Week 11-Lab 2: Worksheet 16: Sections 13.3 & 16.1

They said: "When will I use calculus in engineering?" I said: "My first computing arc length of a helix was when I was a freshman and starting my second semester of college. In Multivariable calculus, we were covering limits and continuity. Unrelated to school, I was put on the spot in front of three experienced civil engineers, one of whom was about to catch a flight to another city to measure the length of a spiral so they can evaluate the cost assessments by contractors. They had measurements of height of each teeth of the spiral and the radius of cylinder containing the spiral."

Arc Length Formula The arc length of the curve parametrized by $\vec{\mathbf{r}}(t)$ for $a \le t \le b$ is

$$\int_a^b \|\vec{\mathbf{r}}'(t)\|\, dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt.$$

The Arc Length Function The length of the portion of the curve $\vec{\mathbf{r}}$ over the interval $[a, t]$ is

$$S(t) = \int_a^t \|\vec{\mathbf{r}}'(\tau)\|\, d\tau$$

The letter S is reserved for arc length. (Here τ is just a dummy variable.)
Note that $S(t)$ is a <u>scalar</u> function of t

Arc Length Parametrization: ①Find the arc length functions. ②Solve for t in (the original parameter) in terms of S. ③ Replace the original parameter in the original parameterization by $t(s)$.

The Del Operator:

The **del** or **nabla** operator[1] ∇ is defined by $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.

Applying ∇ to a scalar function f gives its gradient:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The curl and divergence of a vector field can also be written in terms of ∇:

$$\operatorname{div}(\vec{\mathbf{F}}) = \nabla \cdot \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\operatorname{curl}(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle F_1, F_2, F_3 \rangle = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Conservative Vector Fields:

Let $f(x, y, z)$ be a scalar-valued function. Its gradient is a vector field:

$$\vec{\mathbf{F}} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The function f is called a **(scalar) potential function** for $\vec{\mathbf{F}}$.

A vector field is called **conservative** if it has a potential function,

Conservative fields occur naturally in physics, as force fields in which energy is conserved.

Theorem:

If $\vec{\mathbf{F}}$ is conservative on an open connected domain $\mathcal{R}$, then any two potential functions of $\vec{\mathbf{F}}$ differ by a constant.

Theorem:

If $\vec{\mathbf{F}}$ is a conservative vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\text{Curl}(\vec{\mathbf{F}}) = \vec{0}$.

Finding Scalar Potentials The process for finding scalar potential functions is essentially antidifferentiation, but with a twist.

For $\vec{\mathbf{F}}(x, y) = \langle F_1(x, y),\ F_2(x, y)\rangle$:

1. Find the indefinite integrals $\int F_1(x, y)\, dx$ and $\int F_2(x, y)\, dy$.
   - **The constants of integration are $c_1(y)$ and $c_2(x)$ respectively** (instead of the usual "$+C$"), because if $\frac{\partial}{\partial x}(f(x, y)) = F_1$ then $\frac{\partial}{\partial x}(f(x, y) + c_1(y)) = F_1$ as well.
2. "Match up the pieces" to determine $f(x, y)$.

For $\vec{\mathbf{F}}(x, y, z) = \langle F_1(x, y, z),\ F_2(x, y, z),\ F_3(x, y, z)\rangle$:

1. Find the indefinite integrals $\int F_1\, dx$, $\int F_2\, dy$, and $\int F_3\, dz$.
   Constants of integration: $c_1(y, z)$, $c_2(x, z)$, $c_3(x, y)$.
2. "Match up the pieces" to determine $f(x, y, z)$.

Group Work Portion of Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. **Background Story:** To find the arc length parameterization, find any parameterization, find the arc length function, find inverse function of $s(t)$ to replace t in the original parametrization.

   Question(s):

   (A) Find a parameterization of the circle in the plane $y = 5$ with radius 7 and center $(2, 5, 3)$.

   (B) Circle can be presumed as the curve of an intersection of a cylinder and a plane. Give an example of such surfaces for this circle.

   (C) Find an arc length parameterization of the circle.

Video: https://youtu.be/a2cdLgQtF4c

2. **Background Story:** Helices show up in different area of science and engineering. They also are some of the few types of curves who have closed form arc length functions.

   Question(s):

   One method of creating a helix is drawing equidistant parallel line segments of the same length where each segment starts at the same edge of a rectangular page. Then rolling the page into a cylinder matching each end of line segment with beginning of the next line segment.
   https://www.geogebra.org/m/bnwtjnfa
   We want to create a helix using the above method where the parameterization of the helix is

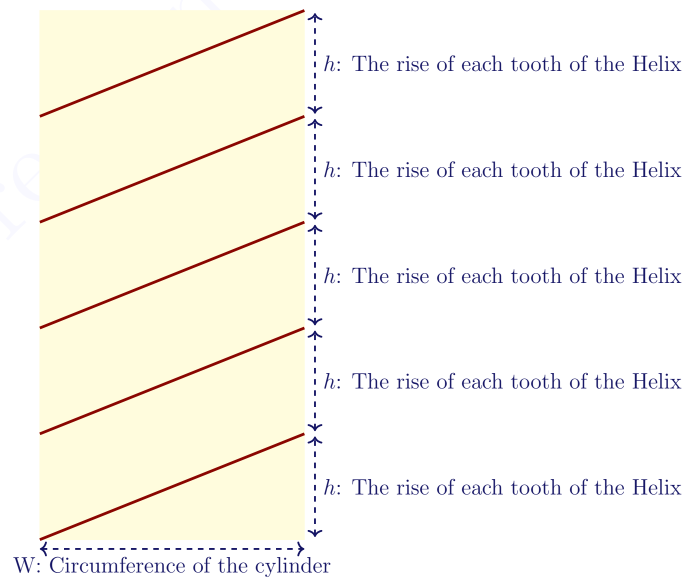
$$\vec{\mathbf{r}}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle \text{ for } 0 \le t \le 10\pi.$$

   (A) What is the rise of each tooth of the helix (The distance between the start points of two consecutive line segments)?

   (B) What is the minimum height of the rectangular page?

   (C) What is the minimum width of the rectangular page?

   (D) Use Pythagorean Theorem to compute the arc length of the Helix.



Rectangular page

3. **Background Story:** In many of the courses requiring multivariable Calculus as prerequisite, it is important to know: ① Whether curl, divergent and gradient are scalar or vector and whether they act on a scalar or a vector function. ② Meaning of curl, divergence and gradient.

   So we repeat these questions from before Midterm 1.

   (i) Fill in the blanks. The gradient takes a ________ function as an input and gives a ________ function as an output.

   (A) scalar, scalar
   (B) scalar, vector
   (C) vector, scalar
   (D) vector, vector

   (ii) Fill in the blanks. The gradient of $f(x, y, z)$ is the direction of ________ in ________.

   (A) greatest increase, z
   (B) greatest increase, f
   (C) greatest decrease, z
   (D) greatest decrease, f

4. **Background Story:** Learn before the lecture. Curl of $\vec{\mathbf{F}}$ is

$$\operatorname{curl}(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle F_1, F_2, F_3 \rangle = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

   or

$$\operatorname{curl}(\vec{\mathbf{F}}) = \left(\frac{\partial \vec{\mathbf{F}}_3}{\partial y} - \frac{\partial \vec{\mathbf{F}}_2}{\partial z} \right) \vec{\mathbf{i}} + \left(\frac{\partial \vec{\mathbf{F}}_1}{\partial z} - \frac{\partial \vec{\mathbf{F}}_3}{\partial x} \right) \vec{\mathbf{j}} + \left(\frac{\partial \vec{\mathbf{F}}_2}{\partial x} - \frac{\partial \vec{\mathbf{F}}_1}{\partial y} \right) \vec{\mathbf{k}}$$

   Where $\vec{\mathbf{F}} = \langle \vec{\mathbf{F}}_1, \vec{\mathbf{F}}_2, \vec{\mathbf{F}}_3 \rangle$.

   Question(s): Show $\operatorname{curl}(\nabla f) = \vec{0}$ when $f(x, y, z)$ is an arbitrary scalar function with continuous second derivatives.

5. **Background Story:** Learn before the lecture. Div of $\vec{\mathbf{F}}$ is

$$\operatorname{div}(\vec{\mathbf{F}}) = \nabla \cdot \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial \vec{\mathbf{F}}_1}{\partial x} + \frac{\partial \vec{\mathbf{F}}_2}{\partial y}, \frac{\partial \vec{\mathbf{F}}_3}{\partial z}$$

   Question(s): Show $\operatorname{div}(\operatorname{curl} \vec{\mathbf{F}}) = 0$ when $\vec{\mathbf{F}}(x, y, z)$ is an arbitrary vector function with continuous second derivatives.

GroupWork Rubrics:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

Individual Portion of Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

6. **Background Story:** Practice on your own.

   Question(s): Let $\vec{F}(x, y, z) = \langle \cos(xy), xz\sin(y), \sin(yz) \rangle$.

   (A) *(1.5 points)* Find curl($\vec{\mathbf{F}}$).

   (B) *(0.5 points)* Is $\vec{\mathbf{F}}$ a gradient vector field? Is $\vec{\mathbf{F}}$ a conservative vector field?

   (C) *(1.5 points)* Find Div($\vec{\mathbf{F}}$).