Week 3-Lab 2: Worksheet 6: Sections 2.8 3.1 and 3.2

They said: "I feel like I understand the material but I still cannot solve the problems." I said: "This is how it goes: You memorize $\Longrightarrow$ you understand $\Longrightarrow$ you analyze $\Longrightarrow$ you evaluate $\Longrightarrow$ you create. Be patient! College mathematics is meant to be intense but accessible to all through hard work. If you are missing some prerequisite, you have to make a bigger time commitment. Keep going! Your positive altitude can help you a lot!" I added: "If you are suffering in other parts of your life. We understand and we try to help as much as we can."

They asked: "Does it get easier?" I said: "have you ever done any rigorous exercise when your body gets really sore?" They said: "Yes!?" I said: "If you continue exercising within the next few days, your body slowly gets used to it and builds muscles and you feel more comfortable if you do vigorous exercise consistently. Build those brain muscles!"

The Intermediate Value Theorem:

If f is continuous on the interval $[a, b]$, then for every value M between $f(a)$ and $f(b)$, there exists at least one value c in (a, b) such that $f(c) = M$.

Existence of Zeros:

If f is continuous on $[a, b]$ and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then f has a zero in (a, b).

This rule is a consequence of the Intermediate Value Theorem.

We can use this rule to approximate zeros, by repeatedly bisecting the interval. (cutting it in half)

With each bisection, we check the sign of $f(x)$ at the midpoint to decide which half to look at next.

Differentiability:

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

Detection of the discontinuity through graphical approach:

- Discontinuity, ① Jump ② Hole ③Vertical Asymptote

  ④ Non-decaying infinite oscillation
- Corners and cusps
- Vertical tangent lines
- Oscillating secant lines.

Derivative Rules:

Suppose that f and g are differentiable functions and c is a constant.

Derivative of Constants

$$\frac{d}{dx}(c) = 0$$

Sum and Difference Rules

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

Constant Multiple Rule

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))$$

The Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Notations for the Derivative

Let $y = f(x)$ be a differentiable function.

	Derivative Function	**Derivative at $x = a$**
Lagrange	$y'(x) = f'(x)$	$f'(a)$
Leibniz	$\frac{d}{dx}(f(x)) = \frac{dy}{dx} = \frac{df}{dx}$	$\frac{d}{dx}(f(x))\Big\rvert_{x=a} = \frac{dy}{dx}\Big\rvert_{x=a}$

Group Work Portion of the Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. **Background Story:** The method of bisection to find a root has multiple parts. First you find an interval which contains a root. Then you bisect the interval multiple times while checking if you have found the root within a margin of error.

   Questions: Consider $f(x) = x^3 - 7x - 3$

   (A) Using the Intermediate Value Theorem, show that $f(x) = x^3 - 7x - 3$ has a root in the interval $[2, 3]$.

   (B) Apply the bisection method to obtain an interval of length $\frac{1}{16}$ containing a root from inside the interval $[2, 3]$.

   (C) Give the root accurate to one decimal place.

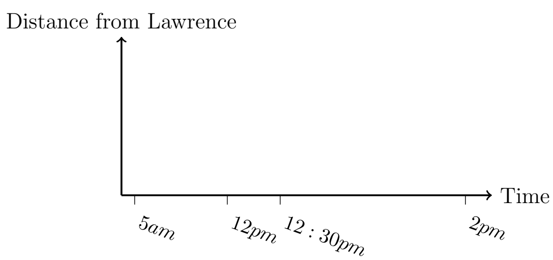
   (D) How many more bisection do you think you need to find the root accurate to two decimal places? (This does not have to be 100% accurate. Give a ballpark figure?)

2. **Background Story: The runner's paradox:** Here is an example very close to what we did in lecture. The parts are supposed to answer most of your questions.

Questions: On Tuesday, Lynne starts her trip on Interstate 70 at 5am from Exit 388 in Lawrence. She arrives at her destination and exits Interstate 70 through Exit 279 in Denver at 2pm the same day. She returns to Lawrence, on Friday, by entering Interstate 70 through Exit 279 at 5am and arriving at Exit 388 at 2pm.

(A) Let $T(t)$ be the function of Lynne's distance from Exit 388, at time t, on **Tuesday**, and $F(t)$ be the function of Lynne's distance from Exit 388, at time t, on **Friday**. Are T and F continuous functions?

(B) On Tuesday, Lynne stops at a gas station and sits in a fast food restaurant at noon for 30 minutes. Describe the graph of T during that 30 minutes as precisely as possible.

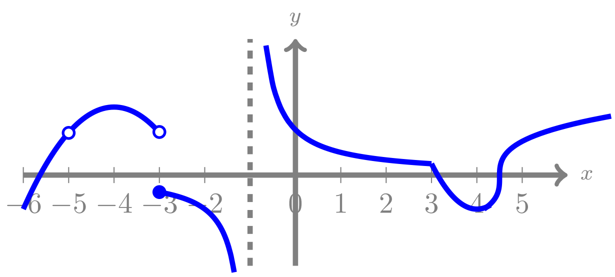


(C) Using the Intermediate Value Theorem, show that there is a point on Interstate 70 that Lynne will cross at **exactly the same time** of day on both days.

(D) Does any of your arguments included the Lynne's instantaneous velocity? Does any of your arguments included Lynne's average velocity?

3. **Background Story:** Recognition non-differentiable points from the graph of a function involves knowing ① Points of discontinuity ② Corners and cusps ③ Vertical tangent lines ④ Oscillating secant lines. Meanwhile recognize that all discontinuous functions are non-differentiable but not all non-differentiable functions are non-continuous.

   Questions: The graph of f is shown. State, with reasons, the x-values at which f is **not** differentiable.



4. **Background Story:** This explain why we look at the differentiability of piecewise defined functions. In this question notice that one condition is being connected and the other one is being a smooth ride. That is, one condition relates to

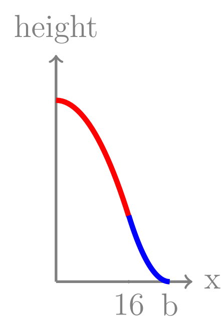
$$\lim_{x\to 16^-} f(x) = \lim_{x\to 16^+} f(x) = f(x)$$

and the other relates to

$$\lim_{x\to 16^-} f'(x) = \lim_{x\to 16^+} f'(x).$$

Use the information you learned in Worksheet 4, Questions 3 and 5 to to find these equations.

Questions: You are designing part of a roller coaster with two pieces of rails that are part of two parabolas. The first piece starts at the top of the track at height 40 meter and ends after it travels a horizontal distance 16 meters. The height of the first piece is modeled by equation $y = 40 - 0.1x^2$ as a function of horizontal distance, x, in meters. The next piece starts at the end of the first piece and ends on the ground level as seen in the picture. The height of the second piece is modeled by equation $y = a(x-b)^2$ where a and b are constants.



(A) Use the given formula for each piece to fill in the blanks and write the piecewise defined function that models your design.

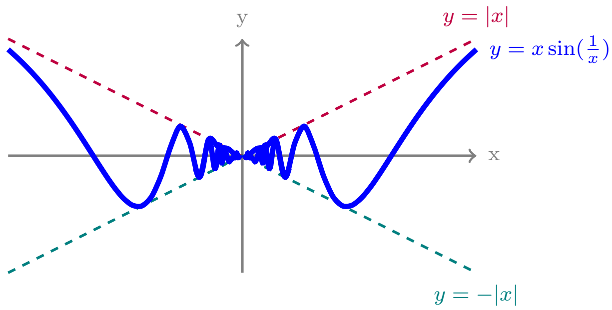
$$f(x) = \begin{cases} \boxed{} & \text{if } 0 \le x \le 16 \\ \boxed{} & \text{if } 16 < x \le b \end{cases}$$

(B) The first requirement for this design is that the two pieces are connected at horizontal distance $x = 16$ meters. What does that mean in mathematical terms for function f? (So far we have studied continuity and differentiability.) Write this condition as an equation of a and b.

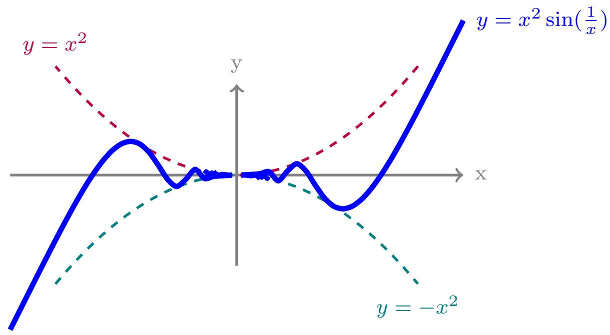
(C) The second requirement for this design is that the track is smooth everywhere including at the connection $x = 16$ m. Again choose either differentiability or continuity for this condition. Write this condition in term of an equation of a and b.

(D) Solve the system of two equations and two variables given in Parts (B) and (C) to find a and b.

5. **Background Story:** In class, we learned that $f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous, by the Squeeze Theorem, but not differentiable at $x = 0$.



In this work, we consider the function $g(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.



Questions:

(A) Form the quotient $\dfrac{g(x) - g(0)}{x - 0}$ for $x \neq 0$ and simplify.

(B) Find the $\lim\limits_{x \to 0} \dfrac{g(x) - g(0)}{x - 0}$. (Hint: You will need to use the Squeeze Theorem. Is this limit "the limit definition of derivative"? At what point?)

(C) Is g differentiable at $x = 0$? If yes, what is $g'(0)$?

Click: https://ggbm.at/quzevdeg for a visual explanation.

Video: https://www.youtube.com/watch?v=3zc2k-ee8UE

GroupWork Rubrics:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

Individual Portion of the Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2^{nd} lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

GroupWork Rubrics day 2:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

6. *(2 points)* **Background Story:** Now that you know some applications, practice the method to creating differentiable piecewise-defined functions. Remember for a function to be differentiable, they need to also be continuous. So set two conditions for the cut-off points.

   Questions: For which values a and b is the function h differentiable everywhere?

$$h(x) = \begin{cases} ax^2 + 2 & x \leq 3 \\ x^2 - x + b & x > 3 \end{cases}$$

Video: https://mediahub.ku.edu/media/Math+125-Piecewise+defined.mp4/1_g7qrdt1p

7. **Background Story:** Lets practice the simple derivative rules. (Section 3.2)

Questions: Let f and g be two differentiable functions such that

$f(4) = 8$	$f'(4) = -1$	$f(17) = 11$	$f'(17) = -5$
$g(4) = 5$	$g'(4) = -6$	$g(17) = 3$	$g'(17) = 2$

Evaluate the following derivatives or say it can not be computed.

(A) *(0.75 point)* $\frac{d}{dx}\left(15f(x) - g(x)\right)\Big|_{x=17}$

(B) *(0.75 point)* $\frac{d}{dx}\left(\frac{x^2 + f(x)}{-15}\right)\Big|_{x=4}$.

8. *(3.5 points)* **Background Story:** Lets start getting ready for Gateway exams. These are the questions from the first pool of questions focusing on Power Rule. Take the derivative of each function. Only five questions out of ten will be graded.

(A) $f(x) = -\frac{2}{3}x^3 + x^2 + 12x + 9$

(F) $f(x) = -x^{\frac{3}{4}} + x^{-\frac{3}{4}}$

(B) $f(x) = x^{\frac{7}{3}} - 8x^{\frac{4}{3}} + 56$

(G) $f(t) = 2t^3 + 6t - \frac{4}{t^2}$

(C) $g(t) = t^{\frac{2}{3}} - t^{-\frac{1}{4}} + \pi$

(H) $f(x) = x^{\frac{5}{4}} - 10x^{\frac{1}{4}} + 1$

(D) $f(x) = \frac{2}{3}x^{\frac{3}{2}} - \left(\sqrt[3]{4}\right) x + \frac{2}{x^2}$

(I) $f(x) = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$

(E) $h(r) = 3r^2 + 4r + \frac{1}{r}$

(J) $p(x) = 16x^3 + \frac{17}{\sqrt{x}} - 10x^{3.1416} + \pi^2$

Videos for these Gateway questions:

1. Question 1: https://mediahub.ku.edu/media/MATH+125+-+001/1_x3turvix
2. Question 2: https://mediahub.ku.edu/media/MATH+125+-+002/1_kxyufjsp
3. Question 3: https://mediahub.ku.edu/media/MATH+125+-+003/1_3r7v1zba
4. Question 4: https://mediahub.ku.edu/media/MATH+125+-+004/1_7hm3dac3
5. Question 5: https://mediahub.ku.edu/media/MATH+125+-+005/1_ene5d09j
6. Question 6: https://mediahub.ku.edu/media/MATH+125+-+006/1_tsb2sbg4
7. Question 7: https://mediahub.ku.edu/media/MATH+125+-+007/1_0xek7qgs
8. Question 8: https://mediahub.ku.edu/media/MATH+125+-+008/1_ip19udzl
9. Question 9: https://mediahub.ku.edu/media/MATH+125+-+009/1_cyu0zs11
10. Question 10: https://mediahub.ku.edu/media/MATH+125+-+010/1_g23etq5y