3.6: Zeros of Polynomial Functions

Theorems and Rules for Finding Roots

- **Fundamental Theorem of Algebra**: a polynomial function with degree greater than 0 has at least one complex zero. (Note that real numbers are subsets of complex numbers.)
- **Linear and Quadratic Factors Theorem**: Every polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors with real coefficients.
- **Conjugate Zeros Theorem**: If a polynomial P has real coefficients and if the complex number z is a zero of P, then the complex conjugate $\bar{z}$ is also a zero of P.
- **Linear Factorization Theorem**: Allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where c is a complex number
- **Factor Theorem**: $x-k$ is a factor of a polynomial P if and only if $P(k)=0$.
- **Remainder Theorem**: If polynomial $P(x)$ is divided by $x-k$, then the remainder is the value $P(k)$.
- **Rational Zero Theorem**: If the polynomial $P(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ has integer coefficients, then every possible rational zero of P is of the form: $\dfrac{p}{q}$ where :

  p is a factor of the constant coefficient a_0 and q is a factor of the leading coefficient, a_n.
- **Descartes' Rule of Signs**: a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $P(x)$ and $P(-x)$.

  Let P be a polynomial with real coefficients.

  (1) The number of positive real zeros of $P(x)$ is either equal to the number of changes in sign in $P(x)$ or less than that by an even whole number.

  (2) The number of negative real zeros of $P(x)$ is equal to the number of changes in sign of $P(-x)$ or less than that by an even whole number.

Watch this example: https://mediahub.ku.edu/media/t/1_3ko6ffj7.

1. Find a polynomial of degree 4, with integer valued coefficients, whose roots include $3, -3, 2-i$.

2. Find the remainder of dividing $x^{5002}+x^{304}+5$ by $x+1$.

3. List all possible rational zeros of $P(x) = 5x^5 - 18x^4 - 6x^3 + 91x^2 - 60x + 9$ given by the Rational Zeros Theorem.

4. Use all theorems to find all rational roots of $P(x) = x^6 - 5x^5 + 8x^4 - 3x^3 - 7x + 6$.

   Hint: (A) List all possible rational zeros. Try to reduce the number of possibilities by any theorem. (B) Plug each number you listed in (A) in $P(x)$ to check if it is a zero.

5. Let $p(x) = x^4 + 2x^3 + x^2 + 12x + 20$

   (a) List all of the rational numbers that could be zeros of $p(x)$ according to the rational zeros theorem. Then use the Factor or the Remainder theorem to find at least one rational zero.

   (b) Use long division/synthetic division and the quadratic formula to find all of the zeros of $p(x)$. List the zeros along with their multiplicities.

   (c) Write $p(x)$ in its fully factored form. (That is, all factors should be linear complex factors.)

6. *(Optional)* By Descartes' rule of signs, how many real zeros does $p(x) = 5x^5 - 18x^4 - 6x^3 + 91x^2 - 60x + 9$ have?

Example Videos:

1. https://mediahub.ku.edu/media/t/1_3ko6ffj7