7.4: Sum-to-Product and Product-to-Sum Formulas

- Derivation of Formulas:

(1)
$$\left.\begin{aligned}\sin(x+y) &= \sin(x)\cos(y)+\cos(x)\sin(y)\\ \sin(x-y) &= \sin(x)\cos(y)-\cos(x)\sin(y)\end{aligned}\right\}\underset{\text{Add}}{\Longrightarrow} \sin(x+y)+\sin(x-y)=2\sin(x)\cos(y)$$

$$\underset{\text{Divide by 2}}{\Longrightarrow} \boxed{\sin(x)\cos(y)=\frac{1}{2}\left(\sin(x+y)+\sin(x-y)\right)} \text{ (I)}$$

$$\underset{\text{Or let } x=\frac{u+v}{2} \text{ and } y=\frac{u-v}{2}}{\Longrightarrow} \boxed{\sin(u)+\sin(v)=2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)} \text{ (II)}$$

(2)
$$\left.\begin{aligned}\sin(x+y) &= \sin(x)\cos(y)+\cos(x)\sin(y)\\ \sin(x-y) &= \sin(x)\cos(y)-\cos(x)\sin(y)\end{aligned}\right\}\underset{\text{Subtract}}{\Longrightarrow} \sin(x+y)-\sin(x-y)=2\cos(x)\sin(y)$$

$$\underset{\text{Divide by 2}}{\Longrightarrow} \cos(x)\sin(y)=\frac{1}{2}\left(\sin(x+y)-\sin(x-y)\right) \text{ (III)}$$

$$\underset{\text{Or let } x=\frac{u+v}{2} \text{ and } y=\frac{u-v}{2}}{\Longrightarrow} \boxed{\sin(u)-\sin(v)=2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)} \text{ (IV)}$$

(3)
$$\left.\begin{aligned}\cos(x+y) &= \cos(x)\cos(y)-\sin(x)\sin(y)\\ \cos(x-y) &= \cos(x)\cos(y)+\sin(x)\sin(y)\end{aligned}\right\}\underset{\text{Add}}{\Longrightarrow} \cos(x+y)+\cos(x-y)=2\cos(x)\cos(y)$$

$$\underset{\text{Divide by 2}}{\Longrightarrow} \boxed{\cos(x)\cos(y)=\frac{1}{2}\left(\cos(x+y)+\cos(x-y)\right)} \text{ (V)}$$

$$\underset{\text{Or let } x=\frac{u+v}{2} \text{ and } y=\frac{u-v}{2}}{\Longrightarrow} \boxed{\cos(u)+\cos(v)=2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)} \text{ (VI)}$$

(4)
$$\left.\begin{aligned}\cos(x+y) &= \cos(x)\cos(y)-\sin(x)\sin(y)\\ \cos(x-y) &= \cos(x)\cos(y)+\sin(x)\sin(y)\end{aligned}\right\}\underset{\text{Subtract}}{\Longrightarrow} \cos(x+y)-\cos(x-y)=-2\sin(x)\sin(y)$$

$$\underset{\text{Divide by 2}}{\Longrightarrow} \boxed{\sin(x)\sin(y)=\frac{-1}{2}\left(\cos(x+y)-\cos(x-y)\right)} \text{ (VII)}$$

$$\underset{\text{Or let } x=\frac{u+v}{2} \text{ and } y=\frac{u-v}{2}}{\Longrightarrow} \boxed{\cos(u)-\cos(v)=-2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)} \text{ (VIII)}$$

In light of formula (I), formula (III) does not render any new information regarding product-to-sum computation.

A List of Formulas to Use in Problems

- **Product-to-Sum or Difference Identities:**

$$\sin(u)\cos(v) = \frac{1}{2}\Big(\sin(u+v)+\sin(u-v)\Big)$$
$$\cos(u)\cos(v) = \frac{1}{2}\Big(\cos(u+v)+\cos(u-v)\Big)$$
$$\sin(u)\sin(v) = \frac{1}{2}\Big(\cos(u-v)-\cos(u+v)\Big)$$

- **Sum-to-Product Identities:**

$$\sin(x)+\sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin(x)-\sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos(x)+\cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos(x)-\cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

1. **Sum to product examples:**

   (A) Write the sum $\sin(3t) - \sin(5t)$ as a product.

   (B) Write the sum $\cos(7t) - \cos(5t)$ as a product.

2. **Product to sum examples:**

   (A) Write the product $\cos(7x)\cos(3x)$ as a sum.

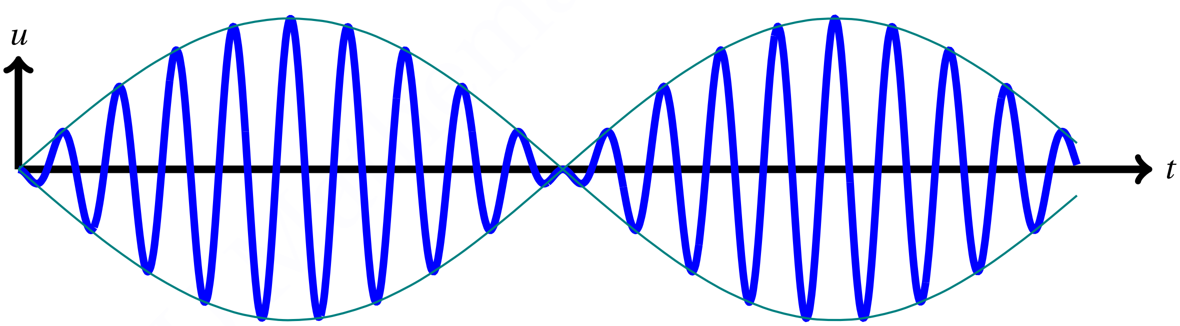
   (B) Write the product $\sin(7x)\cos(5x)$ as a sum.

3. Simplify $\dfrac{\sin(10x) - \sin(2x)}{\cos(10x) + \cos(2x)}$.

4. **Electrical Engineering:** One application of sum-to-product rule is to compute the amplitude modulation of two signals. That is, compute the addition of two signals. When the product is found, one factor of signal is asymptotically present in amplitude and the frequency of oscillation follows the frequency of the signal with higher frequency (whichever has a bigger B.)

   In this phenomena the amplitude changes periodically. This is used in am radio (**amplitude modulation**).[1] It looks like the below graph when B_1 and B_2 are "close" in value.

   In here, the blue graph is $u(t) = \cos(10t) - \cos(9t)$. The green graphs are the enveloping graphs which are only asymptotes. Find the period of either of the green graphs and period of the blue graph.



[1]That is, the signal transfers using amplitude modulation.

Video:

- https://mediahub.ku.edu/media/t/1_53rp4nf0