Worksheet 15: Section 4.5

I said: "Here are some errors that may show deeper misunderstanding of trig functions. ① If you write ~~(sin)~~, it may mean that you don't know sine function has an input. ② If you use a ~~ratio of sides~~ as an input of trig function instead of an angle, it may show you don't know that input of trig functions is an angle and output of them is a ratio of sides. ③ If you are trying to connect two sides and an angle using ~~Pythagorean theorem~~, you may have to review both Pythagorean and Trig functions.
Additionally, make sure to know trig values of common angles on unit circle and application of your trig and arc functions. "

L'Hôpital's Rule

If f and g are differentiable near $x = a$ and either

(i) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, or
(ii) $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$,

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This is, LHR can be used when the limit is of the $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

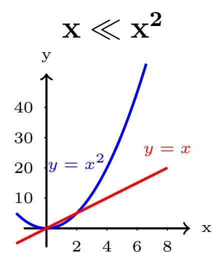
The rule applies equally for one-sided limits.

Comparing Growth Of Functions

We say that f grows faster than g if

(i) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, or equivalently

(ii) $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.



(Notation: $g \ll f$.)

Some important cases (that can all be checked using L'Hôpital's Rule):

(a) $x^n \ll e^x$ for all n.
(b) In fact, $x^n \ll a^x$ for all n and all $a > 1$.
(c) $\log_a(x) \ll x^n$ for all n and all $a > 0$.

Some examples:

Logarithm with LHR	Form 0^0 or ∞^0 or 1^∞			Take ln and convert to $\frac{0}{0}$ or $\frac{\infty}{\infty}$
LHR may be only method	$\frac{\text{exponential}}{\text{polynomial}}$ or $\frac{\text{polynomial}}{\text{exponential}}$ Form $\frac{\infty}{\infty}$ or $\frac{0}{0}$	$\frac{\text{log}}{\text{polynomial}}$ or $\frac{\text{polynomial}}{\text{log}}$ Form $\frac{\infty}{\infty}$ or $\frac{0}{0}$		Exponential: # applications= order of polynomial Log: Simplify after the first application
LHR may perform better	$\frac{\text{polynomial}}{\text{polynomial}}$ form $\frac{0}{0}$	Containing $\frac{\sin(mx)}{\sin(nx)}$ form $\frac{0}{0}$		Polynomial: Apply until not form $\frac{0}{0}$ Sine: Separate before applying.
LHR may perform worse	$\frac{\text{polynomial}}{\text{polynomial}}$ form $\frac{\infty}{\infty}$	$\sqrt{\text{polynomial}} - \sqrt{\text{polynomial}}$ Form $\infty - \infty$		
LHR may be useless	$\frac{\sqrt{\text{polynomial of degree } 2n}}{\text{polynomial of degree } n}$	$\frac{\text{polynomial of degree } n}{\sqrt{\text{polynomial of degree } 2n}}$		

Indeterminate Forms Involving Exponents

For limits of the forms 1^∞, 0^0, or ∞^0, the key step is to rewrite

$$\lim_{x\to c} \blacksquare \quad = \quad \lim_{x\to c} e^{\ln(\blacksquare)} \quad = \quad \lim_{x\to c} \exp(\ln(\blacksquare)) \quad = \quad \exp\left(\lim_{x\to c} \ln(\blacksquare)\right).$$

- The second equality is valid by the Limit Laws (see §2.4) because e^x is continuous everywhere.
- The "exp" notation avoids humongous expressions in superscript.

This technique changes the limit into one of form $0 \cdot \infty$:

$$\begin{array}{ccccccc} 1^\infty & \to & e^{\ln|1^\infty|} & \to & e^{\infty\cdot\ln(1)} & \searrow & \\ 0^0 & \to & e^{\ln\left|0^0\right|} & \to & e^{0\cdot\ln(0)} & \to & e^{0\cdot\infty} \\ \infty^0 & \to & e^{\ln\left|\infty^0\right|} & \to & e^{0\cdot\ln(\infty)} & \nearrow & \end{array}$$

Group Work Portion of the Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. **Background Story:** Compare LHR of the form $\frac{\infty}{\infty}$ and the form $\frac{0}{0}$.

   Questions:

   (A) Evaluate $\lim\limits_{x \to 0} \frac{e^{7x} - 1 - 7x}{x^2}$.

   (B) Evaluate $\lim\limits_{x \to \infty} \frac{e^{7x} - 1 - 7x}{x^2}$.

2. **Background Story:** Either of the following can define an exponential value. Compute those exponential values.

   Questions:

   (a) Let a be a constant. Evaluate $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$.

   (b) Let a be a constant. Evaluate $\lim_{x\to 0^+}(1+x)^{a/x}$.

3. **Background Story:** In calculus II, you will compute limits as $n \to \infty$ of complex expressions. That requires knowing how to group factors together so the limit exists and knowing which expression grows faster in each group.

   (A) Can you use LHR to evaluate the limit $\lim_{x\to\infty} \frac{e^x}{x^n}$, for any whole number $n > 0$, using LHR? How many times? What is the value of the limit?

   (B) Which functions grows faster? (Look up the definition on the first page.)

   (C) Using the growth comparison in Part (B), what is $\lim_{x\to\infty} \frac{e^x + x^3}{x^5 + 4x^3}$?

   (D) Show that $x^3 << e^{x^2}$. (Look up the definition on the first page.)

GroupWork Rubrics:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

Individual Portion of the Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Please show your work and leave comments; your grade is based upon the steps you take completing a problem, not solely on the final answer.

4. *(3.5 points)* Evaluate $\lim_{x \to 0^+} \left(1 + 3x\right)^{\frac{1}{\sin(x)}}$.