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Betz Invariants and Generalization of Vorticity Moment Invariants

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The widely used Betz invariants can be categorized as invariants of vorticity moments or impulses in plane motion for mostly inviscid vortex flow. These invariants have served as a foundation of vortex roll-up process analysis as well as benchmarks to estimate numerical simulation errors for vortex flow. The analysis starts from a more general integral divergence relation in rational mechanics to extend the analysis for both inviscid and viscous vortex flow cases, with or without finite boundaries. Conditions for the existence of the Betz invariants and their extensions to include viscous effects are discussed. These extensions can be used as analytical means for theoretical modeling of complicated vortex systems, especially in flows with viscous effects and finite boundaries. The goal of this study is to clearly establish when the Betz invariants and the generalized vorticity moment invariants are applicable. The approach uses rigorous mathematical analysis, and the physical interpretations of the results are discussed. A trailing vortex pair is used as an example to illustrate the applications of the theoretical study.

I. Introduction

VORTICITY moments, with their several lower orders also defined as vorticity impulses, have been studied mostly in three dimensions. However, it is nontrivial to extend the three-dimensional vorticity moment theorems to plane motion.[1] The underlying reason for the difference is in the fact that vorticity is no longer a vector in two dimensions. A simple example is that in plane motion the total area integration of vorticity is the circulation, as a result of the contour integral residual, whereas in three-dimensional flow the total volumetric integration of vorticity is zero (for bounded vorticity).

Truesdell[2] compared the two-dimensional vorticity integrals of Poincaré and Hamel. Poincaré's results apply to motions of the whole plane and show that constancy exists for the following vorticity moments in incompressible inviscid flow:

$$\Gamma \equiv \int_A \zeta \, \mathrm{d}A \tag{1}$$

$$\Gamma_y \equiv \int_A y\zeta \, \mathrm{d}A \tag{2}$$

$$\Gamma_z \equiv \int_A z\zeta \, \mathrm{d}A \tag{3}$$

$$\Gamma_r \equiv \int_A r^2\zeta \, \mathrm{d}A \tag{4}$$

where the two-dimensional vorticity is defined as $\zeta = \partial w/\partial y - \partial v/\partial z$, A is the material area where the vorticity moments are considered, and $r^2 = y^2 + z^2$. The constancy of these integrals was used by Betz[3] to establish the so-called Betz invariants. The Betz invariants have been widely used as a guideline for predictions of vortex roll-up processes after lifting wings.

Poincaré extended the validity of the integrals to motions of homogeneous viscous incompressible fluid filling the whole plane and showed[2] that the first three integrals remain constant and the fourth one satisfies

$$\frac{\mathrm{d}\Gamma_r}{\mathrm{d}t} = 4\nu\Gamma \tag{5}$$

where ν is the kinematic viscosity. Hamel investigated the vanishing of the vorticity integrals for viscous incompressible fluid and found that for a closed domain with no slipping on all of the boundaries the following condition exists[2]:

$$\int_A \psi\zeta \, \mathrm{d}A = 0 \tag{6}$$

where ψ is a harmonic function in A, which is a finite domain with all of the boundaries no slipping. Donaldson and Bilanin[4] listed the four integrals under viscous flow conditions in aircraft wake vortex cases.

In this paper we start from an integral divergence relation derived by Howard[5] to generalize the rates of change of vorticity moments. One purpose is to investigate the validity of the Betz method by using rigorous mathematical analysis. The physical insight offered by the analytical results is explained in detail. The other is to extend them to the types of boundaries that are not categorized in the theorems by either Poincaré or Hamel, i.e., an infinite domain with one symmetry boundary and an infinite domain with one symmetry boundary and one no-slip boundary.

These domains typically appear in two-dimensional aircraft vortex analysis, and therefore a trailing vortex pair is used among the following discussion as an example to demonstrate the applications of the theoretical results presented in this paper. In the aircraft wake vortex case, the two-dimensional theory is used as an approximation to be applied to the three-dimensional steady far-wake problem. This approximation is based on the assumption of the existence of similarity solution in a plane (the y–z plane in Fig. 1) normal to the flight direction (the x direction). In this sense the time t should be viewed as x/V_∞, where V_∞ is the airplane flying speed. The unsteady effects in the two-dimensional theory are thus indicated as the freestream convection effects.

II. Vorticity Moment Rates in Two Dimensions

By using the two-dimensional vorticity transport equation, Howard[5] obtained the rates of change of vorticity moments for plane flow as

$$\int_A f\frac{\partial\zeta}{\partial t}\,\mathrm{d}A = \int_A \nabla_{yz}\cdot[(\nu\nabla_{yz}\zeta - \boldsymbol{q}\zeta)f - \nu\zeta\nabla_{yz}f]\,\mathrm{d}A + \int_A(\boldsymbol{q}\cdot\nabla_{yz}f)\zeta\,\mathrm{d}A + \int_A \nu(\Delta_{yz}f)\zeta\,\mathrm{d}A \tag{7}$$

where f is a continuously differentiable function of (y, z) and $\boldsymbol{q} = (v, w)$ is the velocity vector. The Nabla operator in the y–z plane is defined as

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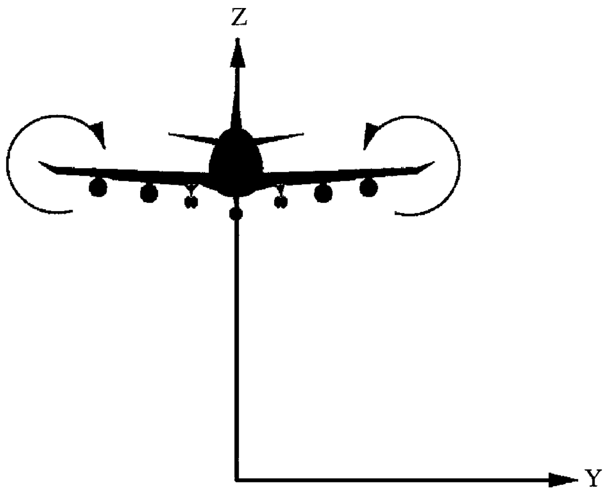

Fig. 1 Illustration of the coordinate system used in the discussion with an example of aircraft trailing vortices.

$$\nabla_{yz} = \frac{\partial}{\partial y}\boldsymbol{e}_y + \frac{\partial}{\partial z}\boldsymbol{e}_z, \qquad \Delta_{yz} = \nabla^2_{yz} = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The coordinate system following aeronautical convention is chosen so that the vorticity is in the x direction in the three-dimensional sense. We can write the first term on the right-hand side (RHS) of Eq. (7) as a contour integral using Green's theorem:

$$\int_A f\frac{\partial \zeta}{\partial t}\,\mathrm{d}A = \oint_L \boldsymbol{B}^c\cdot \mathrm{d}\boldsymbol{l} + \int_A (\boldsymbol{q}\cdot\nabla_{yz} f)\zeta\,\mathrm{d}A + \int_A \nu(\Delta_{yz} f)\zeta\,\mathrm{d}A \tag{8}$$

where $\boldsymbol{B} = (\nu\nabla_{yz}\zeta - \boldsymbol{q}\zeta)f - \nu\zeta\nabla_{yz} f$ and $\boldsymbol{B}^c = (-B_z, B_y)$ and L is the boundary curve of A. By specifying different expressions of f, the rates of change of different vorticity moments can be obtained. The following four rates of change of vorticity moments are discussed, corresponding to the Poincaré integrals and the Betz invariants, with $f = 1$, y, z, and r:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int_A \zeta\,\mathrm{d}A = \int_A \frac{\partial\zeta}{\partial t}\,\mathrm{d}A \tag{9}$$

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int_A y\zeta\,\mathrm{d}A = \int_A y\frac{\partial\zeta}{\partial t}\,\mathrm{d}A \tag{10}$$

$$\frac{\mathrm{d}\Gamma_z}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int_A z\zeta\,\mathrm{d}A = \int_A z\frac{\partial\zeta}{\partial t}\,\mathrm{d}A \tag{11}$$

$$\frac{\mathrm{d}\Gamma_r}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int_A r^2\zeta\,\mathrm{d}A = \int_A r^2\frac{\partial\zeta}{\partial t}\,\mathrm{d}A \tag{12}$$

Equation (9) is the rate of change of the zeroth-order moment of vorticity, which is the total circulation change rate. Equations (10) and (11) are the rates of change of the two first-order moments with respect to y and z coordinates, respectively. Equation (12) is the rate of change of the second-order polar vorticity moment with respect to the origin. Following Saffman's[6] definition of I_y, I_z, and A_r, which are the y-direction impulse, the z-direction impulse, and the angular impulse, respectively, then $I_y = \Gamma_z$, $I_z = -\Gamma_y$, and $A_r = -1/2\Gamma_r$.

III. Flow in an Infinite Domain with Bounded Vorticity

In this case the initial vorticity is bounded or decays exponentially in radius r, and consequently for all of the vorticity in r for $t \geq 0$

$$\zeta(\boldsymbol{r}, t) = \mathcal{O}(e^{-kr}) \tag{13}$$

for a positive constant k. From the Biot–Savart theorem, when $r \to \infty$,

$$|\boldsymbol{q}| = \mathcal{O}(r^{-1}) \tag{14}$$

Because the contour integral (the first term) on the RHS of Eq. (8) is at the infinite radius for an infinite domain, it can be shown that the contour integral is zero for $f = 1$, y, z, and r^2 because of Eqs. (13) and (14).

For $f = 1$ the second and third terms on the RHS of Eq. (8) vanish; hence,

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0 \tag{15}$$

That is, the total circulation is a constant.

For $f = y$ the contour integral and the third term on the RHS of Eq. (8) vanish. The second term, which becomes $\int_A v\zeta\,\mathrm{d}A$, can be proved to be zero by integrating by parts and invoking incompressibility and the velocity decay condition, Eq. (14). It is thus proved that

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t} = 0 \tag{16}$$

With a similar procedure it can be shown that

$$\frac{\mathrm{d}\Gamma_z}{\mathrm{d}t} = 0 \tag{17}$$

Therefore, impulses in both the y direction and the z direction are invariants in this case.

It can also be proved that the total momentum can be expressed as converged integrations

$$\int_A v\,\mathrm{d}A = \Gamma_z = I_y \tag{18}$$

$$\int_A w\,\mathrm{d}A = -\Gamma_y = I_z \tag{19}$$

That means that all of the impulses created by vorticity go to the momentum of the fluid and they are invariants all of the time. The convergence of the area integrations of velocity is not mathematically guaranteed because the velocity is in the order of r^{-1} for the case of $\Gamma \neq 0$ (Refs. 6 and 7). A mathematically sufficient condition is that the integrand is in the order of $r^{-(2+p)}$, where $p > 0$. However, both the y moment and the z moment of vorticity can be mathematically proved to be converged quantities, because ζ is an exponentially decaying function when $r \to \infty$. For a single vortex located at the origin, Γ_y and Γ_z are zero. Therefore, the total y-direction momentum and total z-direction momentum are zero, too, although the total kinetic energy of a single vortex is infinite, diverging at a rate of $\ln r$ (when $r \to \infty$). On the other hand, the total θ-direction momentum is also infinite, diverging at a rate of r.

For the polar vorticity moment the second and third integral terms on the RHS of Eq. (8) are

$$2\int_A (yv + zw)\zeta\,\mathrm{d}A + 4\nu\int_A \zeta\,\mathrm{d}A = 2\int_A \boldsymbol{r}\cdot\boldsymbol{q}\zeta\,\mathrm{d}A + 4\nu\Gamma \tag{20}$$

It can be proved that

$$\int_A \boldsymbol{r}\cdot\boldsymbol{q}\zeta\,\mathrm{d}A = 0 \tag{21}$$

Hence,

$$\frac{\mathrm{d}\Gamma_r}{\mathrm{d}t} = 4\nu\Gamma \tag{22}$$

Equation (22) shows for inviscid flow the vorticity polar moment or the angular impulse is an invariant, and along with Eqs. (15–17) there are four invariants: the total circulation, the y- and z-direction impulses, and the angular impulse. This model, first introduced by Betz,[3] has been widely used in vortex methods for calculations of aircraft wing vortex roll-up processes.[4,8]

It can be seen from Eq. (22) that the appearance of viscosity invalidates the existence of the angular vorticity moment invariant (unless the total circulation is zero, the case being discussed later), which causes the only difference between inviscid and viscous flow in this case. For viscous flow the total circulation and y and z vorticity moments are invariants. Because $\Gamma = \Gamma_0$, we can have, from Eq. (22),

$$\Gamma_r = \Gamma_{r0} + 4\nu\Gamma_0 t \tag{23}$$

Equations (15–17) and (22) for viscous flow are exactly the same as obtained by Poincaré in 1893 (Ref. 2). It can be seen that the vorticity polar moment increases with time infinitely. The infiniteness is caused by the assumption that a net, single-sign circulation exists in an infinite domain. Because of the diffusion of the vorticity, the vorticity polar moment becomes larger and larger. At the same time the total amount of vorticity integration (circulation) is a nonzero constant. Those two effects can lead to an infinitely increasing vorticity polar moment. In a highly diffusive field (low-Reynolds-number flow) the vorticity polar moment increases very quickly. However, as stated in Betz' Theorem 5 (Ref. 3), it is physically impossible to create a net nonzero circulation in an infinite domain, because it requires an infinite amount of kinetic energy and momentum with velocity field behaving as in Eq. (14). In aircraft vortex wakes vortices with opposite signs are created at the port side and the starboard side of the wing. Near no-slip boundaries vortices are always accompanied by opposite sign vorticity produced on the boundaries. Therefore, for a physically possible flow in an infinite domain, the total circulation throughout the whole domain remains zero. Thus, $\Gamma_0 = 0$, and $\Gamma_r = \Gamma_{r0}$. That is, the vorticity polar moment is an invariant for flow in an infinite domain. [With $\Gamma_0 = \Gamma = 0$ from Eq. (15), it can be shown[6,7] that $|\boldsymbol{q}| = \mathcal{O}(r^{-2})$ for $r \to \infty$.]

For inviscid flow, because there is no viscous diffusion, the polar moment is obviously conserved. The existence of the four Betz invariants are related with the infinite domain assumption. The following sections will show that modifications are required if finite boundary effects exist.

IV. Flow with One Symmetry (Slip) Boundary

We assume that there is a symmetry boundary at $y = 0$ and all of the other boundaries are the infinite boundaries. The boundary conditions at $y = 0$ are $v|_{y=0} = 0$, $\zeta|_{y=0} = 0$, and $\partial w/\partial y = 0$. It needs to be pointed out that because the flow has one symmetry boundary at $y = 0$ (i.e., the velocity field is symmetric) the vorticity field is antisymmetric with respect to $y = 0$. The vorticity at one side of the symmetry boundary is opposite to that at the other side. The total circulation including both sides of the symmetry boundary is zero. Thus we have $|\boldsymbol{q}| = \mathcal{O}(r^{-2})$, as discussed in Sec. III.

In the discussion in this section, only half of the flow domain ($y > 0$) is considered, contrary to the flowfield in Sec. III. When one side of an aircraft wing is considered, the wake vortex field is the case here as shown in Fig. 1, with the symmetry line at the fuselage. In this case the Betz invariants have been used to establish a foundation for vortex roll-up analysis after aircraft wings. Donaldson and Bilanin[4] also listed the vorticity moment rates in viscous flow for this case. Here we start from our general formulation [Eq. (8)] to discuss the validity of these invariants based on the mathematical analysis.

For $f = 1$ only the contour integral in Eq. (8) is left. The contour integral vanishes at the three infinite boundaries as a result of the vorticity and velocity decay conditions at infinity, leaving

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = -\nu \int_{-\infty}^{\infty} \left.\frac{\partial \zeta}{\partial y}\right|_{y=0} \mathrm{d}z \tag{24}$$

Equation (24) is the same as Eq. (1.17) in Ref. 4, which shows that the circulation is conserved for inviscid flow, as stated in the Kelvin theorem. For viscous flow the circulation change at one side of a symmetric flowfield is caused by the vorticity gradient on the symmetry boundary where viscous diffusion effects occur because the RHS of Eq. (24) can be rewritten as $-\nu \int_{-\infty}^{\infty} (\Delta_{yz} w)_{y=0}\,\mathrm{d}z$. The diffusion leads to cancellation of the opposite sign vorticity diffused from each side toward the symmetry boundary. For the vortex pair case in Cantwell and Rott,[9] an asymptotic circulation for the Oseen vortex near $t \to \infty$ has been derived as

$$\Gamma_{\mathrm{as}}(t) = \Gamma_y / \sqrt{4\pi\nu t} \tag{25}$$

where Γ_y can be shown later to be a constant with only one symmetry boundary. It is not just coincident that the circulation reduces when the vortex core size [as represented in the Oseen vortex as $\sqrt{(4\nu t)}$] increases. The direct relation between the circulation decay and the vortex core size is because the vorticity is concentrated inside the vortex core. At $t \to \infty$ the core grows to the region near the symmetry boundary. Thus, the core growth directly causes the cancellation of the vorticity, resulting in circulation decay in Eq. (24). For high-Reynolds-number flow, because of the slow growth of the vortex core, cancellation of vorticity happens at a very slow pace, and therefore the circulation can be approximated to be constant. It is noticed that the circulation invariant is independent of any translational motion of a frame of references in which the vortex system is described. Such a motion influences the shape of the streamlines around vortices.[8]

Based on the momentum equation in the z direction, Eq. (24) can also be rewritten as

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = -\int_{-\infty}^{\infty} \left.\frac{\partial w}{\partial t}\right|_{y=0} \mathrm{d}z \tag{26}$$

which means that the circulation decay is caused by the overall unsteadiness of the z-direction component of velocity on the symmetry boundary. For inviscid flow the integration of the unsteadiness turns out to be zero (which is equal to $-\partial\phi_{y=0}/\partial t|_{z=-\infty}^{\infty}$, where ϕ is the velocity potential). For viscous flow there is a net overall unsteadiness. The sign of this overall unsteadiness can be determined by $\partial\zeta/\partial y$ at $y = 0$. For a vortex pair $\partial\zeta/\partial y$ has the same sign as the circulation of the right side vortex because of the antisymmetry of the vorticity field. In a more general case the net effect is that the integration of the vorticity gradient on the symmetry boundary always has the same sign as the total circulation in that half-infinite domain. That is, the circulation always decays in viscous flow with only one symmetry boundary. The pace of the decay depends on the Reynolds number.

For $f = y$, with the boundary conditions and $\int_A v\zeta\,\mathrm{d}A = 0$ (which can be proved in a similar way as in the infinite domain case), we have

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t} = 0 \tag{27}$$

That is, the y moment of vorticity is an invariant so that the z direction impulse is conserved, even for viscous flow with a symmetry boundary. This is a basic condition that the heuristic model in Cantwell and Rott[9] was based on. The z-direction vorticity impulse can also be proved to be equal to the total momentum in the z direction. For vortex pairs

$$\Gamma_y = -\int w\,\mathrm{d}A = \Gamma(t)b(t) \tag{28}$$

where $b(t)$ is the instantaneous half-span of the vortex pair and the negative sign means that the net impulse is toward the minus z direction. In the preceding expression, because Γ, b, and vorticity are inertial-frame-of-reference indifferent but velocity is not, w is the velocity induced by the vortex system and vanishes at infinity. Because the circulation monotonically decays with time, the vortex spacing must increase monotonically with time to keep the impulse being a constant.

It is also shown that

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t} = -\int_A \frac{\partial w}{\partial t}\,\mathrm{d}A = 0 \tag{29}$$

with a symmetry boundary. In comparison with Eq. (26), it can be seen that the unsteadiness of w cancels out when integrated throughout the whole domain, resulting in a constant y moment of vorticity. This constant is the second Betz invariant. The existence of this invariant is inertial-frame-of-reference independent, although the absolute value of Γ_y can change in different reference systems.

For $f = z$ the terms left are

$$\frac{\mathrm{d}\Gamma_z}{\mathrm{d}t} = -\nu \int_{-\infty}^{\infty} z \left.\frac{\partial \zeta}{\partial y}\right|_{y=0} \mathrm{d}z + \int_A w\zeta\,\mathrm{d}A \tag{30}$$

where the first term is from the contour integral and the second term is from the first area integral on the RHS of Eq. (8). Integrating by parts and invoking the boundary conditions and incompressibility, it can be shown that

$$\int_A w\zeta \, \mathrm{d}A = -\frac{1}{2}\int_0^\infty w^2|_{y=0} \, \mathrm{d}z$$

Hence,

$$\frac{\mathrm{d}\Gamma_z}{\mathrm{d}t} = -\int_{-\infty}^\infty \left(\frac{w^2}{2} + \nu z\frac{\partial \zeta}{\partial y}\right)_{y=0} \mathrm{d}z \tag{31}$$

The preceding equation agrees with Eq. (1.19) in Ref. 4. It shows that the z moment of vorticity is a more complicated case. It needs to be mentioned again that in Eq. (31) w is the velocity induced by the vortex system. It does not and cannot include velocity as a result of a moving frame of references. Therefore, it is not an invariant for the case pointed out in Rossow's[8] Fig. 5(a) plotted in a moving frame of references.

It is noted that Γ_z, or the y-direction vorticity impulse, is not equal to the total momentum in the y direction, in contrast to the y moment of vorticity, which is equal to the (negative) total momentum in the z direction. It is caused by the fact that

$$\Gamma_z = \int_A v \, \mathrm{d}A - \int_{-\infty}^\infty zw|_{y=0} \, \mathrm{d}z \tag{32}$$

The first integral, the total momentum in the y direction, has actually been proved to be constant all of the time[9] for flow with a symmetry boundary. Based on that, we can have

$$\frac{\mathrm{d}\Gamma_z}{\mathrm{d}t} = -\int_{-\infty}^\infty z\left.\frac{\partial w}{\partial t}\right|_{y=0} \mathrm{d}z \tag{33}$$

By using the z-direction momentum equation and comparing with Eq. (31), we obtain

$$\int_{-\infty}^\infty p|_{y=0} \, \mathrm{d}z = 0 \tag{34}$$

That is, there is no total pressure force on the symmetry boundary, which is intuitive because there is no change of the total momentum in the y direction. This condition has been used in Cantwell and Rott[9] to calculate the vortex drift.

It can be seen from Eq. (31) that even for inviscid flow Γ_z is not an invariant. Thus, using the z moment of vorticity as an invariant in the Betz model needs a second thought. The concentric streamline assumption[8] has to be satisfied. That is, the influence from the vortices at the other side of the symmetry plane is neglected. Then it would be equivalent to the case when the symmetry plane is infinitely far away from the centroid of the vortex system, and therefore w is approximated to be zero at $y=0$. Only under such a restrictive condition is Γ_z an invariant.

For $f = r^2$ we have

$$\frac{\mathrm{d}\Gamma_r}{\mathrm{d}t} = -\nu\int_{-\infty}^\infty z^2\left.\frac{\partial \zeta}{\partial y}\right|_{y=0} \mathrm{d}z + 2\int_A \boldsymbol{r}\cdot\boldsymbol{q}\zeta \, \mathrm{d}A + 4\nu\Gamma \tag{35}$$

where the first term is from the contour integral and the second and third terms are from the second and third terms on the RHS of Eq. (8), respectively. It can be proved, using the similar integrating-by-parts method in Sec. III and the boundary conditions, that

$$\int_A \boldsymbol{r}\cdot\boldsymbol{q}\zeta \, \mathrm{d}A = -\int_{-\infty}^\infty z\left.\frac{w^2}{2}\right|_{y=0} \mathrm{d}z \tag{36}$$

The convergence of Eq. (36) is guaranteed with $|\boldsymbol{q}| = \mathcal{O}(r^{-2})$ and $\zeta = \mathcal{O}(e^{-kr})$ at $r \to \infty$. Substituting Eq. (36) into Eq. (35), we have

$$\frac{\mathrm{d}\Gamma_r}{\mathrm{d}t} = -\int_{-\infty}^\infty z\left(\nu z\frac{\partial \zeta}{\partial y} + w^2\right)_{y=0} \mathrm{d}z + 4\nu\Gamma \tag{37}$$

Thus, the rate of change of vorticity polar moment is not an invariant, even for inviscid flow. However, in the Betz model[3] a different moment can be defined (which Saffman[6] used to define a radius of gyration):

$$\Gamma_{\bar{r}} \equiv \int_A [(y-\bar{y})^2 + (z-\bar{z})^2]\zeta \, \mathrm{d}A \tag{38}$$

where $\bar{y}$ and $\bar{z}$ are the coordinates of the vorticity centroid, defined as Γ_y/Γ and Γ_z/Γ, respectively. Hence,

$$\frac{\mathrm{d}\Gamma_{\bar{r}}}{\mathrm{d}t} = \frac{\mathrm{d}\Gamma_r}{\mathrm{d}t} - 2\bar{y}\frac{\mathrm{d}\Gamma_y}{dt} - 2\bar{z}\frac{\mathrm{d}\Gamma_z}{\mathrm{d}t} + (\bar{y}^2 + \bar{z}^2)\frac{\mathrm{d}\Gamma}{\mathrm{d}t} \tag{39}$$

For inviscid flow, because both Γ_y and Γ are invariants, we have from Eqs. (31) and (37)

$$\frac{\mathrm{d}\Gamma_{\bar{r}}}{\mathrm{d}t} = -\int_{-\infty}^\infty (z-\bar{z})w^2|_{y=0} \, \mathrm{d}z \tag{40}$$

If $w|_{y=0}$ is symmetric with respect to $(z-\bar{z})$, then the integral in Eq. (40) is zero, and $\Gamma_{\bar{r}}$ is an invariant. Such symmetry occurs at the far field of the aircraft wake where a pair of counter-rotating wake vortices can be approximated with Oseen vortices. This symmetry condition is more restrictive than necessary when Eq. (40) is rewritten in the following form after using the Bernoulli equation:

$$\frac{\mathrm{d}\Gamma_{\bar{r}}}{\mathrm{d}t} = \frac{2}{\rho}\int_{-\infty}^\infty (z-\bar{z})p|_{y=0} \, \mathrm{d}z \tag{41}$$

where p_∞ is assumed zero. Betz[3] discussed the condition when $\Gamma_{\bar{r}}$ remains constant. He combined one part of the vortices into one group and the other into another group under the condition that the total circulation of the one group be equal and opposite to that of the other. That case is a typical flowfield with a symmetry boundary. He stated that when the two groups are separate to a certain extent and closed in themselves the force from the other group almost always passes very close to the vorticity centroid of this group and we can then consider $\Gamma_{\bar{r}}$ approximately as constant. Under large separation assumption both Γ_z and Γ_r can be invariants, too, as already discussed. However, the large separation assumption is more restrictive than just having force going through the vorticity centroid. Therefore, assuming constancy of $\Gamma_{\bar{r}}$ is a better approximation than assuming constancy of Γ_z.

V. Flow with One Slip Boundary and One No-Slip Boundary

We put a no-slip boundary at $z=0$ in addition to the symmetry boundary at $y=0$ and analyze only the flowfield in the first quadrant in Fig. 1. The boundary conditions are $v|_{y=0}=0$, $\zeta|_{y=0}=0$ (for the symmetry boundary); $v|_{z=0}=0$, $w|_{z=0}=0$ (for the no-slip boundary). The other two boundaries are infinite boundaries. Because no-slip boundaries only occur in viscous flow, the following discussion is concentrated on how viscous effects are approximated in the theoretical treatment near the no-slip boundary. The flowfield with two slip boundaries (at both $y=0$ and $z=0$) is not discussed in detail, and the reader is referred to Ref. 6 for more discussion. A pair of aircraft wake vortices approaching the ground (at $z=0$) is an example of the flow structures under discussion in this section.

Using Eq. (8) and following the similar procedures in preceding sections, it can be shown that

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = -\nu\int_0^\infty \left.\frac{\partial \zeta}{\partial y}\right|_{y=0} \mathrm{d}z - \nu\int_0^\infty \left.\frac{\partial \zeta}{\partial z}\right|_{z=0} \mathrm{d}y \tag{42}$$

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t} = -\nu\int_0^\infty y\left.\frac{\partial \zeta}{\partial z}\right|_{z=0} \mathrm{d}y \tag{43}$$

$$\frac{\mathrm{d}\Gamma_z}{\mathrm{d}t} = -\int_0^\infty \left(\frac{w^2}{2} + \nu z\frac{\partial \zeta}{\partial y}\right)_{y=0} \mathrm{d}z + \nu\int_0^\infty \zeta\Big|_{z=0} \mathrm{d}y \tag{44}$$

$$\frac{\mathrm{d}\Gamma_r}{\mathrm{d}t} = -\int_0^\infty z\left(\nu z\frac{\partial \zeta}{\partial y} + w^2\right)_{y=0} \mathrm{d}z + 4\nu\Gamma \tag{45}$$

The convergence conditions of the integrations are satisfied because of Eq. (13) and the $\mathcal{O}(r^{-2})$ decay of velocity at infinity (with the symmetry of the flow). Equations (42–45) show complicated behavior of the rates of vorticity moments when a no-slip boundary is included. However, we can still deduce simplified theoretical results if the flow is at high Reynolds numbers and no separation occurs on the no-slip boundary. Under these conditions viscous diffusion on $y = 0$ can be neglected, and Prandtl's boundary-layer assumptions can be employed at $z = 0$.

For the high-Reynolds-number cases, when the initial spacing between the two vortices in the wake vortex pair is large in comparison with the vortex core, viscous diffusion on the symmetry boundary can be neglected at early time stages. Then,

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = -\nu \int_0^\infty \left.\frac{\partial \zeta}{\partial z}\right|_{z=0} \mathrm{d}y = \frac{1}{\rho}\int_0^\infty \left.\frac{\partial p}{\partial y}\right|_{z=0} \mathrm{d}y \tag{46}$$

Using the boundary-layer assumptions to the order of $\mathcal{O}(Re^{-0.5})$, we get $\partial p/\partial y|_{z=0} = -\rho(\partial V/\partial t + 1/2\partial V^2/\partial y)$, where V is the outer flow inviscid velocity on $z = 0$. Therefore,

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = -\frac{\partial}{\partial t}\int_0^\infty V\,\mathrm{d}y = \left.\frac{\partial \phi}{\partial t}\right|_{z=0,\,y=0} \tag{47}$$

where ϕ is the velocity potential of the outer inviscid flow. Hence,

$$\Gamma = -\int_0^\infty V\,\mathrm{d}y + C_0 \tag{48}$$

where C_0 is an integration constant. Furthermore, from the definition of Γ

$$\Gamma = -\int_0^\infty w|_{y=0}\,\mathrm{d}z \tag{49}$$

Comparing Eqs. (48) and (49) shows that

$$C_0 = -\int_0^\infty w|_{y=0}\,\mathrm{d}z + \int_0^\infty V\,\mathrm{d}y \tag{50}$$

which is exactly equal to the circulation in the first quadrant with the bottom boundary located at the outer edge of the unseparated thin boundary layer. (In the aircraft wake vortex problem this is the circulation generated by the aircraft). It is now shown that this quantity is a constant for high-Reynolds-number flow without diffusion on the symmetry boundary and when the no-slip boundary layer is not separated. It is not a surprising result. It can be seen from Eq. (50) that C_0 behaves the same as in an inviscid flow with two symmetry boundaries. According to Sec. IV, it is an invariant.

Equation (48) means that the total circulation is equal to the inviscid circulation minus the circulation from the secondary vorticity generated inside the boundary layer. The amount of the boundary-layer circulation is equal to $\int_0^\infty V\,\mathrm{d}y$. In the problem of a vortex pair approaching the ground, $V > 0$; therefore, $\Gamma < C_0$. The difference between C_0 and Γ is caused by friction effects of the no-slip boundary. This statement is only true if the viscous boundary layer does not separate. If there is flow separation on $z = 0$, which does happen in the problem of a vortex pair approaching close to the ground,[10,11] Eq. (42) has to be used.

The once conserved Γ_y in the preceding two cases is no longer an invariant with a no-slip boundary (which can still be proved equal to the negative total momentum in the z direction). From momentum equations and Eq. (43) it can be shown that

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t} = -\frac{1}{\rho}\int_0^\infty \left. p\right|_{z=0} \mathrm{d}y \tag{51}$$

which means that the z-direction vorticity impulse or total momentum change is caused by the total force in the z direction caused by the pressure exerted on the wall surface (assuming $p_\infty = 0$).

Without separation on the no-slip boundary, it can be proved that

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t} = \int_0^\infty \frac{\partial \phi|_{z=0}}{\partial t}\,\mathrm{d}y + \frac{1}{2}\int_0^\infty V^2\,\mathrm{d}y \tag{52}$$

For the problem of a vortex pair approaching the ground, we can have, using the potential flow theory,

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t} = -\frac{\mathrm{d}y_0}{\mathrm{d}t}\int_0^\infty V\,\mathrm{d}y + \frac{1}{2}\int_0^\infty V^2\,\mathrm{d}y \tag{53}$$

where y_0 is the y coordinate of the center of the first quadrant vortex. Following Saffman,[6]

$$\frac{\mathrm{d}y_0}{\mathrm{d}t} = \frac{1}{2C_0}\int_0^\infty V^2\,\mathrm{d}y \tag{54}$$

Hence,

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t} = \frac{\mathrm{d}y_0}{\mathrm{d}t}\left(-\int_0^\infty V\,\mathrm{d}y + C_0\right) \tag{55}$$

Noting Eq. (48), Eq. (55) becomes

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t} = \Gamma\frac{\mathrm{d}y_0}{\mathrm{d}t} = \frac{\Gamma}{C_0}\frac{\mathrm{d}C_{0y}}{\mathrm{d}t} \tag{56}$$

where C_{0y} represents the y moment of vorticity if the boundary at $z = 0$ is also a slip boundary. This means that the rate of change of z-direction vortex impulse or momentum is equal to the (negative) y-direction speed of the vortex center multiplied by the total circulation (including original and secondary vorticity). If Eq. (56) is rewritten as

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t}\bigg/\frac{\mathrm{d}C_{0y}}{\mathrm{d}t} = \frac{\Gamma}{C_0} \tag{57}$$

it shows that the ratio between the two rates of change of y moment of vorticity, one with a no-slip boundary and the other with a slip boundary replacing such a no-slip boundary, is equal to the ratio of the two circulations.

When we consider the z-moment change, Eq. (44) shows that with the boundary-layer assumptions to the same order of accuracy near the boundary layer [$\mathcal{O}(Re^{-0.5})$] we can have

$$\frac{\mathrm{d}\Gamma_z}{\mathrm{d}t} = -\int_0^\infty \left.\frac{w^2}{2}\right|_{y=0} \mathrm{d}z \tag{58}$$

because the viscous term at $z = 0$ is a higher-order term and the diffusion on the symmetry boundary is neglected. Equation (58) is the same as Saffman's[6] inviscid results.

It is thus shown that the motion of the aircraft wake vortex centroid is exactly the same as in the case with two inviscid boundaries if there is no separation on the no-slip boundary. Even in a numerical simulation where a thin boundary layer to the order of $\mathcal{O}(Re^{-0.5})$ was considered, the vortex trajectories did not show rebound (see Ref. 12 for a case with no shear flow). The vortex centroid always has a motion toward the negative z direction. Hence, flow separation is a necessary (although not sufficient) condition to have vortex rebound from the no-slip boundary. However, the centroid of the total vorticity field is different from that of the case with two slip boundaries because the inclusion of a no-slip boundary produces secondary vorticity. Because of the unsteadiness of Γ, the vorticity moment rates do not represent the centroid velocities anymore.

If viscous diffusion on $y = 0$ and around the vortex core is neglected, the vorticity polar moment rate, Eq. (45), looks the same as in inviscid flow. However, with the unsteadiness of the other moments caused by viscous effects $\Gamma_{\bar{r}}$ is not an invariant. Therefore, none of the Betz invariants exist when no-slip boundary effects are included.

As shown earlier, several results presented in this section can be applied to discuss the problem of a trailing vortex pair encountering the ground, as the one shown in Fig. 1. Calculation of the circulation in the domain of $0 < y < \infty$ and $0 < z < \infty$ is difficult because of the unsteady vorticity produced on the no-slip wall. However, if the boundary-layer assumption already discussed in this section is employed the circulation can be calculated by using Eq. (48) or (49). The procedure is explained in the following.

Table 1 Summary of the rates of change of the four vorticity moments under different boundary conditions

Vorticity moment	Infinite domain	One symmetry boundary	One symmetry boundary and one non-slip boundary
Total vorticity	Eq. (15)	Eqs. (24) and (26)	Eqs. (42), (48–50)
y moment	Eqs. (16) and (19)	Eqs. (27) and (29)	Eqs. (43), (51), and (52)
z moment	Eqs. (17) and (18)	Eqs. (31) and (33)	Eqs. (44) and (58)
Polar moment	Eqs. (22) and (23)	Eqs. (37), (40), and (41)	Eq. (45)

In Eq. (48), because V is the outer flow inviscid velocity on $z = 0$, according to the potential flow theory for a vortex pair we have

$$V(y,t) = \frac{4C_0}{\pi}\frac{y_0 z_0 y}{\left[(y-y_0)^2 + z_0^2\right]\left[(y+y_0)^2 + z_0^2\right]} \tag{59}$$

where $[y_0(t), z_0(t)]$ is the instantaneous position of the vortex in the first quadrant. It can be shown that

$$\int_0^\infty V\,\mathrm{d}y = \frac{2C_0}{\pi}\arctan\frac{y_0}{z_0} \tag{60}$$

Hence, from Eq. (48)

$$\Gamma = C_0[1 - (2/\pi)\arctan(y_0/z_0)] \tag{61}$$

If Eq. (49) is used in a similar manner, we can have

$$\Gamma = (2C_0/\pi)\arctan(z_0/y_0) \tag{62}$$

which is equivalent to Eq. (61). It can be seen that when $z_0 \to \infty$ (i.e., when the vortex is infinitely away from the ground) $\Gamma \to C_0$, an obvious observation. When the vortex approaches closer to the ground, z_0 decreases while y_0 increases (because of the effects from its image vortex on the other side of the ground); therefore, y_0/z_0 increases. Because arc tan is a monotonically increasing function, it can be deduced from Eq. (60) that the contribution from the secondary vorticity in the ground boundary layer increases; thus, Γ decreases when the vortex is getting closer to the ground. When $y_0 \to \infty$ or $z_0 \to 0$, $\Gamma \to 0$. The potential theory only allows z_0 to asymptote to $y_0(0)z_0(0)/\sqrt{[y_0^2(0) + z_0^2(0)]}$ when $y_0 \to \infty$. However, even this cannot realistically happen because before the vortex reaches this state the separation of the boundary layer near the ground already occurs, and the resultant secondary vortices cause the vortex rebound.[11] The instantaneous circulation can be fully determined after $y_0(t)$ and $z_0(t)$ are expressed as[13]

$$y_0^2(t) = \frac{2[F + (t+E)^2]}{DF}\left\{1 \mp \left[\frac{F}{(t+E)^2} + 1\right]^{-\frac{1}{2}}\right\} \tag{63}$$

$$z_0^2(t) = \frac{2[F + (t+E)^2]}{DF}\left\{1 \pm \left[\frac{F}{(t+E)^2} + 1\right]^{-\frac{1}{2}}\right\} \tag{64}$$

where

$$D = \frac{1}{y_0^2(t)} + \frac{1}{z_0^2(t)} = \frac{1}{y_0^2(0)} + \frac{1}{z_0^2(0)}$$

$$E = \frac{4\pi}{DC_0}\left\{\frac{Dy_0^2(0) - 2}{\left[Dy_0^2(0) - 1\right]^{\frac{1}{2}}}\right\}$$

$$F = \frac{64\pi^2}{D^2C_0^2}$$

and the upper alternating sign applies when $t < E$ and the lower sign applies when $t > E$.

Another interesting result that can be obtained is the rate of change of y moment of vorticity. According to Eqs. (56) and (62) and invoking the potential theory for the expression of $\mathrm{d}y_0/\mathrm{d}t$, we can obtain

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t} = \frac{C_0^2 y_0^2}{2\pi^2 z_0\left(y_0^2 + z_0^2\right)}\arctan\frac{z_0}{y_0} \tag{65}$$

Based on Eq. (51), the total force in the z direction caused by the pressure exerted on the wall surface can also be calculated from Eq. (65). When $z_0 \to \infty$, the RHS of Eq. (65) is zero, showing that the y moment of vorticity is an invariant, reducing to the case in Sec. IV, Eq. (27).

VI. Conclusions

By using an integral divergence formula for two-dimensional vorticity field, we have shown that the rates of change of vorticity moments related with the Betz invariants (circulation, y and z moments of vorticity and polar vorticity moment) can be obtained under various boundary conditions for both viscous and inviscid incompressible fluid. As a summary, the rates of change of the four vorticity moments are listed in Table 1.

For flow in an infinite domain with bounded vorticity, the invariants for circulation, y- and z-vorticity moments exist even in viscous flow. The fourth one, the vorticity polar moment, remains constant in viscous flow if the total circulation is zero, which is the only physically possible flowfield filling the whole plane. For inviscid flow filling the whole plane, the four Betz invariants are valid exactly. These results are the same as obtained by Poincaré and Betz.

For flow with one symmetry boundary (at $y = 0$), only the y moment of vorticity is an invariant, even for viscous flow. The circulation always decays in this case, and the pace of decay depends on the Reynolds number. The circulation is an invariant for inviscid fluid. The z moment and the polar moment of vorticity are not invariant even for inviscid fluid, unless the influence from the other side of the symmetry plane can be neglected. However, in inviscid flow the polar moment of vorticity with respect to the vorticity centroid is an invariant if the force from the other side of the symmetry plane passes the vorticity centroid.

For flow with one slip boundary (at $y = 0$) and one no-slip boundary (at $z = 0$), none of the Betz invariants exist. The four vorticity moments are discussed for high-Reynolds-number flow with neglected diffusion on the symmetry boundary and without separation on the no-slip boundary. The total circulation is equal to the inviscid circulation minus the unsteady circulation from the secondary vorticity generated inside the boundary layer. The rate of y moment of vorticity, which is equal to the negative z-direction vortex impulse or momentum, is as a result of the total force in the z direction caused by the pressure on the no-slip boundary at $z = 0$. In the case of a trailing vortex pair encountering the ground, it is also equal to the product of the y-direction speed of the vortex center and the total circulation including the contribution of the secondary vorticity induced on the ground. The motion of the trailing vortex centroid is exactly the same as in the case with two inviscid boundaries if there is no separation on the no-slip boundary. The circulation and the rate of change of y moment of vorticity can be determined analytically by utilizing the theoretical results.

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