Read the problems carefully. Solve them according to the requirements and show the key intermediate steps to receive full credits. You may use a simple calculator, but you may use neither books nor notes.
1. (10 points) Tell whether the given equations are linear or nonlinear and determine their order.

(a) \(3y'y = 2t^2\). **Nonlinear, 1st Order**

(b) \((\sqrt{3} - 1)t^3y'' + t^2y' - 3y + \cos t = 0\). **Linear, 2nd order**

(c) \(y' + 2e^y = 0\). **Nonlinear, 1st order**

(d) \(y'' + \cos y = 0\). **Nonlinear, 2nd order**

(e) \(2x''' - 7xx'' + x' + 9x = 0\). **Nonlinear, 3rd order**

2. (15 points) Consider the IVP (initial value problem)

\[ty' = -y + \sin(t), \quad y\left(\frac{\pi}{2}\right) = 0.\]

(a) Find the solution.

**Solution.** Rewrite the equation into

\[y' + \frac{1}{t} y = \frac{1}{t} \sin(t)\]

Then,

\[\mu(t) = e^\int \frac{1}{t} dt = e^{\ln t} = t\]

and

\[y = \frac{1}{t} \left[ \int \sin(t) dt + C \right] = \frac{1}{t} \left[ -\cos(t) + C \right]\]

The IC leads to \(C = 0\). Thus,

\[y = \frac{-\cos(t)}{t}\]

(b) Determine how the solution behaves as \(t \to \infty\).

**Solution.**

\[y(t) \to 0 \text{ as } t \to \infty\]

3. (15 points) Consider the differential equation

\[\frac{dy}{dt} = \frac{\sin(t)}{y^2}.\]

(a) Find the general solution using the technique for separable equations.

**Solution.** Rewrite the DE as

\[y^2 dy = \sin(t) dt, \quad \text{or} \quad \int y^2 dy = \int \sin(t) dt\]

Thus, we get

\[\frac{1}{3} y^3 = -\cos(t) + C \quad \text{or} \quad y = (-3\cos(t) + C)^{\frac{1}{3}}\]

(b) Find the solution satisfying the initial condition \(y(\pi/2) = 0\).

**Solution.** The IC \(y(\pi/2) = 0\) yields \(C = 0\). Thus we have

\[y = -\left(3\cos(t)\right)^{\frac{1}{3}}\]
4. (20 points) Sketch the direction field and draw some integral curves for the differential equation
\[ y' = y^2 - 2y. \]

Based on your results, state the equilibrium solutions and determine their stability.

**Solution.**

<table>
<thead>
<tr>
<th>Slope</th>
<th>Equation</th>
<th>Isoclines</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( y^2 - 2y = 0 )</td>
<td>( y = 0 ) or ( y = 2 )</td>
</tr>
<tr>
<td>1</td>
<td>( y^2 - 2y = 1 )</td>
<td>( y = 1 + \sqrt{2} ) or ( y = 1 - \sqrt{2} )</td>
</tr>
<tr>
<td>-1</td>
<td>( y^2 - 2y = -1 )</td>
<td>( y = 1 )</td>
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</tbody>
</table>

The equilibrium solutions are \( y = 0 \) and \( y = 2 \).

\( y = 0 \) is stable whereas \( y = 2 \) is unstable.

5. (20 points) Consider the autonomous differential equation
\[ y' = (1 - y^2)e^{-y}. \]

Find the equilibrium solutions and sketch some typical integral curves in the \( t - y \) plane. Based on the results you obtain, state the stability of the equilibrium solutions.

**Solution.** Set \( (1 - y^2)e^{-y} = 0 \). We get \( y = +1 \) or \( y = -1 \). Thus, the equilibrium solutions are \( y = +1 \) and \( y = -1 \).

From the figure on the right, it is known that \( y = 1 \) is stable whereas \( y = -1 \) is unstable.

6. (20 points) A person wants to borrow $110,000 to buy a house. The lender will charge interest at an annual rate of 8.25\%. Assuming that interest is compounded continuously and that the borrower will make payment continuously at a constant monthly rate \( k \), can you tell the person how much the monthly payment (i.e. \( k \)) will be in order to pay off the loan in thirty years?

**Solution.** The balance equation and the initial condition are
\[ \frac{dB}{dt} = \frac{0.0825}{12}B - k, \quad B(0) = 110,000 \]

We want to find \( k \) such that \( B(360) = 0 \).

The solution of the IVP is
\[ B(t) = \frac{12k}{0.0825} + \left( 110,000 - \frac{12k}{0.0825} \right) e^{\frac{0.0825}{12}t}. \]

Setting \( B(360) = 0 \) we get
\[ k = \frac{110,000 \times 0.0825}{12} \cdot \left[ e^{\frac{0.0825}{12} \times 30} - 1 \right] \approx \$825.73 \]

8.25\% per year

\( B(t) \): balance at time \( t \)

\( k \): monthly payment