 3.8.** Consider the case of a fractional Brownian motion with Hurst parameter \( H > \frac{1}{2} \). Conditions \( H1), H2), \) and \( H3) \) are satisfied and we obtain

\[ \int_0^a E(M_{\tau_y}) \, dy = a - \lambda HE \left( \int_0^a \int_0^1 M_{\tau_y + (1-z)\tau_y} \left( [z\tau_y + (1-z)\tau_y]^{2H-1} - |z(\tau_y + \tau_y)|^{2H-1} \right) \, dz \, d\tau_y \right). \]

Moreover, \( E(e^{-\alpha \tau_0^2}) \leq e^{-\alpha \sqrt{2\lambda}} \), and therefore, \( E(\tau_0^p) < \infty \) if \( p < H \). According to (3.13), \( E(\tau_0^p) \) is infinite if \( pH > 1/4 \) and (3.12) entails that \( \tau_0 \) has finite negative moments of all orders.

**REFERENCES**


