Applying Combinatorics in the Design of Multiversion Exams

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ARTICLE HISTORY

Compiled July 16, 2023

ABSTRACT

In Spring 2020, the need for redesigning online assessments to preserve integrity became a priority to many educators. Many of us found methods to proctor exams using Zoom and proctoring software. Such exams pose their own issues. To reduce the technical difficulties and cost, many Zoom proctored exam sessions were shortened; to create summative assessments we supplemented the short proctored sessions with short unproctored sessions. In the aftermath of the pandemic, the method of creating a large volume of exam variants is helping us design mastery-based exams. We review the mathematics behind choosing the size of question pools to improve integrity of the exam. Finally we provide LaTeX files for generating exam variants from the pools.

KEYWORDS

Question pools; Size of an Assessment; Integrity of Exams; Creating Mastery-Based Assessments; Exam Generators; Galois Fields; Vector Spaces; Finite Projective Planes; LaTeX; Python.

1. Introduction

One of the traditional methods of mitigation for academic dishonesty is creating assessment variants where each exam has different questions from the others around them. This, typically, entails creating a pool of questions for each learning goal that are in the same level of Bloom's taxonomy. In a lecture hall, the maximum number of students immediately surrounding a test taker is usually considered to be four; thus the test maker can create four versions (exam variants) of the exam and alternate them between test takers and rows. In an online setting this maximum number can

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be the same as the entire student body in the class and for mastery-based learning this number can be as large as the total number of times all students in the class will take and retake the assessment. Often writing that many exam variants for larger class sizes is difficult and that is when randomization of the pools is recommended, [2], and the integrity of the exam becomes dependent on the maximum number of questions that each pair of exam variants have in common. [4], [6], [7].

In this paper we explain the problem using examples, showing how it can be viewed as a combinatorics problem. We describe the mathematics in the general case, and, in Theorem 6, give the existence of sets of assessment variants, overlapping in at most two questions, in terms of the size of the question pools. We provide LaTeX files to generate a maximum number of assessment variants in a few cases.

We are hoping the audience for this paper will include any STEM educator or LaTeX user, and the mathematics is explained for a general audience. To explain the importance of methodical use of question pools, we start with an example.

Figure 1. Example 1: Nature 101, Quiz 1, 4 Learning objectives.
Note that each question is labeled by b_{ij} where i is the pool index and $j+1$ is the order of the question in the corresponding pool.

Learning objective 1: Knowing your Sky Question Pool 1 (B_1)	Learning objective 2: Knowing your Hiking Choices Question Pool 2 (B_2)
(b_{10}) Why is the sky blue? (b_{11}) What is the Big Dipper? (b_{12}) What type of clouds are rain clouds?	(b_{20}) Name 3 national parks. (b_{21}) Name 3 mountain ranges in US. (b_{22}) Name 3 of the Great Lakes.
Learning objective 3: Knowing your Birds Question Pool 3 (B_3)	Learning objective 4: Knowing your Camping Choices Question Pool 4 (B_4)
(b_{30}) Name 3 bird species local to Kansas. (b_{31}) Name 1 bird that remains in Kansas over the winter. (b_{32}) Name 3 yellow birds in Kansas.	(b_{40}) Where is the best camping site near you? (b_{41}) What do you need to pack for camping? (b_{42}) When is the best time to go camping?

Figure 2. Example 1: An example of a student quiz. (Quiz variant $\{b_{10}, b_{22}, b_{30}, b_{41}\}$.)

Exam 1, Nature 101	Name:
1. Why is the sky blue?	3. Name 3 bird species local to Kansas.
2. Name 3 of the Great Lakes.	4. What do you need to pack for camping?

Example 1. A professor is planning to assess students' learning in four different learning objectives. That is, any quiz variant contains four questions each from one of the four distinct learning objectives; see Figure 1. The professor has 9 students; to ensure integrity they want each two students to have quizzes with no common questions. To create 9 quiz variants with no common questions, each pool of questions has to have 9 questions. If the professor allows each pair of quiz variants to have exactly one question in common, they only need to create 3 questions in each pool. For example, the 9 quiz variants can use the following sets of questions: $\{b_{10}, b_{20}, b_{30}, b_{40}\}$, $\{b_{11}, b_{21}, b_{31}, b_{40}\}$, $\{b_{12}, b_{22}, b_{32}, b_{40}\}$, $\{b_{10}, b_{21}, b_{32}, b_{41}\}$, $\{b_{11}, b_{22}, b_{30}, b_{41}\}$, $\{b_{12}, b_{20}, b_{31}, b_{41}\}$, $\{b_{10}, b_{22}, b_{31}, b_{42}\}$, $\{b_{11}, b_{20}, b_{32}, b_{42}\}$, $\{b_{12}, b_{21}, b_{30}, b_{42}\}$. (One exam variant is shown in 2)

2. Preliminaries

Our goal is to use mutually disjoint question pools to create assessment variants for an assessment. Note that we are assuming that each pool of questions contains questions that assess similar or the same learning objectives. The number of questions in an assessment is equal to the number of question pools.

Definition 1. An **assessment variant** is a choice of one question from each question pool.

To preserve the integrity of the exam, we are looking for a set V of assessment variants where each pair of assessment variants have at most $\mathfrak{cq}$ questions in common. The number of students in a class should be less than or equal to the size of V. We show a bound for the number of questions in each pool so that each pair in V have at most one question in common. We show later in Proposition 1 that the smallest question pool affects the number of assessment variants when the maximum number of questions in common is one. We assume all question pools to be of the same size.

2.1. Notation and the Statement of the Main Result

Throughout this work, we are assuming an assessment contains $\mathfrak{s}$ questions and each question Q_i is chosen from a corresponding pool (bank) of questions B_i containing questions for the same learning goal. For now, we are assuming that all pools contain the same number of questions, κ, and we explain why later. We also assume that each pair of question pools are mutually disjoint.

$\mathcal{N}$:	Number of students in class or the number of mastery assessments needed
$\mathfrak{s}$:	Number of questions in the assessment
$\mathfrak{cq}$:	Maximum number of questions in common between any pair of assessment variants
κ :	Number of questions in each pool
Q_i :	i^{th} question of the assessment
B_i :	Question pool associated with Q_i
b_{ij} :	j^{th} question in question pool B_i

We can describe each assessment variant as a function that chooses a member of question pool B_i as the i^{th} question of that assessment: $f(Q_i) = b_{ij}$ for some j, $0 \le j < \kappa$. We construct a set of such functions where the size of the set is greater or equal to $\mathcal{N}$ for given κ, $\mathfrak{s}$, and $\mathfrak{cq}$.

The Main Result (Theorem 6): Let $\kappa = p^m$, where p is a prime number and m is an integer. For $\mathfrak{s} \le \kappa + 1$, let $\{B_1, B_2, \cdots, B_{\mathfrak{s}}\}$ be $\mathfrak{s}$ pairwise disjoint question pools where each question pool, B_i, contains exactly κ questions. Then for $\mathfrak{cq} \in \{1, 2\}$ it is possible to generate $\kappa^{\mathfrak{cq}+1}$ assessment variants, where each pair of assessment variants has at most $\mathfrak{cq}$ questions in common. Moreover when the number of questions in each pool is an integer power of 2, $\kappa = 2^m$, it is possible to generate κ^3 assessment variants with at most two questions in common, $\mathfrak{cq} = 2$, for $\mathfrak{s} \le \kappa + 2$ disjoint question pools.

In Theorem 1 we also show that if each question pool in the pairwise disjoint set $\{B_1, B_2, \cdots, B_{\mathfrak{s}}\}$ has κ questions, to generate $\kappa^{\mathfrak{cq}+1}$ assessment variants with maximum question intersection $\mathfrak{cq}$, the number of pools $\mathfrak{s}$ cannot exceed $\kappa + \mathfrak{cq}$.

Example 2. In Figures 3, 4, 5, 6 and 7, we illustrate five sets of five assessment variants generated from six mutually disjoint question pools B_i where each pool has five distinct questions in it. That is, $\kappa = 5$, $\mathfrak{s} = 6$ and $\mathfrak{cq} = 1$.

Let $F_i = \{f_{ij} \,|\, j \in \{0,1,2,3,4\}\}$, for $i \in \{0,1,2,3,4\}$. Each F_i contains 5 assessment variants and each pair of these variants have only question b_{6i} in common. An instructor can use the set union of these sets, $\bigcup_{i\in\{0,1,2,3,4\}} F_i$, to create an assessment for up to 25 students where each pair of students' exams have only one question in common. These assessment variants each have 6 questions for 6 different learning goals in them.

Now define restrictions of a set of exam variants to one fewer question to be $F_i|_{\{Q_1,Q_2,Q_3,Q_4,Q_5\}} = \{f_{ij}|_{\{Q_1,Q_2,Q_3,Q_4,Q_5\}} | j \in \{0,1,2,3,4\}\}$; here each $F_i|_{\{Q_1,Q_2,Q_3,Q_4,Q_5\}}$ contains five mutually disjoint assessment variants for a 5-question long assessment, $\kappa = 5$, $\mathfrak{s} = 5$, and $\mathfrak{cq} = 0$. $F|_{\{Q_1,Q_2,Q_3,Q_4,Q_5\}} := \bigcup F_i|_{\{Q_1,Q_2,Q_3,Q_4,Q_5\}}$ contains 25 assessment variants with $\kappa = 5$, $\mathfrak{s} = 5$, and $\mathfrak{cq} = 1$.

Figure 3. Images of functions $f_{00}, f_{01}, f_{02}, f_{03}, f_{04}$ make a set F_0 of 5 assessment variants. Each pair of assessment variants has only b_{60} in common.

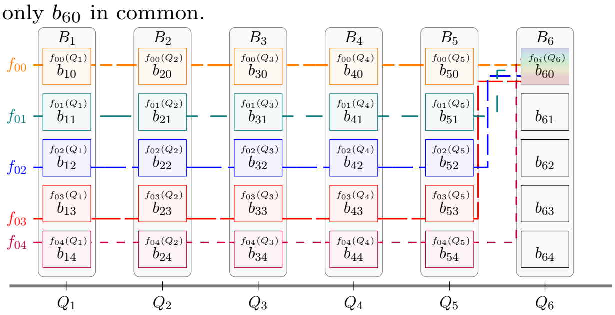


Figure 4. Images of functions $f_{10}, f_{11}, f_{12}, f_{13}, f_{14}$ make a set F_1 of 5 assessment variants. Each pair of assessment variants has only b_{61} in common.

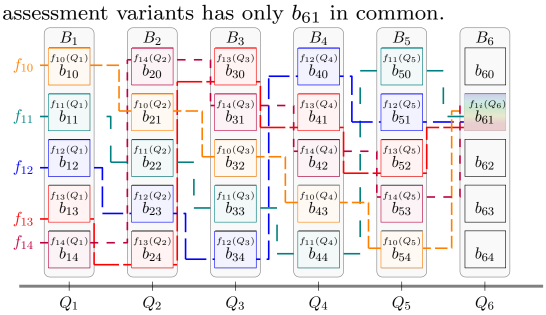


Figure 5. Images of functions $f_{20}, f_{21}, f_{22}, f_{23}, f_{24}$ make a set F_2 of 5 assessment variants. Each pair of assessment variants has only b_{62} in common.

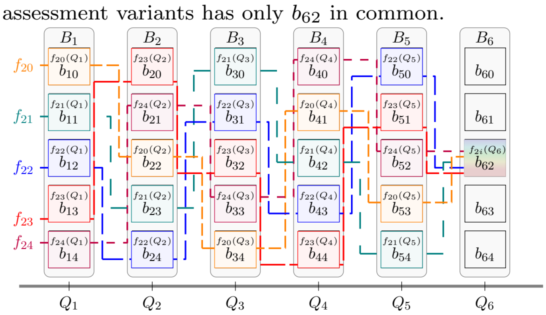


Figure 6. Images of functions $f_{30}, f_{31}, f_{22}, f_{33}, f_{34}$ make a set F_3 of 5 assessment variants. Each pair of assessment variants has only b_{63} in common.

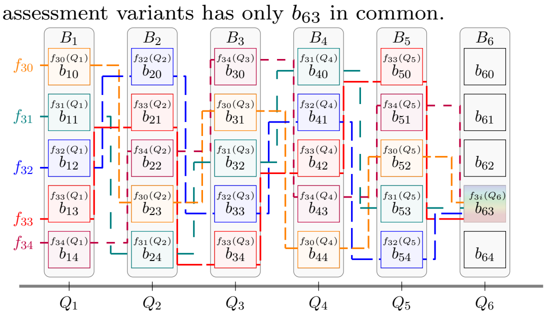
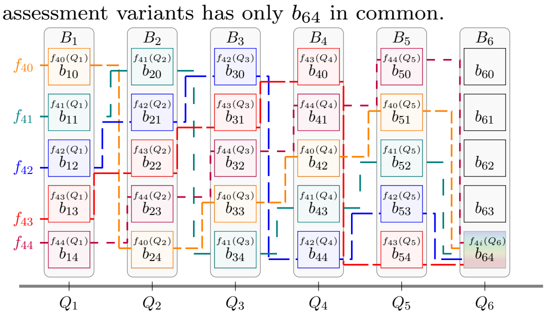


Figure 7. Images of functions $f_{40}, f_{41}, f_{42}, f_{43}, f_{44}$ make a set F_4 of 5 assessment variants. Each pair of assessment variants has only b_{64} in common.



We consider some relations among the parameters $\mathfrak{s}$, $\mathfrak{cq}$, $|B_i|$ and the number of possible assessment variants.

Proposition 1. *If $\mathcal{V}$ is a set of assessment variants with $\mathfrak{cq} = 1$ and $\mathfrak{s} \geq 2$, and two of the associated question pools have size k_1 and k_2, then $|\mathcal{V}| \leq k_1 k_2$.*

Proof. Assume the question pools B_n and B_m, respectively, contain k_1 and k_2 questions. Then each question b_{ni} in B_n can be in at most k_2 assessment variants since all assessment variants containing b_{ni} have to have different questions from B_m. This implies that the maximum number of assessment variants cannot exceed $k_1 \times k_2$. □

Thus, the maximum size of a set of assessment variants with $\mathfrak{cq} = 1$ is bounded by the sizes of the two smallest question pools. So there is no advantage in having question pools of different sizes. From now on we will assume all question pools are the same size κ.

Proposition 2. *Let $\mathcal{V}$ be a set of assessment variants where each question pool is of size κ and $\mathfrak{cq} = c$. Then $|\mathcal{V}| \leq k^{c+1}$.*

Proof. Let $\mathcal{V}$ be a set of assessment variants using question pools of size κ. If $\mathfrak{s} \leq c$, then the set of all possible assessment variants is at most κ^c. So assume $\mathfrak{s} \geq c+1$. Then, since $\mathfrak{cq} = c$, each vector in $B_1 \times B_2 \times \cdots \times B_{c+1}$ can be in only one assessment variant in $\mathcal{V}$. There are κ^{c+1} vectors in $B_1 \times B_2 \times \cdots \times B_{c+1}$, so $|\mathcal{V}| \leq \kappa^{c+1}$. □

Theorem 1. *Let c be a positive integer. Let $\mathcal{V}$ be a set of κ^{c+1} assessment variants, where each question pool is of size κ and $\mathfrak{cq} = c$. Then the number of questions in the assessment is $\mathfrak{s} \leq \kappa + c$.*

Proof. The proof is by induction on c.
Base step $c = 1$. Assume the assessment has $\mathfrak{s}$ questions, each question pool has κ questions, and V is a set of size κ^2 of assessment variants generated from these pools with $\mathfrak{cq} = 1$. Let $A_{i\ell}$ be the set of all assessment variants containing question $b_{i\ell} \in B_i$.

For fixed $j \neq i$ each variant in $A_{i\ell}$ must contain one of the κ question in B_j. Since no two assessment variants share two questions, each of these κ questions can be in at most one variant in $A_{i\ell}$. So $A_{i\ell}$ contains at most κ variants. On the other hand, $V = \bigcup_{\ell \in \{0,1,\cdots,\kappa-1\}} A_{i\ell}$ and V is of size κ^2. Therefore each $A_{i\ell}$ is exactly of size κ, and every b_{jm} $(j \neq i)$ is in exactly one of the variants in $A_{i\ell}$.

Now fix one assessment variant, say $v \in A_{10}$. Define a function $f_v : \{2, 3, \ldots, s\} \to A_{11}$ as follows: $f_v(j)$ is the assessment variant in A_{11} containing the same question b_{jm} from pool B_j as v. The function f_v is well-defined by the statement above. It is one-to-one, because one assessment variant cannot share two questions with v. So $s - 1 \leq \kappa = |A_{11}|$. So $s \leq k + 1$.

Inductive step Assume true for $\mathfrak{cq} < c$. Let $\mathcal{V}$ be a set of κ^{c+1} assessment variants with $\mathfrak{cq} = c$. Let B_i, $1 \leq i \leq \mathfrak{s}$, be the question pools for $\mathcal{V}$. Since there are κ questions in B_1, there exists a question $b_{1i} \in B_1$ that is contained in κ^c assessment variants. Let A_{1i} be the set of assessment variants containing b_{1i}. Define a set of assessment variants by $\mathcal{A} := \{v \setminus \{b_{1i}\} | v \in A_{1i}\}$. The maximum number of common questions for each pair of assessment variants in $\mathcal{A}$ is $c - 1$, $\mathcal{A}$ is of size κ^c, so by induction the number of questions for $\mathcal{A}$ is at most $\kappa + c - 1$. Since we removed the question pool B_1 from $\mathcal{V}$, the number of questions for $\mathcal{A}$ is $\mathfrak{s} - 1$. Thus $\mathfrak{s} \leq \kappa + c$.

□

Definition 2. A **maximal assessment variant formation for $\mathfrak{cq}$ of order κ** is a set of $\kappa^{\mathfrak{cq}+1}$ assessment variants with maximum number of common questions $\mathfrak{cq}$, generated from $\mathfrak{s} = \kappa + \mathfrak{cq}$ question pools where each pool contains κ questions.

Theorem 2. *In any maximal assessment variation formation for $\mathfrak{cq} = 1$, every pair of assessment variants share exactly one question.*

Proof. Suppose f is an assessment variant in a maximal assessment variant formation for $\mathfrak{cq} = 1$. For each ℓ, $1 \leq \ell \leq \kappa+1$, let $f(Q_\ell) = b_{\ell\ell_f}$, that is, $b_{\ell\ell_f}$ is the question from pool B_ℓ in f. As in the proof of Theorem 1, write $A_{\ell\ell_f}$ for the set of all assessment variants containing $b_{\ell\ell_f}$. Since no other assessment variant can share two questions with f, for all $i \neq j$, $A_{ii_f} \cap A_{jj_f} = \{f\}$. Thus distinct $A_{\ell\ell_f} \setminus \{f\}$ are disjoint. So

$$\kappa^2 = |V| \geq \left| \bigcup_{\ell=1}^{\kappa+1} A_{\ell\ell_f} \right| = 1 + \sum_{\ell=1}^{\kappa+1} |A_{\ell\ell_f} \setminus \{f\}| = 1 + (\kappa+1)(\kappa-1) = \kappa^2.$$

So $V = \bigcup_{\ell=1}^{\kappa+1} A_{\ell\ell_f}$. Thus for each $g \in V$, $g \neq f$, g contains $b_{\ell\ell_f}$ for exactly one ℓ. That is, every pair of assessment variants share exactly one question.

□

Our goal now is to see if these upper bounds can be attained. With this goal in mind, we define two mathematical concepts.

Note 1. *For each prime p and positive integer m, there exists a finite field $GF(p^m)$ of size p^m, called the Galois field of order p^m. Addition, subtraction, multiplication and division are defined. In the case of $m = 1$, the field is $\mathbb{Z}_p$, where the operations are arithmetic modulo p. We will represent the elements of $GF(p^m)$ by the integers 0 to $p^m - 1$. [5].*

Definition 3. A **subspace** of a vector space $\mathcal{R}^k$ for field $\mathcal{R}$ is a subset V of $\mathcal{R}^k$ that contains all linear combinations of V. For k finite, this is equivalent to saying that a subspace is the set of all linear combinations of a finite set, which is called a spanning set. A basis for a subspace of $\mathcal{R}^k$ is a smallest spanning set.

We represent Example 2 as a subspace in the vector space $GF(5)^6$, over the Galois field of order 5. Each function f, representing an assessment variant, is given by an ordered list of questions from the six question pools. Corresponding to the function $(b_{1i_1}, b_{2i_2}, b_{3i_3}, b_{4i_4}, b_{5i_5}, b_{6i_6})$ is the element $\langle i_1, i_2, i_3, i_4, i_5, i_6\rangle$ of the vector space $GF(5)^6$. (This is why we chose to index each question pool starting with 0.) The set F_0 is actually a subspace of $GF(5)^6$:

$$\begin{aligned} F_0 &= \{\langle 0,0,0,0,0,0\rangle, \langle 1,1,1,1,1,0\rangle, \langle 2,2,2,2,2,0\rangle, \langle 3,3,3,3,3,0\rangle, \langle 4,4,4,4,4,0\rangle\} \\ &= \{t\langle 1,1,1,1,1,0\rangle | t \in GF(5)\} \end{aligned}$$

The other sets F_i $(1 \le i \le 4)$ are not subspaces, but they are translates ("cosets" in group theory terminology) of F_0. For example,

$$\begin{aligned} F_1 &= \{\langle 0,1,2,3,4,1\rangle, \langle 1,2,3,4,0,1\rangle, \langle 2,3,4,0,1,1\rangle, \langle 3,4,0,1,2,1\rangle, \langle 4,0,1,2,3,1\rangle\} \\ &= \{t\langle 1,1,1,1,1,0\rangle + \langle 0,1,2,3,4,1\rangle | t \in GF(5)\} \end{aligned}$$

In fact, $\bigcup_{0 \le i \le 4} F_i$ is the subspace of $(GF(5))^6$ spanned by $\{\langle 1,1,1,1,1,0\rangle, \langle 0,1,2,3,4,1\rangle\}$, and $\bigcup F_i|_{\{Q_1,Q_2,Q_3,Q_4,Q_5\}}$ is the subspace of $GF(5)^5$ spanned by $\{\langle 1,1,1,1,1\rangle, \langle 0,1,2,3,4\rangle\}$.

3. The Proof for $\mathfrak{cq} = 1$

Our goal in this section is to create the largest number of assessment variants where each pair of assessment variants have at most one question in common. In a subsection below, we connect the maximal assessment variant formations for $\mathfrak{cq} = 1$ to finite projective planes. Meanwhile, a few concrete examples of maximal assessment variant formations for $\mathfrak{cq} = 1$ are given between the theorems.

3.1. Constructing κ^2 Many Assessment Variants

Theorem 3. *Suppose p is a prime number, $m \geq 1$ is an integer and $\kappa = p^m$. Then $F = \{s\langle 1,1,1,\cdots,1,0\rangle + t\langle 0,1,2,\cdots,\kappa-1,1\rangle \mid s,t \in GF(\kappa)\} \subset (GF(\kappa))^{\kappa+1}$ represents a maximal assessment variant formation for $\mathfrak{cq} = 1$ of order κ.*

Proof. We leave the cases $\kappa = 4$ and $\kappa = 5$ as examples and leave the case $\kappa = 3$ as an exercise to the reader; so we can assume $\kappa > 5$.

Let F be the subspace of $GF(\kappa)^{\kappa+1}$ spanned by $\vec{v}_1 = \langle 1,1,1,\cdots,1,0\rangle$ and $\vec{v}_2 = \langle 0,1,2,\cdots,\kappa-1,1\rangle$. Hence F is of size κ^2. It remains to show that $\mathfrak{cq} = 1$.

For each pair $\vec{g}_1, \vec{g}_2 \in F$, $\vec{g}_1 - \vec{g}_2 = s\langle 1,1,1,\cdots,1,0\rangle + t\langle 0,1,2,\cdots,\kappa-1,1\rangle$ for $s,t \in GF(\kappa)$.
If $\vec{g}_1(i) = \vec{g}_2(i)$ and $\vec{g}_1(j) = \vec{g}_2(j)$ for distinct i,j, then $(\vec{g}_1 - \vec{g}_2)|_{\{i,j\}} = \vec{0}$, that is, $(\vec{g}_1 - \vec{g}_2)|_{\{i,j\}} = s\langle 1,1,1,\cdots,1,0\rangle|_{\{i,j\}} + t\langle 0,1,2,\cdots,\kappa-1,1\rangle|_{\{i,j\}} = \langle 0,0\rangle$. We show that any pair of the form $\{\langle 1,1,1,\cdots,1,0\rangle|_{\{i,j\}}, \langle 0,1,2,\cdots,\kappa-1,1\rangle|_{\{i,j\}}\}$ are linearly independent:

Case 1: For $i < j \neq \kappa+1$, then $\vec{v}_1|_{\{i,j\}} = \langle 1,1\rangle$ and $\vec{v}_2|_{\{i,j\}} = \langle i-1, j-1\rangle$ and they are linearly independent.

Case 2: For $i < j = \kappa+1$, then $\vec{v}_1|_{\{i,j\}} = \langle 1,0\rangle$ and $\vec{v}_2|_{\{i,j\}} = \langle i-1, 1\rangle$ and they are linearly independent.

Then s and t have to be both zero. Therefore, $\vec{g}_1 - \vec{g}_2$ has at most one zero entry, i.e., $\mathfrak{cq} = 1$. □

Theorem 2 tells us that in this maximal assessment variant formation every pair of assessment variants share exactly one question. We can see this directly, because for each pair (s_1, t_1) and (s_2, t_2), either $t_1 = t_2$, so that $s_1\langle 1,1,1,\cdots,1,0\rangle + t_1\langle 0,1,2,\cdots,\kappa-1,1\rangle$ and $s_2\langle 1,1,1,\cdots,1,0\rangle + t_2\langle 0,1,2,\cdots,\kappa-1,1\rangle$ share only question $Q_{\kappa+1}$, or $s_1 + t_1 i = s_2 + t_2 i$ has a unique solution i, so that
$s_1\langle 1,1,1,\cdots,1,0\rangle + t_1\langle 0,1,2,\cdots,\kappa-1,1\rangle$ and $s_2\langle 1,1,1,\cdots,1,0\rangle$
$+\, t_2\langle 0,1,2,\cdots,\kappa-1,1\rangle$ share only question Q_{i+1} $(0 \leq i \leq \kappa-1)$.

Example 3. We can generate a maximal assessment variant formation for $\mathfrak{cq} = 1$ of order $\kappa = 4$ in $(GF(4))^5$.

Consider the vector space $(GF(2^2))^5$, where the sum operation is induced by $+$ and scalar multiplication is induced by $\times$ in Table 1. The subspace generated by $\{\langle 0,1,2,3,1\rangle, \langle 1,1,1,1,0\rangle\}$ consists of the following assessment variants.

$\{b_{10}, b_{20}, b_{30}, b_{40}, b_{50}\}$	$\{b_{10}, b_{21}, b_{32}, b_{43}, b_{51}\}$	$\{b_{10}, b_{22}, b_{33}, b_{41}, b_{52}\}$	$\{b_{10}, b_{23}, b_{31}, b_{42}, b_{53}\}$
$\{b_{11}, b_{21}, b_{31}, b_{41}, b_{50}\}$	$\{b_{11}, b_{20}, b_{33}, b_{42}, b_{51}\}$	$\{b_{11}, b_{23}, b_{32}, b_{40}, b_{52}\}$	$\{b_{11}, b_{22}, b_{30}, b_{43}, b_{53}\}$
$\{b_{12}, b_{22}, b_{32}, b_{42}, b_{50}\}$	$\{b_{12}, b_{23}, b_{30}, b_{41}, b_{51}\}$	$\{b_{12}, b_{20}, b_{31}, b_{43}, b_{52}\}$	$\{b_{12}, b_{21}, b_{33}, b_{40}, b_{53}\}$
$\{b_{13}, b_{23}, b_{33}, b_{43}, b_{50}\}$	$\{b_{13}, b_{22}, b_{31}, b_{40}, b_{51}\}$	$\{b_{13}, b_{21}, b_{30}, b_{42}, b_{52}\}$	$\{b_{13}, b_{20}, b_{32}, b_{41}, b_{53}\}$

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\times$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Table 1. Addition and multiplication tables for $GF(4)$.

Example 4. For $\mathfrak{cq} = 1$, the set of all images of elements in F in Example 2 is a maximal assessment variant formation for $\mathfrak{cq} = 1$ of size 25 of order $\kappa = 5$. A LaTeX template to generate an assessment variant formation and a question pool template is given in Figure A1 to be used with the code in Figure C1. The operations $+$ and $\times$ used for $(GF(5))^6$ is induced by $+$ and $\times$ in $\mathbb{Z}_5$.

3.2. How the case $\mathfrak{cq} = 1$ is related to Finite Projective Planes

Definition 4. A projective plane consists of a set of points and a set of lines, each line defined as a subset of the set of points, having the following properties:

(1) For any two distinct points, there is exactly one line that includes both points.
(2) For any two distinct lines, there is exactly one point that is included in both lines.
(3) There are four points such that no line includes more than two of them.

A projective plane P is of order κ if it contains $\kappa^2 + \kappa + 1$ lines and $\kappa^2 + \kappa + 1$ points; furthermore, each point in P belongs to $\kappa + 1$ lines and each line in P contains $\kappa + 1$ points. [1]

Theorem 4. *A maximal assessment variant formation for $\mathfrak{cq} = 1$ of order $\kappa > 3$ exists if and only if a projective plane of order κ exists.*

Proof. Let V be a maximal assessment variant formation for $\mathfrak{cq} = 1$ and of order κ, with question pools B_i $(1 \leq i \leq \kappa + 1)$. Define an arrangement of points and lines as follows. The set of points is $\{\infty\} \cup \bigcup B_i$, and the set of lines is $V \cup \{B_i \cup \{\infty\} : 1 \leq i \leq \kappa + 1\}$. Thus there are $\kappa^2 + \kappa + 1$ points and $\kappa^2 + \kappa + 1$ lines. We show this gives a projective plane.

Axiom 1: If the two questions are from different pools, $b_{i\ell}$ and b_{jm} for $i \neq j$, there is exactly one assessment variant containing $\{b_{i\ell}, b_{jm}\}$ since $\mathfrak{cq} = 1$. If the two questions are in the same pool or one point is infinity, then the pool containing the question(s) is the line.

Axiom 2: If two lines are two distinct assessment variants, then by Theorem 3, they intersect in exactly one point.

If two lines are both (extended) question pools, then infinity is the only common point. If one line is an assessment variant and the other one is a pool, since each assessment variant contains exactly one question from each question pool, the intersection is that question from that pool.

Axiom 3: A set needed for the third axiom is a set of two questions from B_1 and two questions from B_2.

Conversely, let P be a projective plane of order κ, and fix $p \in P$. Let L_1, L_2, $\ldots$, $L_{\kappa+1}$ be the lines containing p. Let $B_i = L_i \setminus \{p\}$; the sets B_i are the question pools. There are κ^2 lines P not containing p; let L be such a line. We claim these are assessment variants for a maximal assessment variant formation with question pools B_i. We check that each point in L is contained in one of the $B_i = L_i \setminus \{p\}$. Each L contains exactly $\kappa + 1$ points and intersects each of the $\kappa + 1$ L_i in exactly one point. Since no point can be in L and in two of the L_i, the intersections $L \cap L_i$ cover all $\kappa + 1$ points of L. This setup provides $\kappa + 1$ question pools, each of size κ and κ^2 exam variants, where each exam variant contains exactly one question from each pool; finally, every pair of question variants have only one common question. □

Corollary 1. *For every power p^m of a prime p with $p^m > 3$, there exists a maximal assessment variant formation for $\mathfrak{cq} = 1$ of order p^m.*

There is no maximal assessment variant formation for $\mathfrak{cq} = 1$ of order $\kappa = 6$ or order $\kappa = 10$.

This is because there exists a projective plane of every prime power order. It is not known whether there exists a projective plane of any order that is not a power of a prime, although it is known that no projective planes of order 6 or 10 exist. So Theorem 4 does not answer the question of whether a maximal assessment variant formation exists for $\mathfrak{cq} = 1$ with order not a prime power.

4. The Proof for $\mathfrak{cq} > 1$

There is a natural way of generating assessment variants from $2\kappa + 2$ disjoint question pools where each pool is of size κ: divide the sets of pools into two equal size sets and generate two maximal assessment variant formations; pair each assessment variant from the first set with exactly one assessment variant from the second set. This method creates κ^2 assessment variants, from $2\kappa + 2$ pools of size κ, where $\mathfrak{cq} = 2$. This method does not give the optimum pool size but it can be desirable with larger $\mathfrak{s}$.

Definition 5. Let $\mathcal{V}_1 = \{v_i | 1 \le i \le N_1\}$ be a set of N_1 distinct assessment variants, and $\mathcal{V}_2 = \{u_i | 1 \le i \le N_2\}$ a set of N_2 distinct assessment variants, $\mathcal{V}_1$ and $\mathcal{V}_2$ have disjoint question pools. The *distinct concatenation* of $\mathcal{V}_1$ and $\mathcal{V}_2$ is the set $\mathcal{C} := \{v_i \cup u_i | 1 \le i \le \min(N_1, N_2)\}$.

Note: $\mathcal{C}$ is a set of exam variants of size $min(\mathcal{N}_1, \mathcal{N}_2)$ with $\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_2$ and $\mathfrak{cq} \le \mathfrak{cq}_1 + \mathfrak{cq}_2$, where $\mathfrak{s}_i$ is the number of questions and $\mathfrak{cq}_i$ is the maximum number of common questions for $\mathcal{V}_i$.

In Theorem 5, we construct an optimal number of exam variants for $\mathfrak{cq} = 2$.

Theorem 5. *Let $\kappa > 3$ be a positive integer power of a prime number. Then*
$G = \{s \underbrace{\langle 1, 1, 1, \cdots, 1, 0\rangle}_{\vec{u}_1} + t \underbrace{\langle 0, 1, 2, \cdots, \kappa - 1, 0\rangle}_{\vec{u}_2} + u \underbrace{\langle 0, 1^2, 2^2, 3^2, \cdots, (\kappa - 1)^2, 1\rangle}_{\vec{u}_3} | s, t, u \in GF(\kappa)\} \subset (GF(\kappa))^{\kappa+1}$
represents a set of κ^3 assessment variants with $\mathfrak{s} = \kappa + 1$ and $\mathfrak{cq} = 2$.

Furthermore, if $GF(\kappa)$ is of characteristic 2 (that is, for all $i \in GF(\kappa)$, $i + i = 0$), then
$H = \{s \underbrace{\langle 1, 1, 1, \cdots, 1, 0, 0\rangle}_{\vec{v}_1} + t \underbrace{\langle 0, 1, 2, \cdots, \kappa - 1, 1, 0\rangle}_{\vec{v}_2} + u \underbrace{\langle 0, 1^2, 2^2, 3^2, \cdots, (\kappa - 1)^2, 0, 1\rangle}_{\vec{v}_3} | s, t, u \in GF(\kappa)\} \subset (GF(\kappa))^{\kappa+2}$
represents a set of κ^3 assessment variants with $\mathfrak{s} = \kappa + 2$ and $\mathfrak{cq} = 2$.

Proof. The proof is similar to the proof of Theorem 3.

First note that the sets $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly independent; therefore, each set spans a vector subspace of size κ^3. It remains to show that each pair of the vectors in G agree in fewer than 3 entries, and that the same holds for H if $GF(\kappa)$ is of characteristic 2.

For each pair $\vec{w}_1$ and $\vec{w}_2$ in H, $\vec{w}_1 - \vec{w}_2 = s\vec{v}_1 + t\vec{v}_2 + u\vec{v}_3$. If $\vec{w}_1(i) = \vec{w}_2(i)$, $\vec{w}_1(j) = \vec{w}_2(j)$ and $\vec{w}_1(k) = \vec{w}_2(k)$, for distinct i, j, k, then $(\vec{w}_1 - \vec{w}_2)|_{\{i,j,k\}} = \vec{0}$. It remains to show $\{\vec{v}_1|_{\{i,j,k\}}, \vec{v}_2|_{\{i,j,k\}}, \vec{v}_3|_{\{i,j,k\}}\}$ is linearly independent for any distinct i, j, k.

Associate with the subspace H the matrix $A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix}$. The set of columns of A is

$$C = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \cup \left\{ \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix} : 1 \le \alpha \le \kappa - 1 \right\},$$

where $\vec{e}_i$ is the standard unit vector. Note that deleting the column $\vec{e}_2$ from A gives the matrix whose rows are the vectors $\vec{u}_i$ associated with G. We wish to show that any three vectors in $C \setminus \{\vec{e}_2\}$ are linearly independent for any field $GF(\kappa)$, and any three vectors in C are linearly independent if $GF(\kappa)$ is of characteristic 2. Clearly, any set of three vectors in C including at least two of the $\vec{e}_i$ is linearly independent. We consider the other cases. Note that a set of three vectors is linearly independent if and only if the 3×3 matrix M with these vectors as columns has nonzero determinant.

Case 1: $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \alpha & \beta \\ 0 & \alpha^2 & \beta^2 \end{bmatrix}$ for distinct $\alpha, \beta \neq 0$. The determinant of M is $\alpha\beta^2 - \beta\alpha^2 = \alpha\beta(\beta - \alpha) \neq 0$.

Case 2: $M = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{bmatrix}$ for distinct $\alpha, \beta \neq 0$. The determinant of M is $\beta - \alpha \neq 0$.

Case 3: $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & \alpha & \beta \\ 0 & \alpha^2 & \beta^2 \end{bmatrix}$ for distinct $\alpha, \beta \neq 0$. The determinant of M is $\beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha)$. This is 0 if and only if $\beta = -\alpha$. If $GF(\kappa)$ is of characteristic 2, then $\beta = -\alpha$ if and only if $\beta = \alpha$. So for distinct nonzero α, β if $GF(\kappa)$ of characteristic 2, the determinant of M is nonzero.

Case 4: $M = \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{bmatrix}$ for distinct $\alpha, \beta, \gamma \neq 0$. The determinant of M (a Vandermonde matrix) is $(\gamma - \alpha)(\gamma - \beta)(\beta - \alpha) \neq 0$.

Thus, in all cases, the three vectors are linearly independent, so any pair of vectors in the subspace agree in fewer than 3 entries. $\square$

Note 2. *For the set of assessment variants in Theorem 5, if we raise the number of learning goals, $\mathfrak{s}$, by one and add a new question pool of κ questions, $\mathfrak{cq}$ will be raised to 3. This is because any additional length 3 vector forms a linearly dependent set with two columns of A.*

Note 3. *Theorem 5 provides an optimal size for a set of assessment variants, where each pair of variants have at most two questions in common, but some pairs have fewer than two questions in common. This is different from a maximal assessment variant formation for $\mathfrak{cq} = 1$ and of order κ, where each pair have exactly one question in common. Using a Python code we checked the set of assessment variants generated in Theorem 5. Table 2 contains the result concerning the intersection of pairs of variants for $\kappa = 7, 8, 9, 11, 13, 17$. Note that for each κ, arrays for κ^3 assessment variants were generated, and $\frac{\kappa^3(\kappa^3-1)}{2}$ distinct pairs of those variants were considered.*

κ	$\mathfrak{s}$	Number of distinct pairs with no common questions	Number of distinct pairs with 1 common question	Number of distinct pairs with 2 common questions
7	8	21609	8232	28812
8	10	50176	0	80640
9	10	104976	29160	131220
11	12	366025	79860	439230
13	14	1028196	184548	1199562
17	18	5345344	707472	6013512

Table 2. Intersection of variants when $\mathfrak{cq} = 2$.

In Theorems 3 and 5, we give bases to generate a set of assessment variants for $\mathfrak{s} = \kappa+1$ questions. If we restrict the basis vectors to vectors of smaller dimensions; the new basis generates a set of assessment variants for $\mathfrak{s} < \kappa + 1$. So from Theorems 1, 3, and 5 we get the following.

Theorem 6. *Suppose p is a prime number, $m \geq 1$ is an integer and $\kappa = p^m$. Then for any $\mathfrak{s} \leq \kappa + 1$, there exists a set of assessment variants of size κ^{c+1} for $\mathfrak{cq} = c$ where $c \in \{1, 2\}$.*

Note 4. *Using Python, we verified that for $\kappa = 7, 8, 9, 11$ a basis in $GF(\kappa)^{\kappa+1}$, similar to those in Theorems 3 and 5, will generate κ^4 assessment variants with $\mathfrak{s} = \kappa + 1$ and $\mathfrak{cq} = 3$. The ratio $\frac{\mathfrak{cq}}{\mathfrak{s}}$ is at least $\frac{1}{4}$ for these cases; this ratio is already considered high for conserving the integrity of the exam, [4]. We do not see any value in expanding the method in Theorem 3 to higher $\mathfrak{cq} > 2$. Instead we use other methods to complement Theorem 3.*

Example 5. An instructor has $\mathcal{N} = 98$ students in a course and is planning to write an unproctored exam with $\mathfrak{s} = 20$ learning goals. How should the instructor choose the number of questions in each question pool?

This instructor has two major constraints for writing this exam: ① the number of questions they are able to write for this assessment, which itself can be related to time constraints, question type constraints and other factors; and ② the maximum number of questions in common for each pair of assessment variants.

Case 1: The instructor doesn't mind writing a large number of questions. Then creating 20 pools of $\kappa = 19$ questions is recommended, that is, $20 \times 19 = 380$ questions in total. These pools will produce $19^2 = 361$ exam variants with $\mathfrak{cq} = 1$, using the basis in Theorem 3.

Case 2: The instructor is trying to minimize the number of question in each pool and $\mathfrak{cq} \leq 2$ is acceptable. In this case, $\kappa = 11$ is the smallest prime power whose square is larger than $\mathcal{N} = 98$. Using distinct concatenation of two assessment variant sets each made from 10 pools of $\kappa = 11$ questions ($20 \times 11 = 220$ questions in total) generates $11^2 = 121$ exam variants with $\mathfrak{cq} = 2$.

Case 3: The instructor is trying to minimize the number of questions in each pool and $\mathfrak{cq} \leq 3$ is acceptable. Then they can divide the learning goals into two sets of sizes 12 and 8; creating 11 questions in each of the first 12 pools and 7 questions in each of the remaining 8 pools. Using Theorem 3, they generate 121 assessment variants with $\mathfrak{cq} = 1$; using Theorem 5, the generate $8^3 = 343$ assessment variants with $\mathfrak{cq} = 2$. The set of distinct concatenation of the two assessment variants is of size 121 with $\mathfrak{cq} = 3$. In this case, the instructor must write a total number of

$12 \times 11 + 8 \times 7 = 188$ questions.

Depending on how the instructor wants to arrange the exam, there are multiple ways of using Theorems 3 and 5 and distinct concatenation to create assessment variants.

5. How to Choose the Number of Questions in Each Pool

When deciding on the number of questions per pool, we typically have two known parameters. The first known parameter is $\mathcal{N}$, the number of assessment variants needed, which is either the number of students in the course or the total number of expected retake exams. The other known parameter is $\mathfrak{s}$, the number of questions or the number of question pools in the assessments. Choose a value for $\mathfrak{cq}$ to achieve a desired ratio $\mathfrak{cq} : \mathfrak{s}$. If $\mathfrak{cq} = 1$, then use the corresponding case below to find the size of each question pool. If $\mathfrak{cq} = 2$, decide if you need to apply distinct concatenation or not. $\mathfrak{cq} = 2$ without distinct concatenation produces a large number of assessment variants and is ideal for large courses. $\mathfrak{cq} = 2$ with distinct concatenation, produces a smaller set of assessment variants and requires reasonably fewer questions in each question pool.

$\mathfrak{cq} = 1$: (Before distinct concatenation.) Find the smallest positive integer power of a prime, $\kappa = p^m$ such that $p^m \geq \mathfrak{s} - 1$ and $p^{2m} > \mathcal{N}$. When $\mathfrak{s}$ is relatively high this process may require writing a high volume of questions; we recommend $\mathfrak{cq} > 1$ with distinct concatenation.

$\mathfrak{cq} = 2$: (Before distinct concatenation.) Find the smallest positive integer power of a prime, $\kappa = p^m$ such that $p^m \geq \mathfrak{s} - \ell$ and $p^{3m} > \mathcal{N}$, where $\ell = 1$ when $p > 2$ and $\ell = 2$ when $p = 2$.

If you decide to use $\mathfrak{cq} > 1$ with distinct concatenation, divide the learning goals into smaller sets. Depending on the number of distinct concatenations you are applying, find the number of questions per pool in each set of learning goals, using the above cases. Generate the sets of assessment variants for each set of learning goals. Then distinctly concatenate the sets of assessment variants.

6. Conclusion and Further Questions

Even when classes fully in-person, large question pools can be used to create pre-assessments, [3]. In this way students can practice concepts before a proctored assessment, with the incentive of earning a few points. This work can be used to effectively create mastery-based assessments. In fact one of the authors and two other colleagues have been using this method to create mastery-based assessments for two courses. We each print large pdf files of assessment variants and pdf files of the solutions for grading purposes. This way, each student can retake the assessment using a new variant. We are planning to discuss some of the classroom practices, appropriate for large question pools, in another paper.

Finally, some open questions:

(1) For what values of κ and $\mathfrak{cq} > 1$, does a maximal assessment variant formation exist?

(2) Let $\mathfrak{s} = \kappa + 1$, and $\mathcal{N} = \kappa^2$; our method renders $\mathfrak{cq} = 1$. The number of ways to choose a maximal assessment variant formation is far less than the number of choices of all sets of assessment variants of size $\kappa+1$. Therefore in case $\mathcal{N} = \kappa+1$, the expected value of $\mathfrak{cq}$ in the randomization method is bigger than the value of $\mathfrak{cq}$ when using the methods of this paper. Are there ways to improve expected value of $\mathfrak{cq}$ for the method of randomization?

(3) When there are major difficulties in creating large enough pools for a few learning goals, what are the best practices for generating assessment variants?

Acknowledgments

The implementation of our paper mastery-based assessments was supported by the University of Kansas Course Transformation Grant in Spring 2022.

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Appendix A. Main LaTeX File for 6 Pools of 5 Questions, One Question in Common

Figure A1. This is the main file for creating a set of 25 assessment variants from 6 pools of 5 questions each, where each pair of assessment variants have only one question in common. Create another file, named Questions, for pools. The code for the pools is given in Figure C1.

```
\documentclass[12pt]{article}
\usepackage{amsmath, amsthm, amssymb}
\usepackage{multicol,pgf,tikz,import,ifthen,framed,aurical}
\usetikzlibrary{math}
\usepackage[margin=0.6in]{geometry}
\begin{document}
\tikzmath{\p=5;\seven=0;\kone=1;\ktwo=2;\kthree=3;\kfour=4;\kfive=5;\ksix=1;}
\tikzmath{\lone=1;\ltwo=1;\lthree=1;\lfour=1;\lfive=1;\lsix=0;}
\foreach \l in {0,1,...,4}
{\foreach \k in {0,1,...,4}
    {\tikzmath{\one=int(Mod(\l*\lone+\kone*\k,\p));}
    \tikzmath{\two=int(Mod(\l*\ltwo+\k*\ktwo,\p));}
    \tikzmath{\three=int(Mod(\l*\lthree+\k*\kthree,\p));}
    \tikzmath{\four=int(Mod(\l*\lfour+\k*\kfour,\p));}
    \tikzmath{\five=int(Mod(\l*\lfive+\k*\kfive,\p));}
    \tikzmath{\six=int(Mod(\l*\lsix+\k*\ksix,\p));}
    \input{Questions}
    }}
\end{document}
```

Appendix B. Main LaTeX File for 6 Pools of 5 Questions, Two Questions in Common

Figure B1. This is the main file for creating a set of 125 assessment variants from 6 pools of 5 questions each where each pair of assessment variants have at most two question in common. Create another file, named Questions, for pools. The code for the pools is given in Figure C1.

```
\documentclass[12pt]{article}
\usepackage{amsmath, amsthm, amssymb}
\usepackage{multicol,pgf,tikz,import,ifthen,framed,aurical}
\usetikzlibrary{math}
\usepackage[margin=0.6in]{geometry}
\begin{document}
\tikzmath{\p=5;\kone=1;\ktwo=1;\kthree=1;\kfour=1;\kfive=1;\ksix=0;
\lone=1;\ltwo=2;\lthree=3;\lfour=4;\lfive=0;\lsix=0;
\mone=int(Mod(int((\lone)^2),\p)); \mtwo=int(Mod(int((\ltwo)^2),\p));
\mthree=int(Mod(int((\lthree)^2),\p));\mfour=int(Mod(int((\lfour)^2),\p));
\mfive=0;\msix=1;}
\foreach \m in {0,1,...,4}
{\foreach \l in {0,1,...,4}
{\foreach \k in {0,1,...,4}
    {\tikzmath{\one=int(Mod(\l*\lone+\kone*\k+(\kone)^2*\m,\p));}
    \tikzmath{\two=int(Mod(\l*\ltwo+\k*\ktwo+(\ktwo)^2*\m,\p));}
    \tikzmath{\three=int(Mod(\l*\lthree+\k*\kthree+(\kthree)^2*\m,\p));}
    \tikzmath{\four=int(Mod(\l*\lfour+\k*\kfour+(\kfour)^2*\m,\p));}
    \tikzmath{\five=int(Mod(\l*\lfive+\k*\kfive+(\kfive)^2*\m,\p));}
    \tikzmath{\six=int(Mod(\l*\lsix+\k*\ksix+(\msix)*\m,\p));}
    \input{Questions}}}}
\end{document}
```

Appendix C. LaTeX File for Storing 6 Pools of 5 Questions

Figure C1. This is a sample of 6 question pools where all pools are in one file. Instruction: Save all in a file name Questions.tex in the same directory as the file in Figure A1. Replace each b_{ij} by a question in pool i.

```
\clearpage\setcounter{page}{1}
\begin{center}{\bf \huge Nature 101 - Prerequisite Quiz}\end{center}
\hrule\vskip 7pt
{\bf \Large Name (in print):} \hskip 5cm {\bf \Large Lab Instructor: }
\vskip 5pt\hrule
\begin{multicols}{2}
\begin{enumerate}\setcounter{enumi}{0}%Question Pool 1 for Learning objective1
    \item\ifthenelse{\one=0}{$b_{10}$}{\ifthenelse{\one=1}{$b_{11}$}
      {\ifthenelse{\one=2}{$b_{12}$}{\ifthenelse{\one=3}{$b_{13}$}
      {\ifthenelse{\one=4}{$b_{14}$}{\ifthenelse{\one=5}{Error!}{}}}}}}
\vskip 2cm%Question Pool 2 for Learning objective2
    \item\ifthenelse{\two=0}{$b_{20}$}{\ifthenelse{\two=1}{$b_{21}$}
      {\ifthenelse{\two=2}{$b_{22}$}{\ifthenelse{\two=3}{$b_{23}$}
      {\ifthenelse{\two=4}{$b_{24}$}{\ifthenelse{\two=5}{Error!}{}}}}}}
\vskip 2cm%Question Pool 3 for Learning objective3
    \item\ifthenelse{\three=0}{$b_{30}$}{\ifthenelse{\three=1}{$b_{31}$}
      {\ifthenelse{\three=2}{$b_{32}$}{\ifthenelse{\three=3}{$b_{33}$}
      {\ifthenelse{\three=4}{$b_{34}$}{\ifthenelse{\three=5}{Error!}{}}}}}}
\end{enumerate}\vskip 2.5cm\columnbreak
\begin{enumerate}\setcounter{enumi}{3}
%Question Pool 4 for Learning objective4
    \item\ifthenelse{\four=0}{$b_{40}$}{\ifthenelse{\four=1}{$b_{41}$}
      {\ifthenelse{\four=2}{$b_{42}$}{\ifthenelse{\four=3}{$b_{43}$}
      {\ifthenelse{\four=4}{$b_{44}$}{\ifthenelse{\four=5}{Error!}{}}}}}}
      \vskip 2cm%Question Pool 5 for Learning objective5
    \item \ifthenelse{\five=0}{$b_{50}$}{\ifthenelse{\five=1}{$b_{51}$}
      {\ifthenelse{\five=2}{$b_{52}$}{\ifthenelse{\five=3}{$b_{53}$}
      {\ifthenelse{\five=4}{$b_{54}$}{\ifthenelse{\five=5}{Error!}{}}}}}}
\vskip 2cm%Question Pool 6 for Learning objective6
    \item \ifthenelse {\six=0}{$b_{60}$}{\ifthenelse {\six=1}{$b_{61}$}
      {\ifthenelse{\six=2}{$b_{62}$}{\ifthenelse{\six=3}{$b_{63}$}
      {\ifthenelse{\six=4}{$b_{64}$}{\ifthenelse{\six=5}{Error!}{}}}}}}
\end{enumerate}\vskip 3.5cm\end{multicols}\pagebreak
```