Worksheet 10: Sections 3.4, 2.6, 3.6 and 3.8

They said: "Oh! That fall break was too short!" I said: "True! But we are glad to see your bright faces again!"

The Distance, Velocity and Acceleration: Kinematics is all about calculus! (In fact, it was one of the original motivations for Newton to develop calculus as a separate branch of mathematics.)

Quantity	Symbol	Calculus	Units
Distance	$s(t)$		distance
Velocity	$v(t)$	$= s'(t)$	distance/time
Acceleration	$a(t)$	$= v'(t) = s''(t)$	distance/time^2
Jerk	$j(t)$	$= a'(t) = v''(t) = s'''(t)$	distance/time^3

We often use $h(t)$ (for height) instead of $s(t)$ when the motion is vertical.

Physics: Applications to Kinematics In physics, polynomials are used to model how gravity affects the height of a projectile. Gravity on Earth provides a constant acceleration of -9.8 m/sec^2 $\approx$ -32 ft/sec^2.

By the power rule, the degree of the height function $h(t)$ is two higher than the degree of acceleration $a(t)$. Since acceleration is constant, it has degree zero, and it follows that the height polynomial is quadratic:

$$\begin{aligned} h(t) = pt^2 + qt + r \quad &\Rightarrow \quad v(t) = h'(t) = 2pt + q \\ &\Rightarrow \quad a(t) = v'(t) = h''(t) = 2p \end{aligned}$$

What do p, q, r signify? Since $a(t) = -9.8$ we have $p = -4.9$. Also, $q = v(0) = v_0$ is the initial velocity of the object, and $r = h(0) = h_0$ is the initial position.

$$\boxed{h(t) = -4.9t^2 + v_0 t + h_0}$$

Trig Limits:

Formulas: $\displaystyle\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$ and $\displaystyle\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta} = 0.$ You may need to regroup.

You may need to multiply by $\dfrac{n}{n}$ if you see trig functions $\sin(n\theta)$ and $\cos(n\theta)$.

You may need to multiply by $\dfrac{\theta}{\theta}$ when the number of factors $\sin(n\theta)$ exceeds the number of factors θ or multiply by conjugate $\dfrac{1+\cos(n\theta)}{1+\cos(n\theta)}$ to convert $1-\cos(n\theta)$ to a sine function.

You may need to use identities such as

$$\tan(u) = \frac{\sin(u)}{\cos(u)} \qquad \csc(u) = \frac{1}{\sin(u)} \qquad \text{and} \qquad 1-\cos^2(u) = \sin^2(u)$$

Inverse Trig Derivatives:

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$$

Group Work Portion of the Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

GroupWork Rubrics day 2:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

1. **Background Story:** The position function of a moving body indicates where that body is at time t. If body is traveling in one direction, the total distance traveled during time $[t_{\text{start}}, t_{\text{end}}]$ will be $|s(t_{\text{end}}) - s(t_{\text{start}})|$. But if the direction of travel changes, the total distance is the sum of distances traveled in each direction.

   Questions: At time t seconds, the position of a body moving along the x-axis is

   $$s(t) = 2t^3 - 15t^2 + 24t \quad \text{meters}$$

   (A) Find the equation of velocity and acceleration of the body.

   (B) Find the acceleration of the body each time the velocity is zero.

   (C) Sketch the graph of the velocity and acceleration of the body.

   In the next two parts we are trying to compute the total distance traveled during the time interval $[0, 2]$.

   (D) Use the graph of $v(t)$ to find an interval from $t = 0$ to $t = 2$ when the object is traveling in the positive direction and an interval(s) from $t = 0$ to $t = 2$ when it traveling in the negative direction.

   (E) Find the total distance traveled by the body from $t = 0$ to $t = 2$. *(Compute the total distance traveled using formula $|s(t_{end}) - s(t_{start})|$ on each of the intervals in Part (D). Then add the two total distances.)*

   Click https://ggbm.at/ycjpbnam for a visual explanation of Total distance.

2. **Background Story:** Use the methods described in the beginning of the worksheet to evaluate the following trig limits. There are multiple videos available in the Sections 2.6 and 3.6 working through similar these limits.

   Questions: Evaluate the **limits**.

   (i) $\displaystyle\lim_{x\to 0} \frac{1-\cos(2x)}{\sin^2(3x)}$

   (ii) $\displaystyle\lim_{x\to 0} \frac{\sin(16x)}{\tan(4x)}$.

3. **Background Story:** This is an implicit differentiation problem involving trig functions.

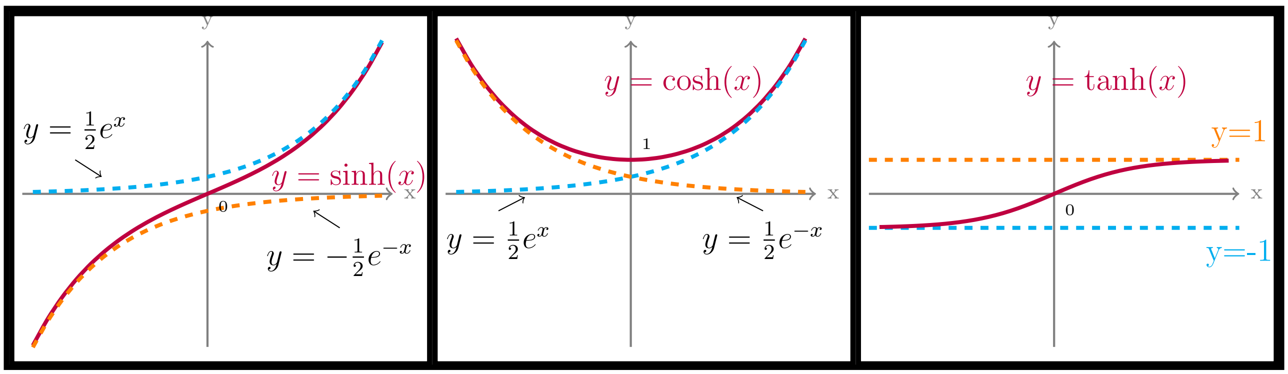
   Questions: Find $\dfrac{dz}{dx}$: $xz = \arctan(z^2)$

4. **Background Story:** Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematical modeling that they have earned their own special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason the are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, **hyperbolic tangent**, etc. The purpose of this problem is to make sense of the relationship between hyperbolic functions and their derivatives.

Questions: The purpose of these questions is for you to familiarize yourself with the concept. Verify both parts of Item (A) and then choose one or two parts from Item (B) to verify.

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)}$$



(A) Verify each of the following hyperbolic identities by replacing the hyperbolic functions on the left of equal sign with the above definition and then simplifying to get to right of equal sign:

(i) $\cosh^2(x) - \sinh^2(x) = 1$

(ii) $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

(B) Verify the hyperbolic derivatives: (Hint: Use Parts (i) and (ii) to Show Part (iii).)

(i) $\dfrac{d}{dx}(\sinh(x)) = \cosh(x)$

(ii) $\dfrac{d}{dx}(\cosh(x)) = \sinh(x)$

(iii) $\dfrac{d}{dx}(\tanh(x)) = \operatorname{sech}^2(x)$

(iv) $\dfrac{d}{dx}(\operatorname{sech}(x)) = -\operatorname{sech}(x)\tanh(x)$

(v) $\dfrac{d}{dx}(\operatorname{csch}(x)) = -\operatorname{csch}(x)\coth(x)$

(vi) $\dfrac{d}{dx}(\coth(x)) = -\operatorname{csch}^2(x)$

GroupWork Rubrics:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

Individual Portion of the Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2^{nd} lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

5. **Background Story:** Use the arc derivative rule on the first page and a chain rule to compute.

   Questions: *(1 point)* Evaluate $\frac{d}{dx}\big(\arctan(5x^3)\big)$

6. **Background Story:** One of the reason we emphasize the trig limits is the over skills that you will learn in simplifying expressions and dealing with limits after that. Those skills will come handy in Calculus II. Here you may need to convert $1 - \cos(5x)$ using a conjugate, multiply by $\frac{5}{5}$ and simplify.

   Questions: *(1 point)* Evaluate $\lim_{x\to 0} \frac{x\sin(5x)}{(1-\cos(5x))}$

7. *(2 points)* Find $\dfrac{dy}{dx}$: $y = \arcsin\left(e^{-\sqrt{x}}\right)$

8. **Background Story:** There may be multiple steps in computing a trig limit of the form $\frac{0}{0}$. Here is an example.

   Questions: *(3 points)* Evaluate $\lim_{x\to 0} \frac{\sin^2(3x)}{\tan^2(4x)}$. (Do not use L'hospital Rule.)