Well-posedness for nonlinear wave equations

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Outline

Nonlinear hyperbolic PDEs

are widely used in modeling wave-like motions, such as water waves, gas dynamics, liquid crystal, traffic flow, etcs. It is well known singularity often happens for nonlinear models. Most quasilinear models exhibit singularities in two classes: Shock/discontinuity and Cusp/peakon.

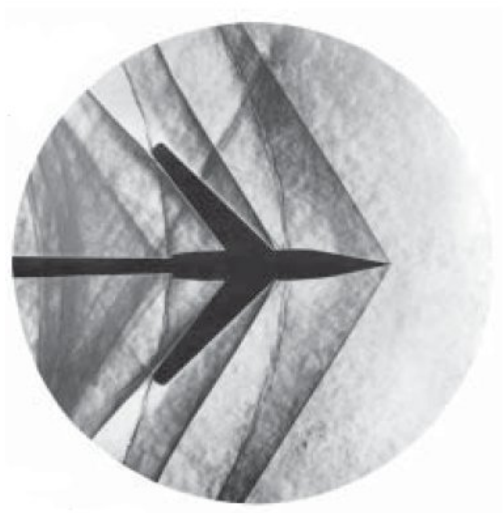
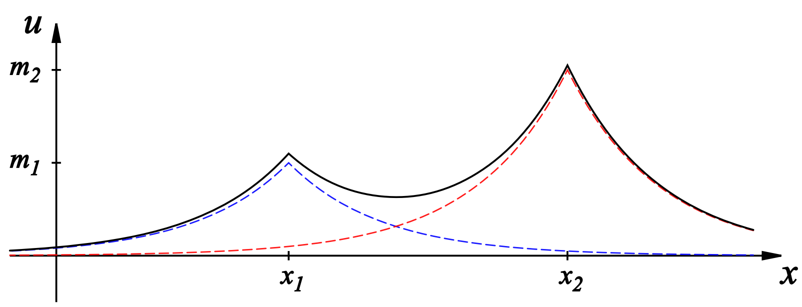


Figure : Shocks near a supersonic body.



Cusp/peakon for water wave.

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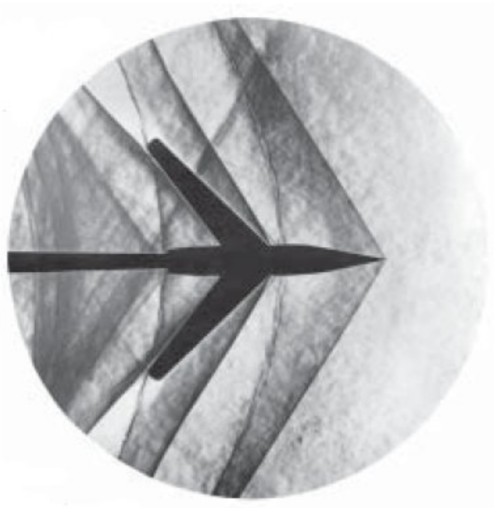

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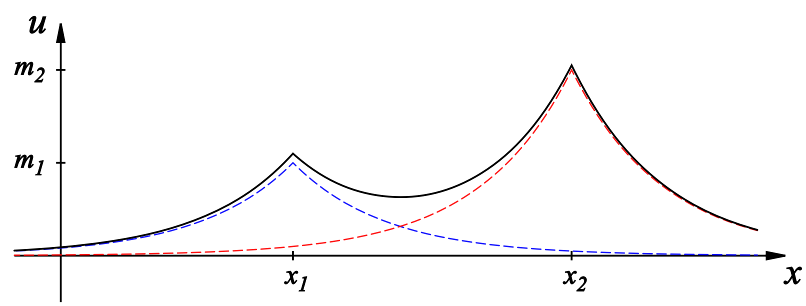


Cusp/peakon for water wave.

First order equations with cusps/peakons solutions

Camassa-Holm equation

a completely integrable model describing the motion of shallow water wave.

$$u_t + (u^2/2)_x + P_x = 0, \qquad \text{with} \qquad P = \frac{1}{2}e^{-|x|} * (u^2 + \frac{u_x^2}{2}).$$

$u(t,x)$ is fluid velocity. $P(t,x)$ is pressure. $(t,x) \in (\mathbb{R}^+, \mathbb{R})$.

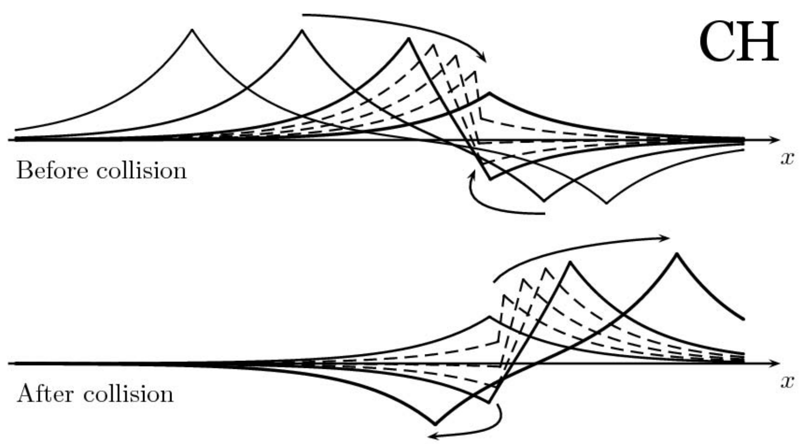


Figure : Interaction of two peakons[1]

The equation can be written as $m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}.$

[1] Figure from http://www.mai.liu.se/ halun/research/shockpeakons/

Second order equations/systems with cusps/peakons solutions:

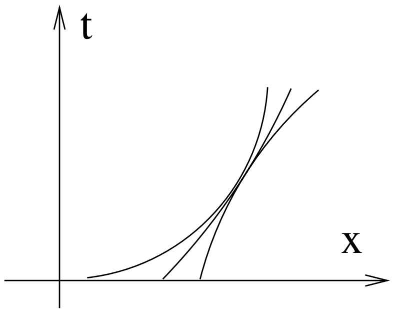
Variational wave equation

$$u_{tt} - c(u)\,(c(u)\,u_x)_x = 0\,.$$

Derived from

$$\frac{\delta}{\delta u}\int_{\mathbb{R}\times\mathbb{R}^+} \frac{1}{2}\left\{u_t^2 - c^2(u)u_x^2\right\} dxdt = 0\,.$$

Liquid crystal, Elasticity...



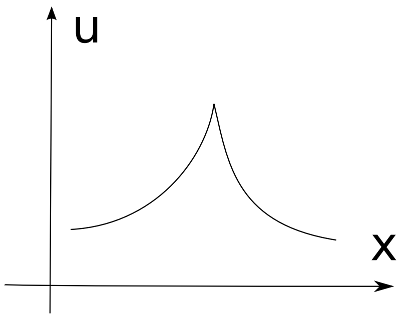
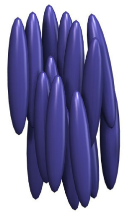


Figure : Gradient blowup

Energy density: $\frac{1}{2}\left(u_t^2 + c^2(u)u_x^2\right)$, so we expect $H^1 \hookrightarrow C^{\frac{1}{2}}$ solution.

Wave type models for nematic liquid crystals



- $\mathbf{n}(t, \mathbf{x})$: A unit vector field describing mean orientation for a nematic liquid crystal, with $(t, \mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^3$.
- When inertia effects dominate viscosity, $\mathbf{n}$ may be modelled by least action principal

$$\frac{\delta}{\delta \mathbf{n}} \int_{\mathbb{R}^3 \times \mathbb{R}^+} \left\{ \tfrac{1}{2} \partial_t \mathbf{n} \cdot \partial_t \mathbf{n} - W(\mathbf{n}, \nabla \mathbf{n}) \right\} \, d\mathbf{x} \, dt = 0, \qquad \mathbf{n} \cdot \mathbf{n} = 1$$

where $W(\mathbf{n}, \nabla \mathbf{n})$ is Oseen-Franck potential energy density:

$$W(\mathbf{n}, \nabla \mathbf{n}) = \tfrac{1}{2}\alpha(\nabla \cdot \mathbf{n})^2 + \tfrac{1}{2}\beta\,(\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + \tfrac{1}{2}\gamma\,|\mathbf{n} \times (\nabla \times \mathbf{n})|^2 + \tfrac{1}{2}\eta\left[\mathrm{tr}(\nabla \mathbf{n})^2 - (\nabla \cdot \mathbf{n})^2\right].$$

- Plannar deformation $\ n = (\cos u(t,x), \sin u(t,x), 0)\ $ with $\ x \in \mathbb{R}$:

$$u_{tt} - c(u)\,(c(u)\,u_x)_x = 0\,, \qquad c^2 = \alpha \sin^2 u + \beta \cos^2 u\,.$$

An asymptotic model (Hunter-Saxton): $\qquad (u_t + u u_x)_x = \tfrac{1}{2} u_x^2\,.$

Global well-posedness of variational wave equation

Variational wave equation: Cauchy problem

$$u_{tt} - c(u)\,(c(u)\,u_x)_x = 0\,,$$

$$u(0,x) = u_0(x) \in H^1, \qquad u_t(0,x) = u_1(x) \in L^2\,.$$

Derived from

$$\frac{\delta}{\delta u}\int_{\mathbb{R}\times\mathbb{R}^+} \frac{1}{2}\left\{u_t^2 - c^2(u)u_x^2\right\} dxdt = 0\,.$$

Energy Law is

$$\tfrac{1}{2}\,(\,u_t^2 + c^2\,u_x^2\,)_t - (\,c^2\,u_t\,u_x\,)_x = 0\,.$$

$c : \mathbb{R} \to \mathbb{R}^+$ is a smooth, bounded uniformly positive function.

Gradient blowup

Denote Riemann variables as

$$\begin{cases} R = u_t + c(u)u_x \\ S = u_t - c(u)u_x. \end{cases}$$

They satisfy

$$\begin{cases} R_t - cR_x = \frac{c'}{4c}(R^2 - S^2), \\ S_t + cS_x = \frac{c'}{4c}(S^2 - R^2). \end{cases}$$

Quadratic terms cause gradient blowup. Glassy-Hunter-Zheng, '95.

Global well-posedness of weak solution

Energy density is $\frac{1}{2}\left(u_t^2 + c^2(u)u_x^2\right)$, so we expect $H^1 \hookrightarrow C^{\frac{1}{2}}$ solution.

Balance laws

$$\begin{cases} (R^2)_t - (c(u)\,R^2)_x \;=\; \frac{c'}{2c}(R^2S - RS^2) \\ (S^2)_t + (c(u)\,S^2)_x \;=\; -\frac{c'}{2c}(R^2S - RS^2) \end{cases} \qquad (1)$$

$$R = u_t + c(u)\,u_x \;\;\&\;\; S = u_t - c(u)\,u_x$$

Weak solutions

$$\iint \Big[\phi_t\, u_t - \big(c(u)\phi\big)_x c(u)\, u_x\Big]\, dxdt \;=\; 0\,, \qquad \phi \in C_c^1$$

Results on conservative solutions

1. Global existence: Bressan-Y.Zheng '06 (when u_0 is absolutely continuous), Holden-Raynaud '11 (remove absolute continuity assumption on u_0).
2. Uniqueness when (1) is satisfied in weak sense: Bressan-GC-Q.Zhang '14(b).
3. Solution is piecewise smooth for generic initial data: Bressan-GC '15(a).
4. Lipschitz continuous dependence on initial data under a Finsler type optimal transport metric: Bressan-GC '15(b).

Existence of dissipative solutions when $c(\cdot)$ is monotonic: P.Zhang-Y.Zheng '03.

Existence

Step 1: Derive a semi-linear system from smooth solution for variables

$$p = \frac{1+R^2}{X_x}, \quad q = \frac{1+S^2}{-Y_x}, \quad A = \arctan R, \quad B = \arctan S, \quad u$$

on characteristic coordinates (X, Y).

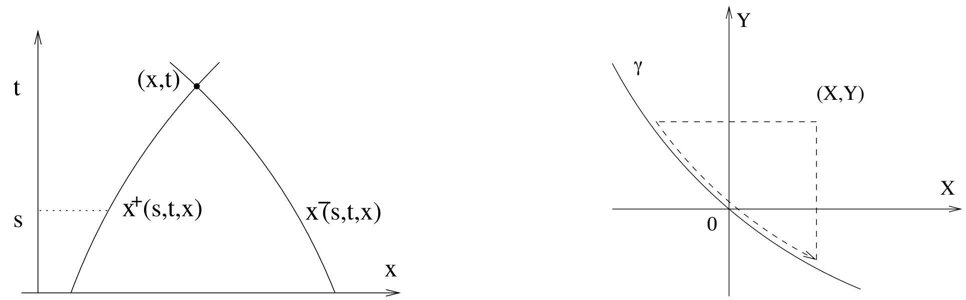


Figure : X and Y are constant along back/forward characteristics, respectively.

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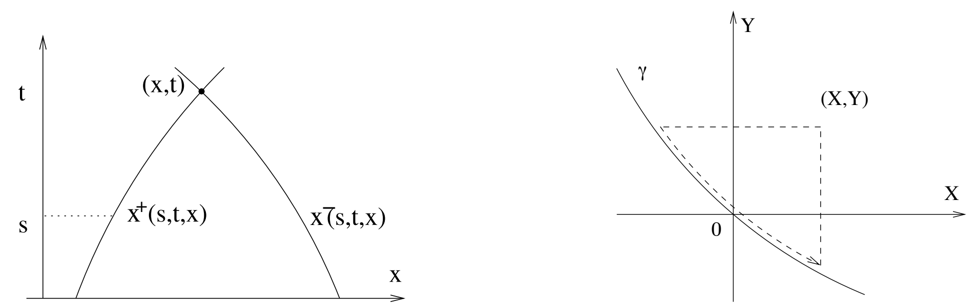


Figure : X and Y are constant along back/forward characteristics, respectively.

Step 2: Now forget smooth solution!

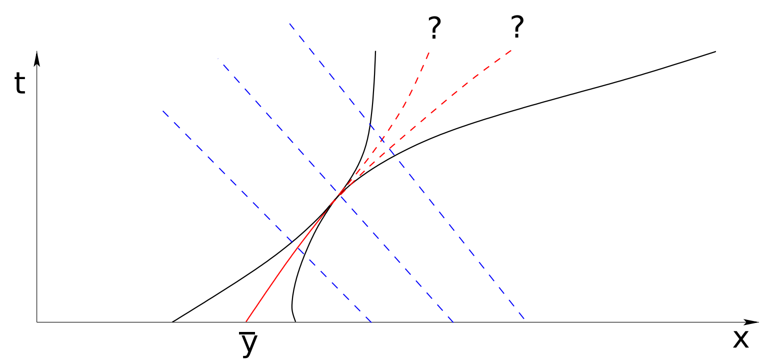
- The semi-linear system exists a global solution, then after an inverse transformation, one obtains weak solution for variational wave equation.
- The transformation is not smooth, so this method can't give uniqueness.

Uniqueness

Weak solutions are in general not unique: Conservative or dissipative solutions. We prove the uniqueness of energy conservative condition.

Step 1: For any solution u, we prove the existence and uniqueness of characteristic under the help of energy conservation.

$$\frac{dx}{dt} = c(u(t,x))\,, \quad u \text{ is Hölder } 1/2 \text{ on } x.$$



Step 2: Show all admissible conservative solutions satisfy a semi-linear system. Because this system has a unique solution, we prove the uniqueness theorem.

An easier equation: Camassa-Holm

$$u_t + (u^2/2)_x + P_x = 0 \quad \text{with} \quad P = \frac{1}{2}e^{-|x|} * (u^2 + \frac{u_x^2}{2}).$$

Theorem (Bressan, GC, Q.Zhang, DCDS, 2014)

For any initial data $u_0 \in H^1(\mathbb{R})$, the Cauchy problem of C-H has a unique weak energy conservative solution.

Definition (Energy conservation condition)

A solution $u = u(t,x)$ is conservative if

$$\int_0^\infty \int \left[u_x^2 \varphi_t + u u_x^2 \varphi_x + 2\left(u^2 - P\right) u_x \varphi \right] dxdt + \int u_{0,x}^2(x) \varphi(0,x)\, dx \;=\; 0$$

for every test function $\varphi \in \mathcal{C}_c^1(\mathbb{R}^2)$.

The existence of conservative solution: Bressan-Constantin '07.

Existence and uniqueness of dissipative solution: P.Zhang-Z.Xin '00, '02.

Step 1: For any conservative solution $u(t,x)$, find a unique characteristic

Define a variable β measuring total energy from $-\infty$ to a characteristic $x(t)$.

$$x(t,\beta) + \int_{-\infty}^{x(t,\beta)} u_x^2(t,\xi)d\xi = \beta$$

Lemma

Let $u = u(t,x)$ be a conservative solution of the Camassa-Holm equation. For any $\bar{y} \in \mathbb{R}$, there exists a unique Lipschitz continuous map $t \mapsto x(t)$ satisfying

$$\frac{d}{dt}x(t) = u(t,x(t)), \qquad x(0) = \bar{y},$$

$$\frac{d}{dt}\int_{-\infty}^{x(t)} u_x^2(t,y)\,dy = \int_{-\infty}^{x(t)} [2u^2u_x - 2u_xP](t,y)\,dy\,,$$

$$u(t,x(t)) - u(\tau,x(\tau)) = -\int_\tau^t P_x(s,x(s))\,ds \qquad 0 \le \tau \le t\,.$$

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$$\frac{d}{dt}\int_{-\infty}^{x(t)} u_x^2(t,y)\,dy = \int_{-\infty}^{x(t)} [2u^2u_x - 2u_xP](t,y)\,dy\,, \tag{2}$$

$$u(t,x(t)) - u(\tau,x(\tau)) = -\int_\tau^t P_x(s,x(s))\,ds \qquad 0 \le \tau \le t\,. \tag{3}$$

$$(1)+(2) \quad \Rightarrow \quad \frac{d\beta(t)}{dt} = G(t,\beta(t))\,, \qquad G_\beta(t,\beta) = \left[u_x + 2u^2u_x - 2u_xP\right] x_\beta$$

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Split (1) and (2) by method of generalized characteristics.

Step 2: We prove solutions satisfy a semi-linear system, then prove uniqueness of VW.

$$\begin{cases} \frac{d}{dt}\beta(t,\bar{\beta}) &= G(t,\beta(t,\bar{\beta})), \\ \frac{d}{dt}x(t,\beta(t,\bar{\beta})) &= u(t,\beta(t,\bar{\beta})), \\ \frac{d}{dt}u(t,\beta(t,\bar{\beta})) &= -P_x(t,\beta(t,\bar{\beta})), \\ \frac{d}{dt}v(t,\beta(t,\bar{\beta})) &= \left(2u^2-2P+1\right)\cos^2\frac{v}{2}-1 \end{cases}$$

where

$$v = \begin{cases} 2\arctan(\frac{u_\beta}{x_\beta}) & \text{when } 0 < x_\beta \leq 1, \\ \pi & \text{when } x_\beta = 0. \end{cases}$$

- By Lipschitz continuity of the right-hand side, we obtain the uniqueness.

Uniqueness for VW: Bressan, GC, Q.Zhang, Arch. Ration. Mech. Anal., 2014.

Difficulty: energy is transferred from forward to backward waves, and vice versa.

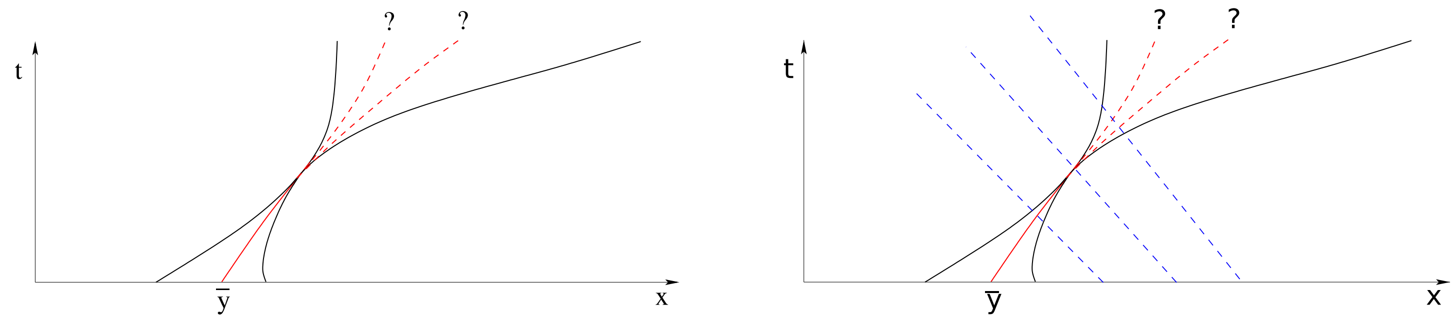

Figure : Left: Camassa-Holm; Right: Variational wave

Ideas

- Use two independent variables α and β for forward and backward waves.
- However, equations of α and β do not have Lipschitz right hand side. The existence is proved by the Schauder's fixed point theorem.
- Uniqueness of α, β: Define a weighted distance by setting

$$d^{(t)}(\alpha_1, \alpha_2) = \int_{\alpha_1(t)}^{\alpha_2(t)} W(t, \alpha)\, d\alpha \;\Rightarrow\; d^{(t)}(\alpha_1, \alpha_2) \le e^{C_0 t}\, d^{(0)}(\alpha_1, \alpha_2).$$

Lipschitz continuous dependence on initial data

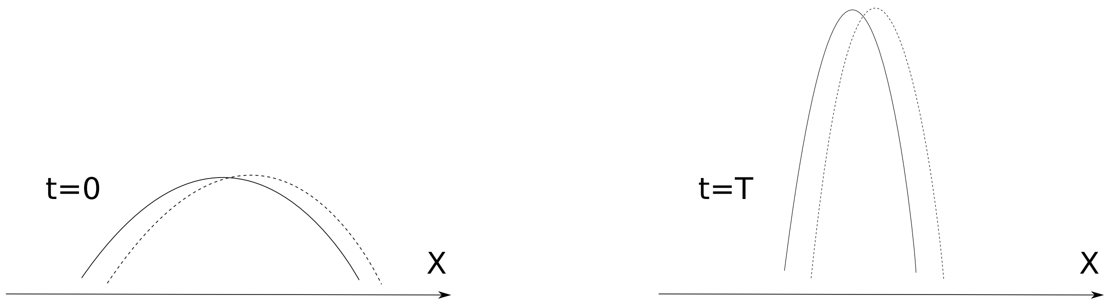

Figure : Profiles of u_x^2 (solid line) and $\tilde{u}_x^2$ (dash line).

Solution is not Lipschitz under H^1 metric.

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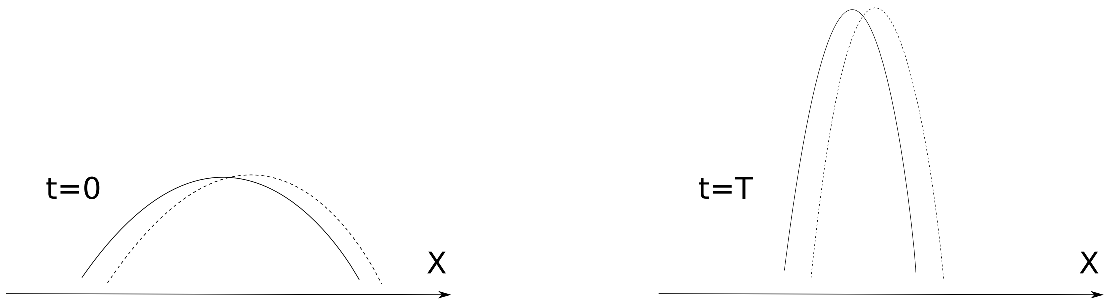

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Idea is to use a transportation metric.

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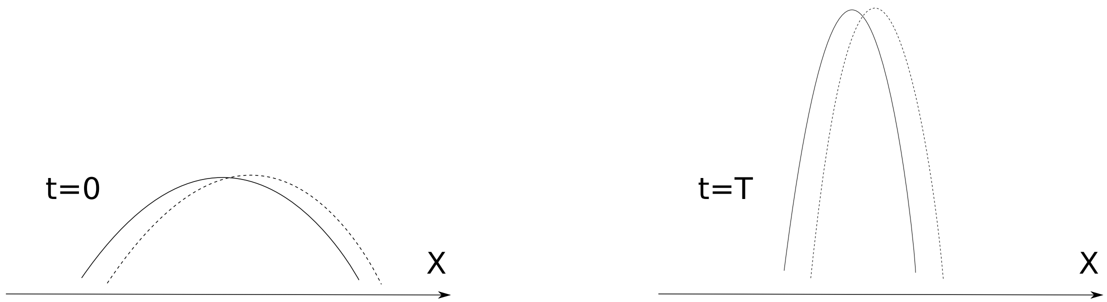

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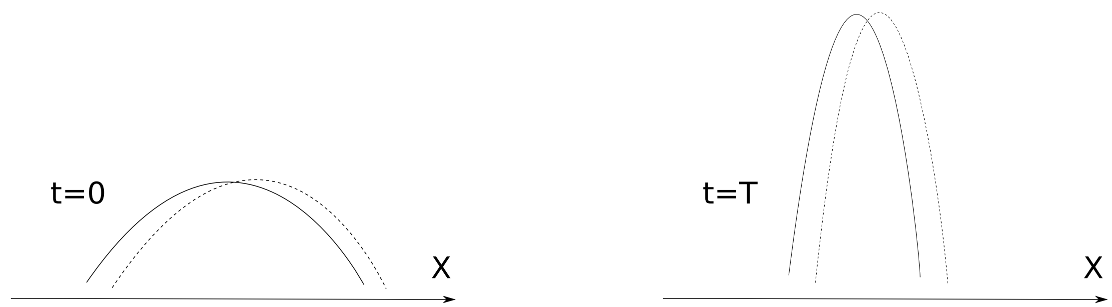
Idea is to use a transportation metric.

- Camassa-Holm: Bressan-Fonte '05, Grunert-Holden-Raynaud '11,

$$d(u,v) \approx \inf_{\psi} \int \left| u(x) - v(\psi(x)) \right| \psi'(x)(1+u_x^2(x))\,dx \cdots$$

for any given time t.

Lipschitz continuous dependence on initial data



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  for any given time t.

- Variational wave equation: Bressan-GC '15.
  Two directions. Direct transport is not working. Need to study gradient flows.

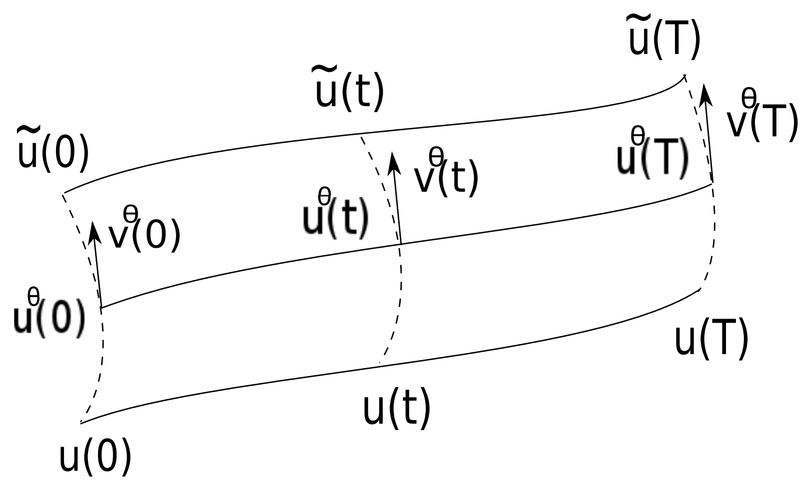
A Finsler type optimal transport metric

$$d^*\Big(\big(u(t),u_t(t)\big),\ \big(\tilde u(t),\tilde u_t(t)\big)\Big)=\inf_{\gamma}\int_0^1 \Big\|v^\theta,r^\theta,s^\theta,u^\theta,R^\theta,S^\theta\Big\|(t)\ d\theta$$

where

$$(v^\theta,r^\theta,s^\theta)=\frac{d}{d\theta}(u^\theta,R^\theta,S^\theta)$$

and $\gamma=(u^\theta,R^\theta,S^\theta)$ connects two solutions $u^{(0)}=u$ and $u^{(1)}=\tilde u$.



How to define $\|v, r, s, u, R, S\|$?

Variation of u, R, S...

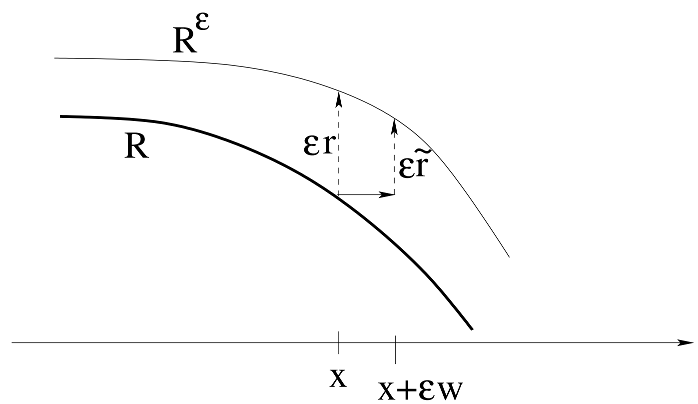


Figure : [Change in R]: $O(\varepsilon)$ term of $R^\varepsilon(x^\varepsilon) - R(x) \approx r + wR_x$

.

Shift components $w(t,x), z(t,x)$: $x^\varepsilon = x + \varepsilon w$ or $x^\varepsilon = x + \varepsilon z$

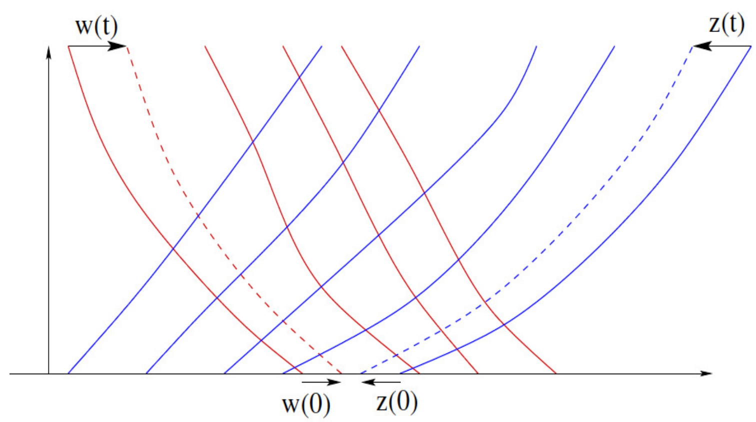


Figure : A characteristic is shifted to a characteristic. Hence w and z satisfy linear equations.

Transport metric:

$$\|v,r,s,u,R,S\| \doteq \inf_{w,z,\ \text{equations}} \|v,r,s,u,R,S,w,z\|$$

where

$$\|v,r,s,u,R,S,w,z\|$$
$$\approx \int_{-\infty}^{\infty} \Big\{[\text{change in } x] + [\text{change in } u] + [\text{change in } \arctan R]\Big\}(1+R^2)W^- dx$$
$$+ \int_{-\infty}^{\infty} [\text{change of the base measure with density } R^2]W^- \, dx$$
$$+ \text{forward direction}\ldots$$

Here $W^-(t,x)$ is a weight (interaction potential).

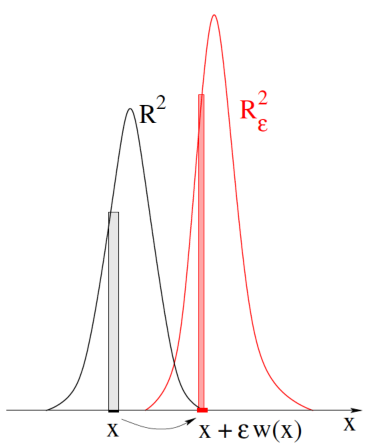


Figure : Change of base measure: $O(\varepsilon)$ term of $R_\varepsilon^2(x_\varepsilon)dx_\varepsilon - R^2(x)dx$

$$
\begin{aligned}
& \Big\| v, r, s, u, R, S, w, z \Big\| \\
= \; & \kappa_1 \int \Big\{ |w| \big(1 + R^2\big) W^- + |z| \big(1 + S^2\big) W^+ \Big\} dx \\
& + \kappa_2 \int \Big\{ |\tilde{r}| W^- + |\tilde{s}| W^+ \Big\} dx \\
& + \kappa_3 \int \Big| v + \tfrac{Rw}{2c} - \tfrac{Sz}{2c} \Big| \Big\{ (1 + R^2)\, W^- + (1 + S^2)\, W^+ \Big\} dx \\
& + \kappa_4 \int \Big\{ \Big| w_x + \tfrac{c'}{4c^2}(w - z) S \Big| W^- + \Big| z_x + \tfrac{c'}{4c^2}(w - z) R \Big| W^+ \Big\} dx \\
& + \kappa_5 \int \Big\{ \Big| R w_x + \tfrac{c'}{4c^2}(w - z) SR \Big| W^- + \Big| S z_x + \tfrac{c'}{4c^2}(w - z) RS \Big| W^+ \Big\} dx \\
& + \kappa_6 \int \Big\{ \Big| 2R\tilde{r} + R^2 w_x + \tfrac{c'}{4c^2} R^2 S (w - z) \Big| W^- + \Big| 2S\tilde{s} + S^2 z_x + \tfrac{c'}{4c^2} S^2 R (w - z) \Big| W^+ \Big\} dx
\end{aligned}
$$

with

$$
\tilde{r} = r + wR_x - \frac{c'}{8c^2}(w - z) S^2, \quad \tilde{s} = r + zS_x - \frac{c'}{8c^2}(w - z) R^2.
$$

Two major difficulties comparing to first order equation

First difficulty: energy transfer

$$\begin{cases} (R^2)_t - (cR^2)_x = \frac{c'}{2c}(R^2S - RS^2) \\ (S^2)_t + (cS^2)_x = -\frac{c'}{2c}(R^2S - RS^2) \end{cases}$$

Add weights $W^{\pm}$ to control crossing terms.

Second difficulty: relative shifts

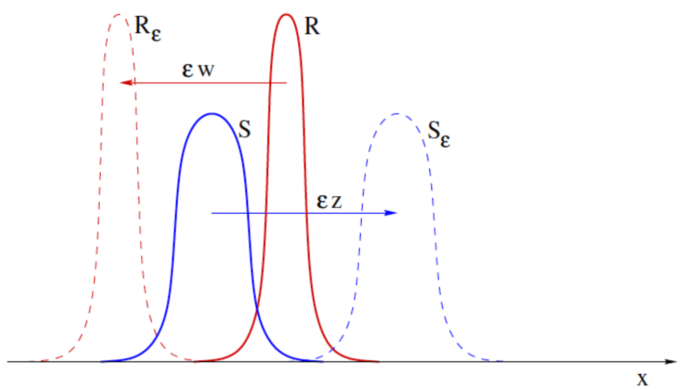


Figure : [Change in R] $= r + wR_x + \frac{c'}{8c^2}(w - z)(R^2 - S^2)$

For smooth solutions on $t \in [0, t^*)$ with possible blowup at t^*,

$$\frac{d}{dt}\|v, r, s, u, R, S\| \leq K \cdot \|v, r, s, u, R, S\| .$$

Constant K is independent of t^*.

Extend the metric to weak solution: Thanks to the generic regularity result in Bressan-GC, Ann. Inst. H. Poincare, 2015.

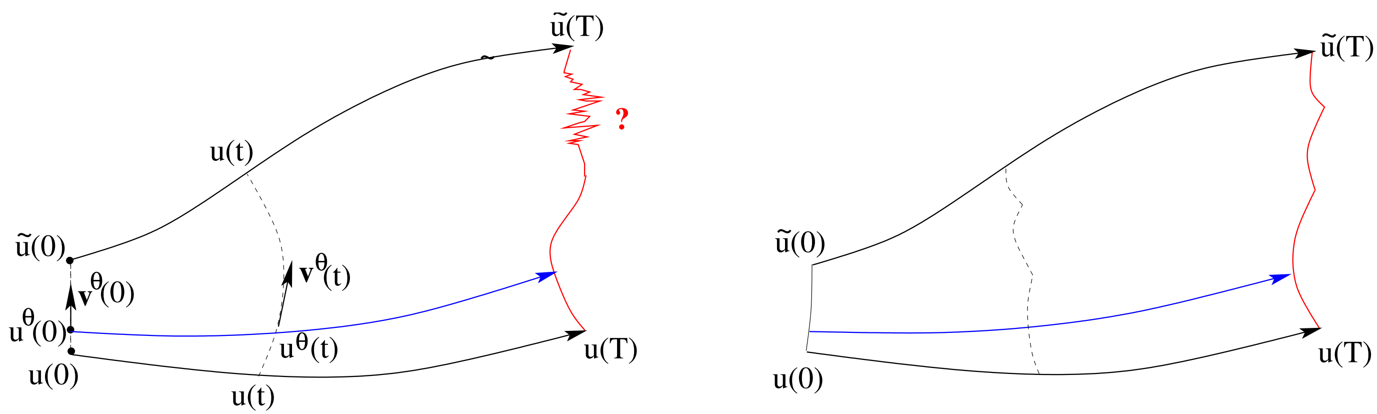


Figure : Solutions with generic initial data are piecewise smooth for any time.

Theorem (Bressan-GC '15(b), submitted)

For any $T_0 > 0$ *and* $E_0 > 0$, *there exists a constant* C, *such that,*

$$d^*\Big(\big(u(t), u_t(t)\big)\,,\;\big(\tilde{u}(t), \tilde{u}_t(t)\big)\Big) \;\leq\; C \cdot d^*\Big(\big(u(0), u_t(0)\big)\,,\;\big(\tilde{u}(0), \tilde{u}_t(0)\big)\Big)$$

for any $t \in [0, T_0]$ *and any energy conservative solutions* u *and* $\tilde{u}$ *with energy less than* E_0 .

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Relation with other metric:

For any time t,

$$\frac{1}{M} \cdot \|u - \tilde{u}\|_{L^1} \leq d^*\big((u, u_t), (\tilde{u}, \tilde{u}_t)\big) \leq M \cdot \big(\|u - \tilde{u}\|_{H^1} + \|u_t - \tilde{u}_t\|_{L^2} + \|u - \tilde{u}\|_{W^{1,1}} + \|u_t - \tilde{u}_t\|_{L^1}\big),$$

and

$$\text{(Wasserstein)} \qquad \frac{1}{M} \sup_{\|f\|_{\mathcal{C}^1} \leq 1} \left| \int f\, d\mu - \int f d\tilde{\mu} \right| \ \leq\ d^*\big((u, u_t), (\tilde{u}, \tilde{u}_t)\big)\,.$$

Here $\mu, \tilde{\mu}$ are the measures with densities $u_1^2 + c^2(u_0)u_{0,x}^2$ and $\tilde{u}_1^2 + c^2(\tilde{u}_0)\tilde{u}_{0,x}^2$ w.r.t. Lebesgue measure.

Applications to other systems

Wave systems modeling liquid crystal

(existence) P.Zhang-Y.Zheng '12, GC-P.Zhang-Y.Zheng, '13,

(uniqueness) GC-J. Huang-Q.Zhang, preprint.

An integrable water wave model (Novikov equation)

$$u_t + u^2 u_x + \partial_x P_1 + P_2 = 0, \qquad u(0,x) = u_0(x),$$

where

$$P_1 := p * \left(\tfrac{3}{2} u u_x^2 + u^3\right), \qquad P_2 := \tfrac{1}{2} p * u_x^3, \qquad p = \tfrac{1}{2} e^{-|x|}.$$

Existence & uniqueness of (Hölder 3/4) solution, GC-R.M.Chen-Y.Liu, submitted.

Key idea is to use both second and fourth order conservation laws to control the cubic nonlinearity in the non-local term.

Thank You!