Week 10-Lab 1: Worksheet 13: Sections 15.6

I said:" Did you know that feeling of learning and actual learning are quite different things?!"
https://www.pnas.org/doi/10.1073/pnas.1821936116

Jacobians and the Change-Of-Variable Formula

The **Jacobian** of the transformation $\vec{\mathbf{G}}(u,v) = (x(u,v), y(u,v))$ is defined as

$$\left|\text{Jac}(\vec{\mathbf{G}})\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\| = abs\left(\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right) = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|$$

The absolute value of Jacobian is the area scaling factor for G. That is, the scaling factor is $\|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|$.

Double Integration with Change of Variables Let $\vec{\mathbf{G}}(u,v) = (x(u,v), y(u,v))$ be a transformation, and let $\vec{\mathbf{G}}(S) = R$. Then

$$\iint_R f(x,y)\,dA_{xy} = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$$

Using this formula does not typically evaluate the integral immediately, but it enables you to convert it into an integral over a geometrically simpler region.

A Useful Fact About Jacobians

If $\vec{\mathbf{F}}$ is the inverse transformation of $\vec{\mathbf{G}}$, that is,

$$\vec{\mathbf{F}}(x,y) = (u,v) \quad \text{and} \quad \vec{\mathbf{G}}(u,v) = (x,y),$$

then

$$\text{Jac}(F) = \text{Jac}(G)^{-1}.$$

This fact is suggested by the notation:

$$\text{Jac}(\vec{\mathbf{G}}) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\text{Jac}(\vec{\mathbf{F}})}$$

Change of Variables for Triple Integrals

Let R be a region in $\mathbb{R}^3$ with coordinates x, y, z.
Let S be a region in $\mathbb{R}^3$ with coordinates u, v, w.
Let G be a transformation that maps S to R:

$$\vec{\mathbf{G}}(u,v,w) = \big(x(u,v,w),\ y(u,v,w),\ z(u,v,w)\big).$$

Then

$$\iiint_R f(x,y,z)\,dV_{xyz} = \iiint_S f\big(\vec{\mathbf{G}}(u,v,w)\big) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV_{uvw}$$

where

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\text{Jac}(\vec{\mathbf{G}})\right| = abs\left(\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}\right).$$

1. Let G^{-1} be the transformation defined by the equations

$$u = 8x - 2y \qquad\qquad v = 4x + 0.25y$$

(A) Find G by solving the system of equations for x and y in terms of u and v.

(B) Find the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

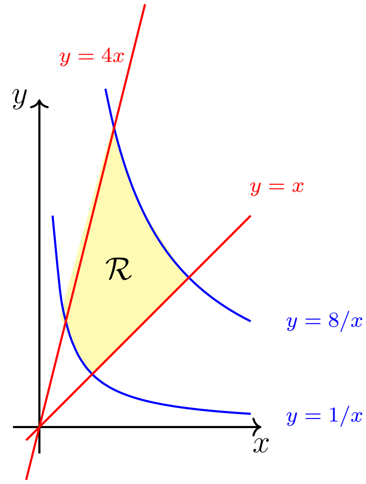
(C) Find the image under G^{-1} of the region in the xy-plane bounded by the x-axis, the y-axis, and $x + y = 1$. Sketch the transformed region in the uv-plane. [1]

https://youtu.be/z5z08DvmgIE

[1]Remember: for linear transformations, you can transform two points on a line segment and then draw the transformed line segment. "Linear transformations transform line to lines."

2. Let $\mathcal{R}$ be the region in the first quadrant bounded by $xy = 1$, $xy = 8$, $y = x$, and $y = 4x$.

(A) Find a transformation G such that the inverse region $G^{-1}(\mathcal{R})$ is rectangular.



(B) What is the $|\text{Jac}(G)|$?

(C) Set up $\displaystyle\iint_{\mathcal{R}} f(x, y)\, dA_{xy}$ using the change of variable in Part (A).

https://youtu.be/fsOIlfmwkSI and https://youtu.be/3ecpMeCWeuI

3. **Background Story:** First try a linear change of variables for triple integrals to transform the ellipsoid to a sphere. Then try a spherical transformation to compute the values for the sphere.

   (A) Find a change of variable that transforms the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ into the sphere

   $$x^2 + y^2 + z^2 = 1.$$

   (B) Find the Jacobian of the the transformation in Part (A).

   (C) Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the change of variable in Part (A).

4. **Background Story:** The polar, cylindrical and spherical conversions are transformations.

$$\begin{aligned}
(x, y) &= G(r, \theta) &&= (r\cos(\theta), r\sin(\theta)) && G : \mathbb{R}^2 \to \mathbb{R}^2 \\
(x, y, z) &= H(r, \theta, z) &&= (r\cos(\theta), r\sin(\theta), z) && H : \mathbb{R}^3 \to \mathbb{R}^3 \\
(x, y, z) &= I(\rho, \phi, \theta) &&= (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) && I : \mathbb{R}^3 \to \mathbb{R}^3
\end{aligned}$$

(A) Compute the Jacobian of polar transformation. $|\text{Jac}(G)| =$

(B) Compute the Jacobian of cylindrical transformation. $|\text{Jac}(H)| =$

(C) Compute the Jacobian of spherical transformation. $|\text{Jac}(I)| =$

GroupWork Rubrics:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

Individual Portion of Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2^{nd} lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

5. **Background Story:** Sometimes a transformation can simplify the domain and simplify the integrand at the same time.

   Questions: *(3.5 points)* Evaluate the integral using a linear change of variables.

$$\iint_{\mathcal{R}} (x+y)e^{x^2-y^2}\, dA$$

   where $\mathcal{R}$ is the polygon with vertices $(2,0)$, $(0,2)$, $(-2,0)$, and $(0,-2)$.[2]

https://youtu.be/J5fAfbkoBjw and https://youtu.be/rL-bacMf4kE

[2]Remember that for linear transformations, you can only transform two points on a line segment and then draw the transformed line segment. Use this fact to draw the region.