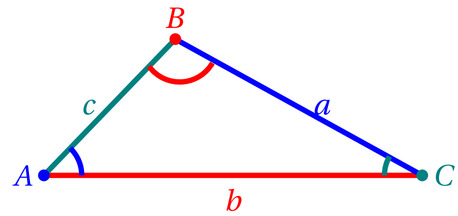
8.1: Non-right Triangles: Law of Sines

- Law of Sines:

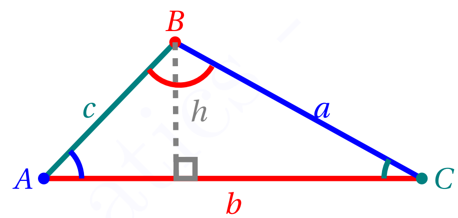


Using the picture above, Law of Sines is the following statement:

$$\frac{\sin(\angle A)}{a} = \frac{\sin(\angle B)}{b} = \frac{\sin(\angle C)}{c}$$

- How it is derived.

First we derive the area formula:



The area of triangle is $\frac{h.b}{2}$. By the definition of sine, $h = a\sin(\angle C)$. Replace h in the area formula to get: Area is $\frac{ab\sin(\angle C)}{2}$ Now this formula is very symmetric with respect to A, B and C. That is, all the other combinations of the formula works:

That is, the area of triangle is equal to

$$\text{Area of Triangle} = \frac{ab\sin(\angle C)}{2} = \frac{ac\sin(\angle B)}{2} = \frac{bc\sin(\angle A)}{2}$$

Now Divide all by abc to get the formula.

- When can it be used?

  ASA: (Angle-Side-Angle) Since any two triangles with congruent ASA are congruent, this gives up to one triangle, up to congruence. To be able to use the formula, first use the Sum of Interior Angles of a Triangle Theorem to solve for the third angle.

  AAS: (Angle-Angle-Side) If two angles of a triangle are congruent, then the third angles are going to be congruent (by the Sum of Interior Angles of a Triangle Theorem). Therefore, AAS implies ASA and any two triangles with AAS are congruent. So AAS gives up to one triangle, up to congruence.

SSA: (Side-Side-Angle) Since SSA is not a congruence condition, it gives, up to, two triangles, up to congruence.

How do we check if a second solution is possible in case of SAA? If the supplementary angle of the arcsine of the angle you found after using the Law of Sines plus the angle that was given is smaller than $180°$, then another triangle exist and one of the angles of that triangle is the supplementary angle.

- When is SSA unique? When are there two triangles with the same SSA? To see different situation with SSA, use this link: https://ggbm.at/t484jnqx. The app at this link shows how to construct any triangle with two fixed side lengths a, c and a fixed angle $\angle A$. Watch how two triangles are built when point A is outside the circle with radius a ($c > a$). Move A on the circle ($c = a$) and inside the circle to see that the triangle becomes unique ($c < a$).
- How to solve with the Law of Sines:
  - You need at least one side length and the angle opposite of the side. Use one equality at a time.
  - In case of SSA, when the first angle is found using arcsine, find the supplementary angle to the angle given. If the supplementary angle plus the given angle is less than $180°$, you will have a second triangle using the supplementary angle of the arcsine of the original angle.

1. Find all angles and sides of a triangle if $\angle A = 30°$, $\angle B = 50°$ and $a = 25$.

2. Find all sides and angles of triangles with given values.

   (a) $a = 20,\ c = 45$ and $\angle A = 125°$

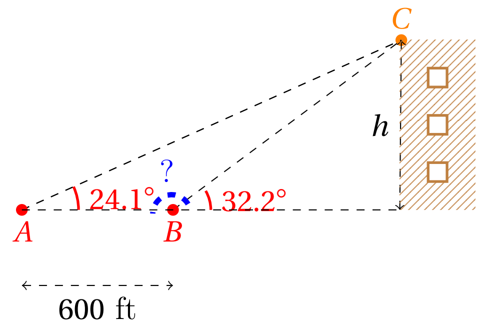
   (b) $b = 248,\ a = 185$ and $\angle A = 43°$.

3. (a) Use the Law of Sines to solve for all possible triangles that satisfy the given conditions.
$a = 23$ and $b = 25$ and $\angle B = 35°$

(b) Use the Law of Sines to solve for all possible triangles that satisfy the given conditions.
$a = 25$ and $b = 23$ and $\angle B = 35°$

(c) Use the Law of Sines to solve for all possible triangles that satisfy the given conditions. $a = 20$, $b = 10$, $\angle A = 120°$

4. **Land surveying, Civil and Environmental Engineering:** From a point A on the ground, the angle of elevation to the top of a building is $24.1°$. From a point B which is 600 feet closer to the building, the angle of elevation is $30.2°$. How tall is the building?



5. Use the Law of Sines to find all possible triangles with $\angle B = 26°$, $b = 9$ and $c = 12$.

Example Videos:

1. https://mediahub.ku.edu/media/t/1_y8wd0m8a