Week 10-Lab 2: Worksheet 14: Section 15.4 and 15.6

Triple Integrals in Cylindrical and Spherical Coordinates

Let $G(u, v, w) = (x, y, z)$ be a transformation with $G(S) = R$. Then

$$\iiint_R f(x, y, z)\, dV_{xyz} = \iiint_S f(G(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dV_{uvw}.$$

Transformations from cylindrical or spherical to rectangular coordinates:

$$G(r, \theta, z) = (r\cos(\theta),\ r\sin(\theta),\ z) \qquad \text{(cylindrical)}$$

$$H(\rho, \phi, \theta) = (\rho\sin(\phi)\cos(\theta),\ \rho\sin(\phi)\sin(\theta),\ \rho\cos(\phi)) \qquad \text{(spherical)}$$

$$\mathrm{Jac}(G) = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \boxed{r}$$

$$\mathrm{Jac}(H) = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{vmatrix} = \boxed{\rho^2\sin(\phi)}$$

Triple Integrals in Cylindrical Coordinates

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \iiint_R f(G(r, \theta, z))\ \boxed{r}\ dr\, d\theta\, dz$$

Triple Integrals in Spherical Coordinates

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \iiint_R f(H(\rho, \phi, \theta))\ \boxed{\rho^2\sin(\phi)}\ d\rho\, d\phi\, d\theta$$

where $G(r, \theta, z) = (x, y, z)$ and $H(\rho, \phi, \theta) = (x, y, z)$.

Or, for short:

Volume Element in $\mathbb{R}^3$

$$dV \ = \ dx\, dy\, dz \ = \ r\, dr\, d\theta\, dz \ = \ \rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta.$$

Moments and Center of Mass: The **mass** of a lamina D with mass density ρ is given by formula $m = \iint_D \rho(x, y) dA$.

The **moments** M_x and M_y of a lamina measure how balanced it is with respect to the x- and y-axes.

$$M_x = \iint_D y\delta(x, y)\, dA \qquad\qquad M_y = \iint_D x\delta(x, y)\, dA$$

where D is the region occupied by the lamina.

The coordinates $(\overline{x}, \overline{y})$ of the center of mass are

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\delta(x, y)\, dA$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\delta(x, y)\, dA$$

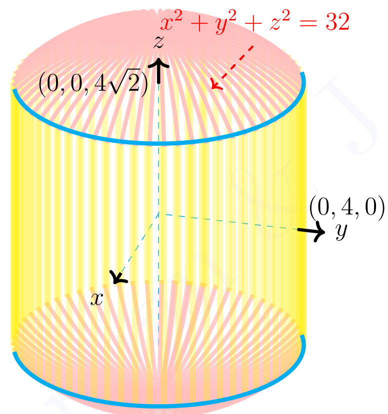
Group Work Portion of Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. **Background Story:** Compare different methods for finding the same integral. Here the set up is more important than the computation.

   Question(s): Let $\mathcal{S}$ be the solid inside both $x^2 + y^2 = 16$ and $x^2 + y^2 + z^2 = 32$.



$$\iiint_{\mathcal{S}} z \, dV$$

   (A) Write an iterated integral for the triple integral in rectangular coordinates.

   (B) Write an iterated integral for the triple integral in cylindrical coordinates.

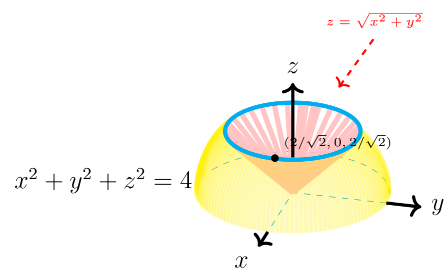
   (C) Write an iterated integral for the triple integral in spherical coordinates.

   (D) Evaluate the triple integral using the iterated integral from (A), (B), or (C).

2. **Background Story:** Now that we covered section 15.5, we can compute the center of density for this lamina.

   Question(s): Find the mass and center of mass of the lamina that occupies the region $\mathcal{D}$ and density $\rho(x, y) = y$ where $\mathcal{D}$ is bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 2$.

3. Set up, but <u>do **NOT** evaluate</u>, a triple integral in spherical coordinates representing the volume of the solid $\mathbb{S}$ above xy-plane and entrapped inside the sphere $x^2 + y^2 + z^2 = 4$ and below the cone $z = \sqrt{x^2 + y^2}$.



4. **Background Story:** The simplest form of a normal distribution is known as the standard normal distribution or unit normal distribution follows the probability distribution $f(x) = Ce^{-x^2/2}$ where $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$. To compute C follow the steps. **This question is recommended but optional.**[1]

   Question(s):

   (A) Compute $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C^2 e^{-\frac{x^2}{2}-\frac{y^2}{2}}\, dx\, dy$, using polar transformation. (This will contain C.)

   (B) Separate $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C^2 e^{-\frac{x^2}{2}-\frac{y^2}{2}}\, dx\, dy$ into a product of two single integrals of the same value. (Separate by functions of x only and y only.)

   (C) Use $\displaystyle\int_{-\infty}^{\infty} Ce^{-x^2/2}\, dx = 1$, in Part (B) to find a value for Part (A). Then solve for C.

[1] https://en.wikipedia.org/wiki/Normal_distribution

GroupWork Rubrics:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

Individual Portion of Worksheet

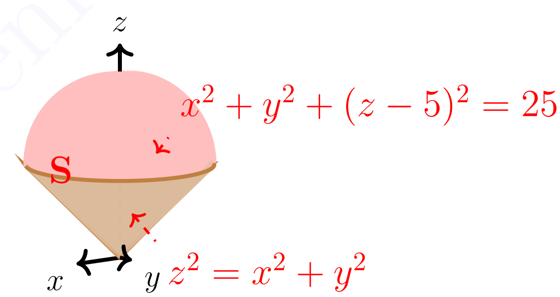
Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

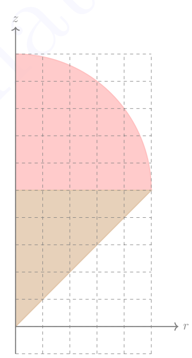
5. **Background Story:** Compare the order **dz dr dθ** to shell method by drawing arrow(s) representing height sample(s) with infinitesimal thickness (Δr) and compare the order **dr dz dθ** to washer method by drawing arrow(s) representing radii sample(s) with infinitesimal thickness (Δz) in rz-plane.

   Question(s):

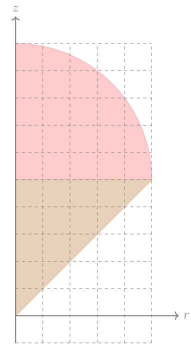
   The solid $\mathcal{S}$ above the surfaces $z^2 = x^2 + y^2$ and below $x^2 + y^2 + (z-5)^2 = 25$.



   (A) (1.5 points) Set up, but do **NOT** evaluate, a triple integral in cylindrical coordinates representing the volume of $\mathcal{S}$ in **dz dr dθ** order. (Comparable to Shell Method.)



(B) (2 points) Set up, but do **NOT** evaluate, a triple integral in cylindrical coordinates representing the volume of $\mathcal{S}$ in $\mathbf{dr\,dz\,d}\theta$ order. (Comparable to Washer Method.)



https://youtu.be/W0nQTT2oF4U