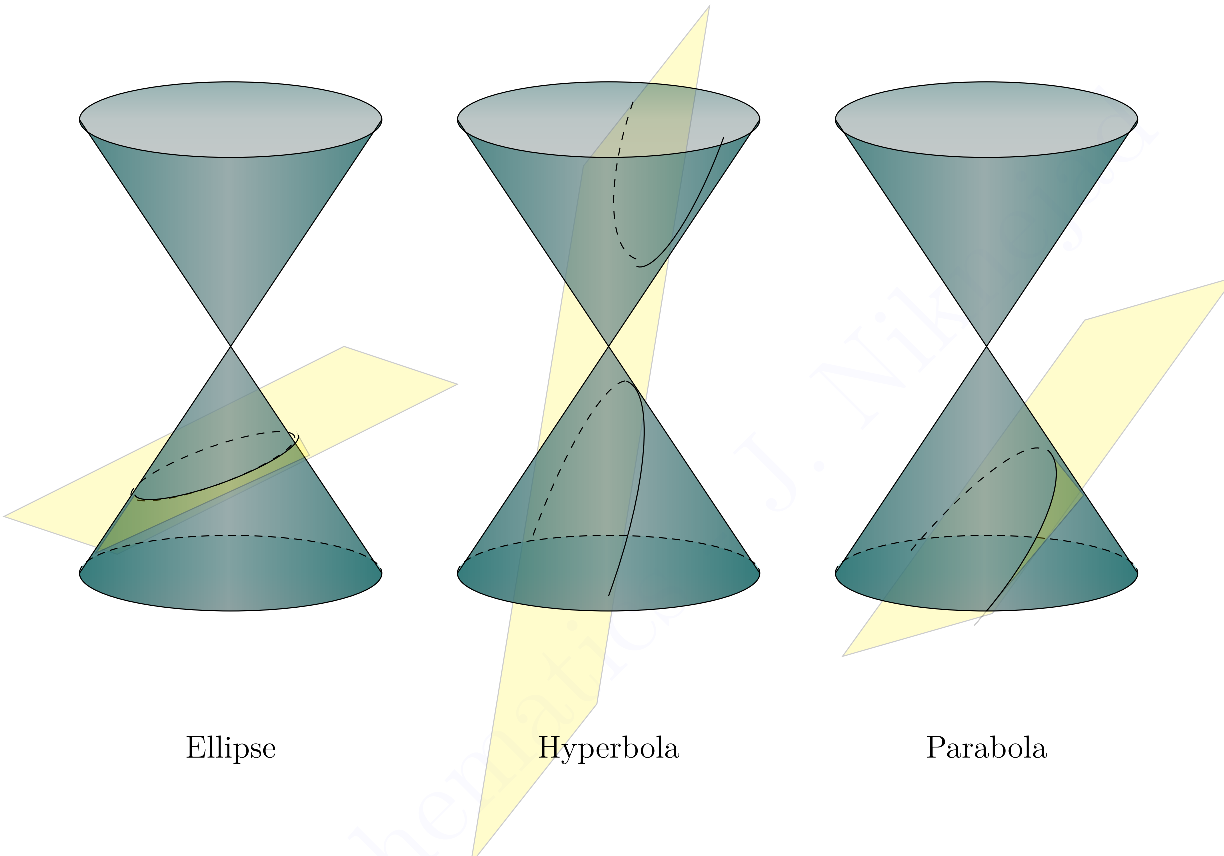
10.1: Ellipses

In the next three sections, we discuss conical sections, starting with ellipses.

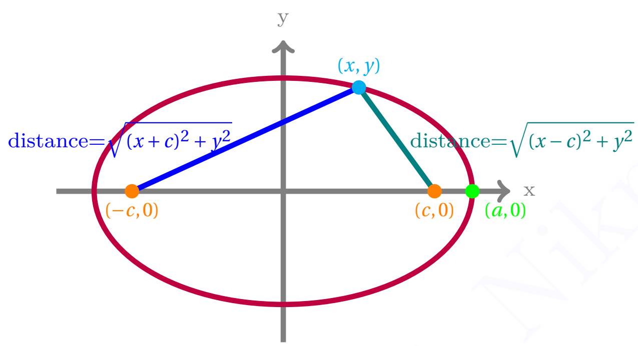
Conical Sections



Ellipses

- Geometric definition: The set of points whose sum of distances from two points (each called a focus and together called **foci**) is constant.

- Using the geometric definition to find a formula



Note that in the picture the sum of distances between the vertex $(a,0)$ from each of the two foci is $2a$ so the sum of the distances between any point on the ellipse and each of the foci should be equal to $2a$.

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a$$

Manipulate the same way you would when solving equations with two radicals. Solving for x or y renders the same answer:

$\underset{\text{Isolate one radical:}}{\Longrightarrow}$ $\sqrt{(x-c)^2+y^2}=2a-\sqrt{(x+c)^2+y^2}$

$\underset{\text{Raise to Power 2:}}{\Longrightarrow}$ $(x-c)^2+y^2=4a^2-4a\sqrt{(x+c)^2+y^2}+(x+c)^2+y^2$

$\underset{\text{Use binomial expansion and simplify:}}{\Longrightarrow}$ $-4xc=4a^2-4a\sqrt{(x+c)^2+y^2}$

$\underset{\text{factor 4 and isolate the radical:}}{\Longrightarrow}$ $a\sqrt{(x+c)^2+y^2}=a^2+cx$

$\underset{\text{Raise to power 2 again:}}{\Longrightarrow}$ $a^2(x+c)^2+a^2y^2=a^4+2a^2cx+c^2x^2$

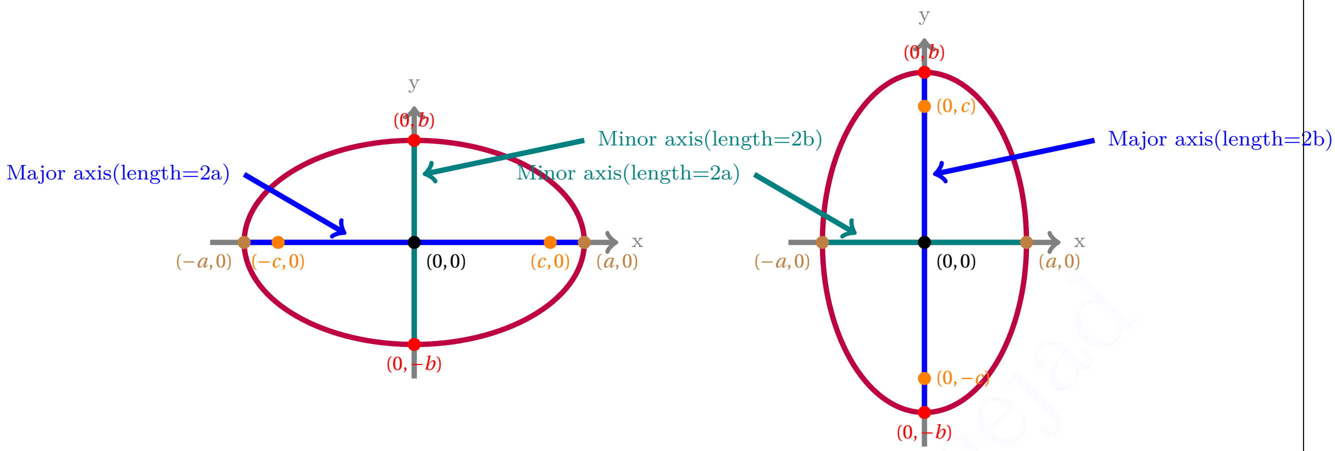
$\underset{\text{Simplify:}}{\Longrightarrow}$ $a^2x^2+a^2c^2+a^2y^2=a^4+c^2x^2$

$\underset{\text{Isolate the terms with variables:}}{\Longrightarrow}$ $(a^2-c^2)x^2+a^2y^2=a^4-a^2c^2$

$\underset{\text{Denote } a^2-c^2 \text{ by } b^2:}{\Longrightarrow}$ $b^2x^2+a^2y^2=a^2b^2$

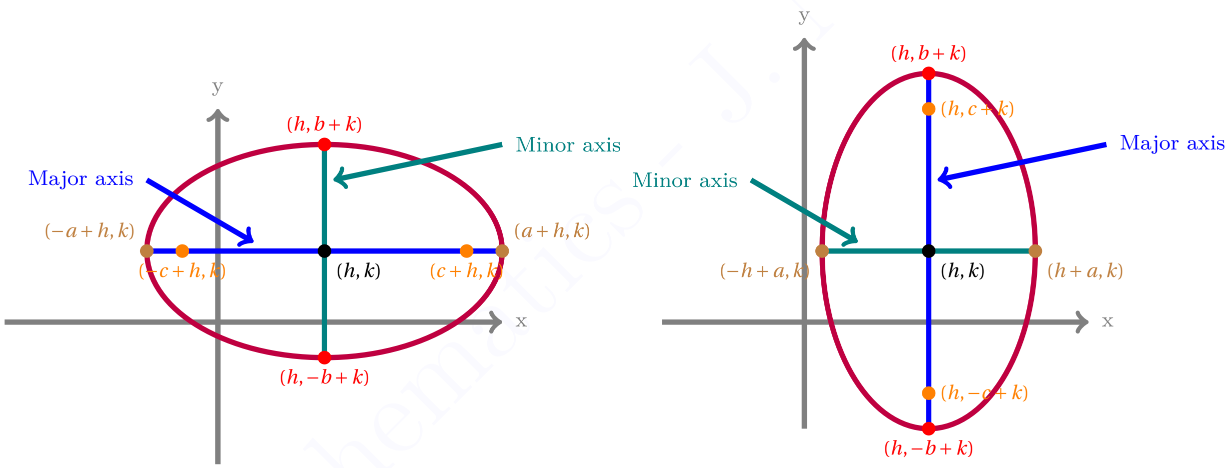
$\underset{\text{Divide by } a^2b^2:}{\Longrightarrow}$ $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

- Graphs of ellipses where axis are vertical or horizontal



Case: $a > b$ Case: $b > a$

Standard Equation: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ Standard Equation: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



Case: $a > b$ and center (h, k) Case: $b > a$ and center (h, k)

Standard Equation: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ Standard Equation: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

- How to find different parameters for an ellipse using its equation:

1. If the equation is anything other than the above equations, reformat to one of the above. Hint: You may need to complete the square.
2. Use the equation to find the horizontal and vertical line containing each axis. If center is $(0, 0)$, then the x-axis and y-axis contain the ellipse's axes. If the center is (h, k), then lines $x = h$ and $y = k$ contain the axes of ellipse.
3. To find endpoints of the horizontal axis, plug in $y = 0$ or $y = k$ and solve for x.
4. To find the endpoints of vertical axis, plug in $x = 0$ or $x = h$ and solve for y.
5. Find c using the equation $c^2 = |a^2 - b^2|$.

Equation of a Circle

By setting $a = b$ in any of standard equations of ellipse, equation of a circle is obtained.

1. Write each of the following ellipse equations in standard form and find a and b. Then find c.

   (A) $25x^2+4y^2=100$ (B) $25x^2+4y^2=1$ (C) $4(x-3)^2+36(y-5)^2=36$

2. Sketch the graph of the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$. Label the vertices and foci.

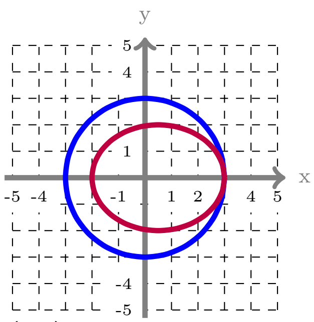
3. Fill in the blank to match each nonlinear system with one of the figures. Do **NOT** solve the systems.

Figure: ___ $\begin{cases} x^2+y^2=9 \\ \dfrac{x^2}{16}+\dfrac{y^2}{4}=1 \end{cases}$

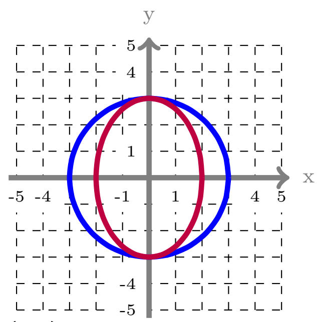
Figure: ___ $\begin{cases} x^2+y^2=9 \\ \dfrac{(x-0.5)^2}{6.25}+\dfrac{y^2}{4}=1 \end{cases}$

Figure: ___ $\begin{cases} x^2+y^2=9 \\ \dfrac{(x-1)^2}{16}+\dfrac{y^2}{4}=1 \end{cases}$

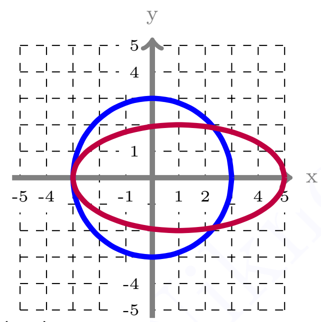
Figure: ___ $\begin{cases} x^2+y^2=9 \\ \dfrac{x^2}{4}+\dfrac{y^2}{9}=1 \end{cases}$



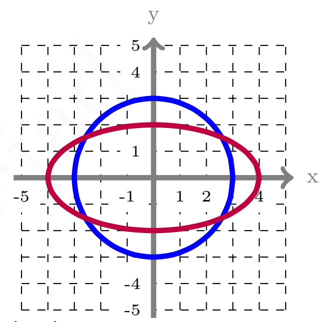
(A) (One solution.)



(B) (Two solutions.)



(C) (Three solutions.)



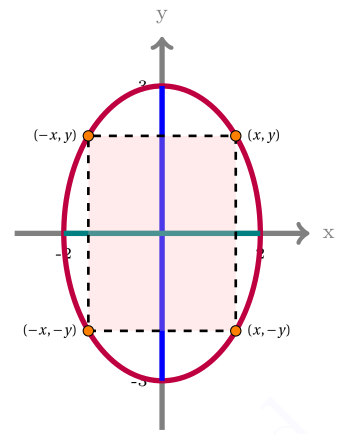
(D) (Four solutions.)

4. Use completing the square to find the standard equation of each of the following ellipses, then find the center of each ellipse.

(A) $4x^2-24x+9y^2+18y+9=0$

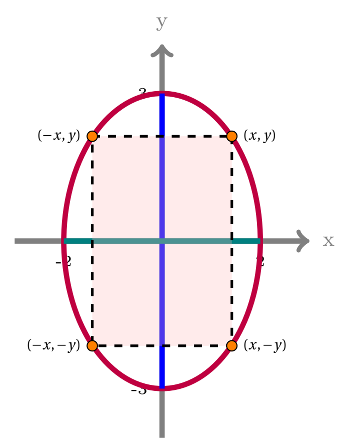
(B) $3x^2-12x+4y^2-8y+8=0$

5. Find the area of a rectangle circumscribed in $\frac{x^2}{4}+\frac{y^2}{9}=1$ as a function of x.

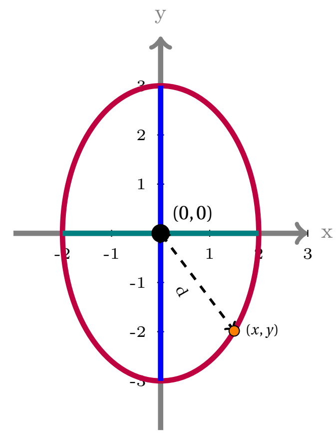


Watch how the area is changing here: https://ggbm.at/dbdd5az7.

6. Use parameterization to find the maximum area of a rectangle circumscribed in $\frac{x^2}{4}+\frac{y^2}{9}=1$. Find the dimensions of such rectangle.



7. Find the distance, d, between a point on the graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and point $(0,0)$ as a function of x.



Standard Equation: $\frac{x^2}{4} + \frac{y^2}{9} = 1$

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4. Example 4: https://mediahub.ku.edu/media/t/1_wwzg1kbh